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POINTWISE ESTIMATES FOR SOLUTIONS TO PERTURBED HASEGAWA-MIMA EQUATIONS

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Abstract. In this article, we study pointwise estimates for solutions to the two-dimensional perturbed Hasegawa-Mima equation based on the analysis of the Green functions for the linearized system. The pointwise estimates not only exhibit the Huygen's principle but also give additional insight on the description of the evolution behavior of the solution.

1. Introduction

The simplest nonlinear model describing the time evolution of drift waves was derived by Hasegawa and Mima [7],

$$\partial_t(1-\Delta)u = -k\partial_{x_2}u + \{u,\Delta u\} + \nu\Delta^3 u, \quad x \in \mathbb{R}^3, \tag{1.1}$$

where u is the electrostatic potential fluctuation and n is the density fluctuation by assuming a Boltzmann relation $n \sim u$, $k = \partial_x \ln n_0$ with n_0 being the background particle density and ν is a positive number. This equation also arises in the context of Rossby waves in the atmospheres of rotating planets [14]. The beautiful structure behind the Hasegawa-Mima equation initiates a lot of mathematical investigations, see [1, 3, 6, 19], and the references therein.

In this article, the two-dimensional perturbed Hasegawa-Mima equation derived by Liang et al [10] is studied as a prototype,

$$\partial_t(u-\Delta u) + k\partial_{x_2}u - \lambda(u-\Delta u) = -\{u,\Delta u\}, \quad (x_1,x_2) \in \mathbb{R}^2, \quad t>0, \tag{1.2}$$

with initial data

$$u(x_1,x_2,0) = u_0(x_1,x_2),$$

where k and λ are constants and $0<\lambda<1$, Δ is the 2D Laplacian, and $\{\cdot,\cdot\}$ denotes the Poisson bracket

$$\{h,g\} = (\partial_{x_1}h)(\partial_{x_2}g) - (\partial_{x_2}h)(\partial_{x_1}g).$$

The term $k\partial_{x_2}u$ describe a drift and $-\lambda(u-\Delta u)$ is the perturbation.

Equation (1.2) is the the simplest model for a two-dimensional turbulent system. It describes the time evolution of drift wave in plasma as mentioned above and the temporal evolution of geostrophic motion. Additionally, it can be reduced to the

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Euler equation for the incompressible homogeneous fluids if we eliminate the term u_t in (1.2). It is well known that Euler equation is a very hot topic in recent years and numerous papers were devoted to its study, such as [2, 13, 15, 18] and references therein.

In recent years, the perturbed Hasegawa-Mima equation has also received much attention, Many authors have contributed to the study of this equation. Guo and Han [5] have obtained the global solutions to the Cauchy problem. In 1998, Grauer obtained the energy estimate for the perturbed Hasegawa-Mima equation [4]. In [20], R. F. Zhang and her collaborator got the existence and the uniqueness of the global solution for the generalized Hasegawa-Mima equation. However, there is little information of the pointwise estimates for this equations to the best of our knowledge. Thus, the principle purpose of this paper is to present the pointwise estimates of solutions to (1.2). In fact, pointwise estimates play a crucial role in the description of the evolution of the solution, as it gives explicit expressions of the time asymptotic behavior of solutions and helps us get the global existence and the L^p estimates of solutions.

The main approach in this article is using the Green function method. We can present solutions of the Cauchy problem for the linearized system by the fundamental solution (i.e. Green function), which can also be applied to write a nonlinear system into an integral system. Actually, using the Green function to study the pointwise estimate for hyperbolic-parabolic systems becomes a very active field of research in recent years. Based on the pointwise estimates, we can not only obtain the decay rate of the solution due to the parabolicity of the system, but also find out the movement of the main part of the solution caused by the hyperbolicity. This method was first introduced by Liu and Zeng in [12] to get the pointwise estimates of solutions to the one dimensional quasilinear hyperbolic-parabolic systems of conservation laws. Later, Hoff and Zumbru ([8] and [9])employed this method to study the Navier-Stokes equation with viscosity.

Liu and Wang [11] also use this method to obtain the pointwise estimates of the solutions to the isentropic Navier-Stokes equations in odd dimensions. The classical Green function method is decomposing the Green function into three parts: lower frequency part, middle part and higher frequency part. Then by Taylor expansion, Fourier analysis and Kirchhoff formulae, we can obtain the pointwise estimates of these three parts, respectively. The pointwise estimates of the Green function shows that the large time behavior of Green function is dominated by the lower frequency waves while the higher frequency waves play a much more significant role in short time. Through the delicate analysis of the Green function, we obtain the explicit expression of the time-asymptotic behavior of the solutions to the linearized system.

Besides, the detailed analysis of the nonlinear terms is also an indispensable part for us because the pointwise estimates of the solutions to the nonlinear system are obtained by combining the analysis of the Green function and the nonlinear terms together. Since there are the third order derivatives in the nonlinear terms, how to deal with the nonlinear term of equation (1.2) becomes a key point in this paper. Usually, if there is only the first order derivative in the nonlinear term, according to the Duhamel principle the solution can be expressed as

$$u = G(\cdot, t) * u_0 - \int_0^t G(t-\tau) * N(\tau)\, \mathrm{d}\tau,$$

Then we can transfer the derivative from the nonlinear term to the Green function by using the integration by parts. Finally, using of pointwise estimates of the Green function we can get the decay estimates of the solutions. However, the order of the derivative of the nonlinear term in our equation is too high that the usual way fails. To overcome this difficulty, a lot of computations and analysis for the nonlinear term will be essential besides the transfer of the derivatives. The main result in this article is stated as follows.

Theorem 1.1. *Suppose $u_0 \in H^{s+l}(\mathbb{R}^2)$, $s = 2$, $l \leq 9$, $E = \max\{\|u_0\|_{H^{s+l}}, \|u_0\|_{L^1}\}$ and*

$$\|u_0\|_{H^{s+l}\cap L^1} \leq E, \quad |D^\alpha u_0| \leq E(1+|x|)^{-m'}, m' > 2, \quad |\alpha| \leq l, \tag{1.3}$$

with E sufficiently small, then when t is large enough, there exists positive constant C such that the solution $u(x,t)$ to (1.2) satisfies

$$|D_x^\alpha u(x,t)| \leq CE(1+t)^{-\frac{2+\nu(|\alpha|)}{2}} B_1(x-bt,t).$$

where $b = (0,k)$,

$$B_N(x,t) = (1+\frac{|x|^2}{1+t})^{-N},$$

$$\nu(|\alpha|) = \begin{cases} |\alpha|, & |\alpha| \leq l-6, \\ 0, & l-6 < |\alpha| \leq l-3. \end{cases}$$

The rest of this article is arranged as follows. The energy estimates and the existence of the global solution to (1.2) will be established in section 2. In section 3, we will get the analysis of the Green function $G(x,t)$ and the pointwise estimates of solutions to (1.2) will follow in section 4. For convenience, we list some important lemmas which are the vital tools for obtaining the pointwise estimates of the Green function as a appendix at the end of this paper.

2. Energy estimates

In this section, we obtain the existence global solutions and the energy estimates for (1.2). We assume that the initial data satisfies $\|u_0\|^2_{H^{s+l}} \leq E$, $s \geq 2$, where $0 < E \ll 1$.

Theorem 2.1. *Suppose the conditions of Theorem 1.1 are satisfied, then for any positive constants m, we have*

$$\|u\|_{H^m} + C_0 \int_0^t \|\nabla u\|_{H^m} \,\mathrm{d}\tau \leq C\|u_0\|_{H^m}. \tag{2.1}$$

where C_0, C are positive constants and independent of t.

Proof. In (1.2), taking the inner product in $L^2(\mathbb{R}^2)$ with $2u$, one obtains

$$(\partial_t(u-\Delta u), 2u) + (k\partial_{x_2}u, 2u) - (\lambda\Delta(u-\Delta u), 2u) = (-\{u, \Delta u\}, 2u),$$

since $(\{u,\Delta u\}, 2u) = 0$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|^2_{L^2} + \|\nabla u\|^2_{L^2}) + 2\lambda(\|\nabla u\|^2_{L^2} + \|\Delta u\|^2_{L^2}) = 0.$$

By integration, we obtain

$$\|u\|^2_{L^2} + \|\nabla u\|^2_{L^2} + 2\lambda\int_0^t (\|\nabla u\|^2_{L^2} + \|\Delta u\|^2_{L^2}) \,\mathrm{d}\tau = \|u_0\|^2_{L^2} + \|\nabla u_0\|^2_{L^2}. \tag{2.2}$$

Similarly, multiplying (1.2) by $-2\Delta u$ and integrating the result over $\mathbb{R}^2$ for the space variable (x_1, x_2), we have

$$\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + 2\lambda \int_0^t (\|\Delta u\|_{L^2}^2 + \|\nabla \Delta u\|_{L^2}^2)\,\mathrm{d}\tau = \|\nabla u_0\|_{L^2}^2 + \|\Delta u_0\|_{L^2}^2, \tag{2.3}$$

by noticing that

$$\int_{\mathbb{R}^2} \{u, \Delta u\}\Delta u \,\mathrm{d}x = 0, \quad t \geq 0,$$

(2.2) and (2.3) imply

$$\|u\|_{H^2}^2 + \int_0^t C_0 \|\nabla u\|_{H^2}^2 \,\mathrm{d}t \leq C\|u_0\|_{H^2}^2.$$

Now we aim to obtain an estimates for $\|u\|_{H^3}$. Differentiating (1.2) with respect to (x_1, x_2), one obtains

$$\partial_t \nabla(u - \Delta u) + k\partial_{x_2}\nabla u + \{u, \nabla\Delta u\} + \{\nabla u, \Delta u\} - \lambda\nabla\Delta(u - \Delta u) = 0. \tag{2.4}$$

Taking the inner product of $-2\nabla\Delta u$ and (2.4) in $L^2(\mathbb{R}^2)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\Delta u\|_{L^2}^2 + \|\nabla\Delta u\|_{L^2}^2) + 2\lambda(\|\nabla\Delta u\|_{L^2}^2 + \|\Delta^2 u\|_{L^2}^2) = 2\int_{\mathbb{R}}^2 \nabla\Delta u \cdot \{\nabla u, \Delta u\}\,\mathrm{d}x, \tag{2.5}$$

where we have used the following facts

$$\int \nabla\Delta u \cdot \partial_{x_2}\nabla u \,\mathrm{d}x = 0, \quad 2\int \nabla\Delta u \cdot \{u, \nabla\Delta u\}\,\mathrm{d}x = 0.$$

For the term $2\int \{\nabla u, \Delta u\} \cdot \nabla\Delta u \,\mathrm{d}x$, we have

$$2\int \{\nabla u, \Delta u\} \cdot \nabla\Delta u \,\mathrm{d}x \leq C\|D^2 u\|_{L^\infty}\|\nabla\Delta u\|_{L^2}^2,$$

where $D^2 = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}$, $|\alpha_1| + |\alpha_2| = 2$. Thanks to the Hölder inequality and the Gagliardo-Nirenberg inequality, one has

$$\|D^2 u\|_{L^\infty} \leq C\|\Delta u\|_{L^2}^{1/2}\|\Delta^2 u\|_{L^2}^{1/2}, \quad \|\nabla\Delta u\|_{L^2} \leq C\|\Delta u\|_{L^2}^{1/2}\|\Delta^2 u\|_{L^2}^{1/2}.$$

The above two inequalities and Young's inequality tell us that

$$\|D^2 u\|_{L^\infty}\|\nabla\Delta u\|_{L^2}^2 \leq C\|\Delta u\|_{L^2}^{\frac{3}{2}}\|\Delta^2 u\|_{L^2}^{\frac{3}{2}} \leq \lambda\|\Delta^2 u\|_{L^2}^2 + C\|\Delta u\|_{L^2}^6. \tag{2.6}$$

Hence, (2.5) and (2.6) imply

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\Delta u\|_{L^2}^2 + \|\nabla\Delta u\|_{L^2}^2) + 2\lambda\|\nabla\Delta u\|_{L^2}^2 + \lambda\|\Delta^2 u\|_{L^2}^2 \leq C\|\Delta u\|_{L^2}^6. \tag{2.7}$$

According to (2.3), we obtain

$$\|\Delta u\|_{L^2}^2 \leq \|u_0\|_{H^2}^2 \leq E, \quad E \ll 1, \tag{2.8}$$

$$\int_0^t \|\Delta u(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \leq C(\|\nabla u_0\|_{L^2}^2 + \|\Delta u_0\|_{L^2}^2). \tag{2.9}$$

The conclusion follows when we combine (2.7), (2.8) and (2.9); i.e.,

$$\|\Delta u\|_{L^2}^2 + \|\nabla\Delta u\|_{L^2}^2 + \lambda\int_0^t (\|\nabla\Delta u\|_{L^2}^2 + \|\Delta^2 u\|_{L^2}^2)\,\mathrm{d}\tau \leq C\|u_0\|_{H^3}.$$

Consequently,

$$\|u\|_{H^m} + C_0 \int_0^t \|\nabla u(\tau)\|_{H^m}\,\mathrm{d}\tau \le C\|u_0\|_{H^m}, \quad 1 \le m \le 3.$$

Now we will use mathematical induction to prove (2.1) for any positive m. Assume that (2.1) holds for the case $m = M > 3$, now we consider $m = M + 1$, Differentiating (1.2) with respect to (x_1, x_2),

$$\partial_t D^\alpha(u - \Delta u) + k\partial_{x_2} D^\alpha u - \lambda D^\alpha \Delta(u - \Delta u) + D^\alpha\{u, \Delta u\} = 0, \tag{2.10}$$

where $|\alpha| = M - 1$.

Taking the scalar product of $-2D^\alpha \Delta u$ and (2.10), and then integrating the result over $\mathbb{R}^2$ for the space variable (x_1, x_2), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|D^\alpha \nabla u\|_{L^2}^2 + \|D^\alpha \Delta u\|_{L^2}^2) + 2\lambda(\|D^\alpha \Delta u\|_{L^2}^2 + \|D^\alpha \nabla \Delta u\|_{L^2}^2)\\
&\le 2\int_{\mathbb{R}^2} \Big(\sum_{\substack{\beta_1+\beta_2=\alpha\\ 1\le|\beta_1|\le M-2,}} \{D^{\beta_1}u, D^{\beta_2}\Delta u\} + \{u, D^\alpha \Delta u\} + \{D^\alpha u, \Delta u\}\Big) \cdot D^\alpha \Delta u\,\mathrm{d}x,
\end{aligned}$$

noticing that

$$\int_{\mathbb{R}^2} \{u, D^\alpha \Delta u\} \cdot D^\alpha \Delta u\,\mathrm{d}x = 0, \quad \forall t \ge 0,$$

we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} \Big(\sum_{\substack{\beta_1+\beta_2=\alpha\\ 1\le|\beta_1|\le M-2,}} \{D^{\beta_1}u, D^{\beta_2}\Delta u\} + \{u, D^\alpha \Delta u\} + \{D^\alpha, \Delta u\}\Big) \cdot D^\alpha \Delta u\,\mathrm{d}x\\
&\le 2 \sum_{\substack{\beta_1+\beta_2=\alpha\\ 1\le|\beta_1|\le M-2,}} \|\nabla D^{\beta_1} u\|_{L^\infty} \|\nabla D^{\beta_2}\Delta u\|_{L^2} \|D^\alpha \Delta u\|_{L^2}\\
&\quad + 2\|\nabla D^\alpha u\|_{L^4} \|\nabla \Delta u\|_{L^4} \|D^\alpha \Delta u\|_{L^2}.
\end{aligned}$$

By the Gagliardo-Nirenberg inequality and the Sobolev inequality, we have

$$\begin{gathered}
\|\nabla D^{\beta_1} u\|_{L^\infty} \le C\|\nabla D^{\beta_1} u\|_{H^2},\\
\|\nabla D^\alpha u\|_{L^4} \le C\|\nabla D^\alpha u\|_{L^2}^{1/2} \|\nabla D^\alpha u\|_{H^1}^{1/2},\\
\|\nabla \Delta u\|_{L^4} \le C\|\nabla \Delta u\|_{L^2}^{1-\frac{1}{2(M-2)}} \|\nabla \Delta u\|_{H^{M-2}}^{\frac{1}{2(M-2)}}.
\end{gathered}$$

According to the induction assumption and Young's inequality, we conclude that

$$\|u\|_{H^{M+1}} + \int_0^t \|\nabla u(\tau)\|_{H^{M+1}}\,\mathrm{d}\tau \le C\|u_0\|_{H^{M+1}}.$$

Thus, by mathematical induction, (2.1) is true for any positive m. □

Based on the above discussion, we can extend the local solution and get the global solution by using the energy estimates (2.1). Therefore, we have the following result.

Theorem 2.2. *Suppose $u_0 \in H^{s+l}$, $s = 2$, l is a positive integer, and $\|u_0\|_{H^{s+l}}$ is sufficient small, then there exists a global solution $u(x,t)$ to* (1.2). *Furthermore, when $M = s + l - 1$, $u(x,t)$ satisfies* (2.1), *i.e.*

$$\|u\|_{H^m} + C_0 \int_0^t \|\nabla u\|_{H^m}\,\mathrm{d}\tau \le C\|u_0\|_{H^m}.$$

where C_0, C are positive constants and independent of t.

3. Pointwise estimates of Green function

The linearized system of (1.2) about the constant state u^*, taken to be 0 without loss of generality, is

$$\partial_t(u-\Delta u)+k\partial_{x_2}u-\lambda(u-\Delta u)=0. \tag{3.1}$$

The Green function $G(x,t)$ satisfies

$$\begin{gathered}\partial_t(G-\Delta G)+k\partial_{x_2}G-\lambda(G-\Delta G)=0,\\ G(x,0)=\delta(x),\end{gathered}$$

where $\delta(x)$ is the Dirac function. By the Fourier transform, we obtain

$$\hat{G}(\xi,t)=\mathrm{e}^{\eta t},\quad \text{where } \eta=-\lambda|\xi|^2-\frac{\sqrt{-1}k\xi_2}{1+|\xi|^2}.$$

Our goal in this section is to derive the pointwise estimates for the Green function $G(x,t)$ defined above. In order to get the estimates of $G(x,t)$, we divide G into the lower frequency part G_1, the middle frequency part G_2 and the higher frequency part G_3, where

$$\begin{gathered}\hat{G}_i=\chi_i(\xi)\hat{G}(\xi,t),\quad (i=1,2,3);\\ \chi_1(\xi)=\begin{cases}1, & |\xi|<\epsilon,\\ 0, & |\xi|>2\epsilon;\end{cases}\\ \chi_3(\xi)=\begin{cases}1, & |\xi|>2R,\\ 0, & |\xi|<R;\end{cases}\\ \chi_2(\xi)=1-\chi_1(\xi)-\chi_3(\xi).\end{gathered}$$

which are smooth cut-off functions for the fixed constants $0<\epsilon<1$ and $R>2$.

First of all, we estimate the lower frequency part G_1.

Lemma 3.1. *For any positive integer N, if ϵ is small enough, there exists constant $C>0$, such that*

$$|D_x^\alpha G_1|\le Ct^{-\frac{2+|\alpha|}{2}}B_N(x-bt,t),$$

where $b=(0,k)$ and $x=(x_1,x_2)$.

Proof. When $|\xi|$ is sufficiently small by the Taylor expansion, we obtain

$$-\frac{\sqrt{-1}k\xi_2}{1+|\xi|^2}=-\sqrt{-1}k\xi_2+O(|\xi|^2).$$

Hence,

$$\begin{aligned}\hat{G}_1&=\chi_1(\xi)\mathrm{e}^{-\lambda|\xi|^2t}\mathrm{e}^{-\sqrt{-1}k\xi_2t}(1+O(|\xi|^2)t)\\ &=\chi_1(\xi)\mathrm{e}^{-\lambda|\xi|^2t}(1+O(|\xi|^2)t)\mathrm{e}^{-\sqrt{-1}k\xi_2t}\\ &=\hat{A}\mathrm{e}^{-\sqrt{-1}k\xi_2t},\end{aligned}$$

where $\hat{A}=\chi_1(\xi)\mathrm{e}^{-\lambda|\xi|^2t}(1+O(|\xi|^2)t)$. By properties of fourier transform, we have

$$G_1=A*\mathcal{F}^{-1}(\mathrm{e}^{-\sqrt{-1}k\xi_2t})=A(x_1,x_2-kt,t),$$

and

$$D_x^\alpha G_1 = D_x^\alpha A(x_1, x_2 - kt, t). \tag{3.2}$$

Moreover, by the expansion of $\hat{A}$, we have

$$|\partial_t^l D_\xi^\beta (\xi^\alpha \hat{A}(\xi,t))| \le C|\xi|^{|\alpha|-|\beta|+2l}(1+O(|\xi|^2)t)^{|\beta|+1}\mathrm{e}^{-\lambda|\xi|^2 t}.$$

Thanks to Lemma 5.1,

$$|D_x^\alpha A(x,t)| \le C_N t^{-\frac{2+|\alpha|}{2}} B_N(x,t).$$

Thus, by (3.2), we have

$$|D_x^\alpha G_1| \le C_N t^{-\frac{2+|\alpha|}{2}} B_N(x-bt,t),$$

with $b = (0,k)$, which is our conclusion. □

About the middle frequency part $\hat{G}_2$, we have the following Lemma.

Lemma 3.2. *Suppose $|\xi| \in (\varepsilon, 2R)$, then there exists a positive b, such that*

$$|D_\xi^\beta \hat{G}_2(\xi,t)| \le C(1+t)^{|\beta|}\mathrm{e}^{-bt}. \tag{3.3}$$

Proof. If $|\xi| \in (\varepsilon, 2R)$, then there exists a positive b, such that

$$|\hat{G}_2(\xi,t)| = |\chi_2(\xi)\hat{G}(\xi,t)| \le C\mathrm{e}^{-bt}. \tag{3.4}$$

By (3.4), this lemma holds when $|\beta| = 0$. Now we use mathematical induction to prove this Lemma. Suppose we have

$$|D_\xi^\beta \hat{G}_2(\xi,t)| \le C(1+t)^{|\beta|}\mathrm{e}^{-bt} \tag{3.5}$$

for $|\beta| \le l-1$. By the Fourier transform,

$$\begin{gathered}
\partial_t(D_\xi^\beta \hat{G}_2) + \frac{k\sqrt{-1}\xi_2}{1+|\xi|^2} D_\xi^\beta \hat{G}_2 + \lambda|\xi|^2 D_\xi^\beta \hat{G}_2 = -F(\xi), \\
D_\xi^\beta \hat{G}_2(\xi,0) = D_\xi^\beta \chi_2(\xi),
\end{gathered}$$

where

$$\begin{aligned}
F(\xi) = &\sum_{|\beta_1|+|\beta_2|=|\beta|, |\beta_1|\neq 0} D_\xi^{\beta_1}(\frac{k\sqrt{-1}\xi_2}{1+|\xi|^2}) D_\xi^{\beta_2}\hat{G}_2 \\
&+ \sum_{|\gamma_1|+|\gamma_2|=|\beta|, |\gamma_1|\neq 0} D_\xi^{\gamma_1}(\lambda|\xi|^2) D_\xi^{\gamma_2}\hat{G}_2.
\end{aligned}$$

Note that

$$D_\xi^\beta \hat{G}_2 = D_\xi^\beta \chi_2(\xi)\hat{G}(t,\xi) - \int_0^t \hat{G}(\xi, t-\tau)F(\tau)\,\mathrm{d}\tau,$$

according to (3.5), we have

$$|D^\beta \hat{G}_2| \le C\int_0^t \mathrm{e}^{-b(t-s)}(1+t)^{|\beta|-1}\mathrm{e}^{-bs}\,\mathrm{d}s + C\mathrm{e}^{-bt} \le C\mathrm{e}^{-bt}(1+t)^{|\beta|},$$

which implies that (3.3) is valid for $|\alpha| = l$. By induction, this lemma is proved. □

Additionally, when $1 \le |\beta| \le l$,

$$|x^\beta D^\alpha G_2(x,t)| \le C\Big|\int_{\mathbb{R}^2} e^{\sqrt{-1}x\cdot\xi} D_\xi^\beta(\xi^\alpha \hat{G}_2)\,d\xi\Big| \le C(1+t)^{|\beta|}e^{-bt}. \tag{3.6}$$

When $|x|^2 \le 1+t$, setting $|\beta| = 0$, we have

$$|D_x^\alpha G_2(x,t)| \le Ce^{-bt} \le C\frac{2^N}{(1+\frac{|x|^2}{1+t})^N}e^{-bt} \le Ce^{-bt}B_N(x,t)$$

When $|x|^2 > 1+t$, setting $|\beta| = 2N$, one has

$$|D_x^\alpha G_2(x,t)| \le Ce^{-bt}\frac{1}{|x|^{2N}} \le C\frac{2^N}{(1+\frac{|x|^2}{1+t})^N}e^{-bt} \le Ce^{-bt}B_N(x,t)$$

Therefore, the following Remark holds.

Remark 3.3. For any fixed ε and R, there exists positive b such that for $|\alpha| \ge 0$

$$|D_x^\alpha G_2(x,t)| \le Ce^{-bt}B_N(x,t).$$

Now we consider the higher frequency part G_3. When $|\xi| \ge 2R$ and ξ is large enough, we obtain

$$\eta = -\lambda|\xi|^2 - \frac{\sqrt{-1}k\xi_2}{1+|\xi|^2} = -\lambda|\xi|^2 + \sum_{j=1}^m a_j|\xi|^{-(2j-1)} + O(|\xi|^{-(2m+1)}),$$

So,

$$\hat{G}_3(\xi,t) = \chi_3 e^{-\lambda|\xi|^2 t}\big(1 + \sum_{j=1}^m p_i(t)q_j(\xi) + p_{m+1}(t)O(|\xi|^{-(2m+1)})\big),$$

where p_j and q_j satisfy

$$|p_j(t)| \le C(1+t)^j, \quad 1 \le j \le m+1, \quad q_j(\xi) = |\xi|^{-(2j-1)}, \quad 1 \le j \le m\,.$$

Denote

$$\hat{F}_\alpha(\xi,t) = \chi_3(\xi)e^{-\lambda|\xi|^2 t}\sum_{j=1}^{|\alpha|+2} p_j(t)q_j(\xi).$$

We have the following lemma.

Lemma 3.4. *When R is sufficient large, there exists a positive a, such that*

$$|D_x^\alpha(G_3 - F_\alpha)(x,t)| \le Ce^{-at}B_N(x,t)$$

Proof. Notice that the dimension we consider is $n = 2$. Hence, we have the estimate

$$\begin{aligned} |x^{2\beta}D_x^\alpha(G_3 - F_\alpha)| &\le \int |\partial_\xi^{2\beta}(\xi^\alpha(\hat{G}_3 - \hat{F}_\alpha))|\,d\xi \\ &\le \int |\xi|^{|\alpha|-2|\beta|-2((|\alpha|+2)+1)}e^{-\lambda|\xi|^2 t}\,d\xi \le Ce^{-at}, \end{aligned} \tag{3.7}$$

where we have used the following fact: If $|x|^2 \le t+1$, let $|\beta| = 0$ and let $|\beta| = N$ when $|x|^2 > t+1$, then

$$|D_x^\alpha(G_3 - F_\alpha)(x,t)| \le Ce^{-at}\min\{1, ((1+t)/|x|^2)^N\}.$$

Additionally, we have

$$1 + \frac{|x|^2}{1+t} \le 2\begin{cases} 1, & |x|^2 \le t+1, \\ \frac{|x|^2}{1+t}, & |x|^2 > t+1, \end{cases}$$

one has

$$\min\{1, ((1+t)/|x|^2)^N\} \le \frac{2^N}{(1+(|x|^2/(1+t)))^N}.$$

By (3.7) and the above inequality, we have

$$|D_x^\alpha (G_3 - F_\alpha)(x,t)| \le C\mathrm{e}^{-at} B_N(x,t)$$

□

By Lemma 3.1, Remark 3.3, Lemma 3.4 and Remark 5.3 we obtain the following result.

Theorem 3.5. *For any $x \in \mathbb{R}^2$, $t > 0$ and $l \le 9$ there exists a constant $C_\alpha > 0$, such that*

$$|D_x^\alpha (G(x,t) - F_\alpha(x,t))| \le C_\alpha (1+t)^{-\frac{2+|\alpha|}{2}} B_N(x - bt, t) \quad |\alpha| \le l,$$

where $b = (0, k)$.

4. Pointwise estimates of the nonlinear equations

In this section, we obtain pointwise estimates for the nonlinear equation

$$\begin{gathered} \partial_t(u - \Delta u) + k\partial_{x_2} u - \lambda\Delta(u - \Delta u) + \{u, \Delta u\} = 0, \\ u(x,0) = u_0(x). \end{gathered} \tag{4.1}$$

According to the Duhamel principle,

$$u = G(\cdot,t) * u_0 - \int_0^t G(t-\tau) * \{u, \Delta u\}\, \mathrm{d}\tau;$$

therefore,

$$D_x^\alpha = D_x^\alpha G(\cdot,t) * u_0 - \int_0^t D_x^\alpha G(t-\tau) * \{u, \Delta u\}\, \mathrm{d}\tau = I_1 + I_2$$

Next, we obtain estimates for I_1 and I_2 respectively. Suppose that the initial data u_0 satisfies the condition

$$|D_x^\alpha u_0(x)| \le CE(1+|x|^2)^{-m'}, \quad m' > 2,\ E \ll 1. \tag{4.2}$$

So we have

$$I_1 = D_x^\alpha (G(\cdot,t) - F_\alpha(\cdot,t)) * u_0 + D_x^\alpha F_\alpha(\cdot,t) * u_0 =: I_{11} + I_{12}.$$

By Theorem 3.5 and Lemma 5.7, we have the estimate

$$\begin{aligned} |I_{11}| &\le CE(1+t)^{-\frac{2+|\alpha|}{2}} \int_{\mathbb{R}^2} B_N(|x - bt - y|, t)(1+|y|^2)^{-m'}\, \mathrm{d}y \\ &\le CE(1+t)^{-\frac{2+|\alpha|}{2}} B_N(x - bt, t). \end{aligned}$$

Since $\|u_0\|_{L^1} \le E$, we have $\|\hat{u}_0\|_{L^\infty} \le \|u_0\|_{L^1} \le E$. Hence there exists a positive p, such that

$$\begin{aligned} |x^\beta D_x^\alpha F_\alpha(\cdot,t) * u_0| &\le \int |\partial_\xi^\beta \xi^\alpha \hat{F}_\alpha \hat{u}_0|\, \mathrm{d}\xi \\ &\le CE\mathrm{e}^{-pt} \int |\xi|^{|\alpha| - |\beta| - (2(|\alpha|+2)-1)}\, \mathrm{d}\xi \\ &\le CE\mathrm{e}^{-pt} \int |\xi|^{|\alpha| - |\beta| - 2|\alpha| - 3}\, \mathrm{d}\xi \le CE\mathrm{e}^{-pt}. \end{aligned}$$

Let $|\beta| = 0$ or $|\beta| = 2N$, one gets

$$|I_{12}| = |D_x^\alpha F_\alpha(\cdot,t) * u_0| \le CE\mathrm{e}^{-\frac{pt}{2}} B_N(x,t).$$

By the estimates of I_{11} and I_{12}, we obtain the estimates of I_1.

Lemma 4.1. *For $|\alpha| \le l$ and sufficient large t, we have*

$$|I_1| \le CE(1+t)^{-\frac{2+|\alpha|}{2}} B_N(x-bt,t).$$

Now we establish the estimates of I_2:

$$\begin{aligned} I_2 &= -\int_0^t D^\alpha G(t-\tau) * \{u, \Delta u\}\,\mathrm{d}\tau \\ &= \int_0^t D^\alpha \partial_{x_2} G(t-\tau) * h_1(\tau)\,\mathrm{d}\tau - \int_0^t D^\alpha \partial_{x_1} G(t-\tau) * h_2(\tau)\,\mathrm{d}\tau =: I_{21} + I_{22}, \end{aligned}$$

where $h_1(x,t) = u_{x_1}\Delta u$ and $h_2(x,t) = u_{x_2}\Delta u$. For I_{21}, we have the following decomposition

$$\begin{aligned} I_{21} &= \int_0^t D^\alpha \partial_{x_2} G(t-\tau) * h_1(\tau)\,\mathrm{d}\tau \\ &= \int_0^t D_x^\alpha \partial_{x_2} (G-F_\alpha)(t-\tau) * h_1(\tau)\,\mathrm{d}\tau + \int_0^t D_x^\alpha \partial_{x_2} F_\alpha(t-\tau) * h_1(\tau)\,\mathrm{d}\tau \\ &=: I_{211} + I_{212}. \end{aligned}$$

Denote

$$\varphi_\alpha(x,t) = (1+t)^{\frac{2+\gamma(|\alpha|)}{2}} (B_1(x-bt,t))^{-1},$$

where

$$\nu(|\alpha|) = \begin{cases} |\alpha|, & |\alpha| \le l-6, \\ 0, & l-6 < |\alpha| \le l-3. \end{cases}$$

and

$$M(t) = \sup_{0\le\tau\le t,\ |\alpha|\le l-3}\ \max_{x\in\mathbb{R}^2} |D^\alpha u(x,\tau)|\varphi_\alpha(x,\tau).$$

According to the definition of $h_1(x,t)$ and Theorem 2.2, we have

$$|D^\alpha h_1(x,t)| \le \begin{cases} M^2(t)(1+t)^{-2-\frac{|\alpha|+3}{2}} B_2(x-bt,t), & |\alpha| \le l-8 \\ M^2(t)(1+t)^{-\frac{5}{2}} B_2(x-bt,t), & |\alpha| = l-7, l-6 \\ (M^2(t)+CEM)(1+t)^{-\frac{3}{2}} B_1(x-bt,t), & |\alpha| = l-5, \\ CEM(t)(1+t)^{-\frac{3}{2}} B_1(x-bt,t), & l-5 < |\alpha| \le l-3. \end{cases}$$

When $|\alpha| \le l-8$, by Theorem 3.5, we have

$$\begin{gathered} |D_x^\alpha (G-F_\alpha)(x,t)| \le C(1+t)^{-\frac{2+|\alpha|}{2}} B_1(x-bt,t), \\ |D_x^\alpha h_1(x)| \le CM^2(t)(1+t)^{-2-\frac{|\alpha|+3}{2}} B_2(x-bt,t). \end{gathered}$$

By Lemma 5.5, it is easy to see that

$$|I_{211}| \le CM^2(t)(1+t)^{-\frac{2+|\alpha|}{2}} B_1(x-bt,t).$$

If $|\alpha| = l-7$, one has

$$|I_{211}| = |\int_0^t D^\alpha \partial_{x_2} (G-F_\alpha)(t-\tau) * h_1(\tau)\,\mathrm{d}\tau|$$

$$= |\int_0^t D^{\alpha'}(D^{\beta}\partial_{x_2}G(t-\tau)) * h_1(\tau)\,\mathrm{d}\tau|,$$

where $|\alpha'| = l-8$, $|\alpha'|+|\beta| = l-7$. Since

$$\begin{aligned}|D^{\alpha'}(D^{\beta}\partial_{x_2}(G-F_\alpha)(t-\tau))| &\le C(1+t)^{-\frac{2+l-6}{2}}B_1(x-bt,t)\\ &\le C(1+t)^{-\frac{2+l-7}{2}}B_1(x-bt,t).\end{aligned}$$

Furthermore,

$$\begin{aligned}|D^{\alpha'}h_2(x)| &\le CM^2(t)(1+t)^{-2-\frac{l-8+3}{2}}B_2(x-bt,t)\\ &\le CM^2(t)(1+t)^{-2-\frac{l-7}{2}}B_2(x-bt,t)\end{aligned}$$

Using Lemma 5.5, one gets

$$\begin{aligned}&|\int_0^t D^{\alpha'}(D^{\beta}\partial_{x_2}(G-F_\alpha)(t-\tau)) * h_1(\tau)\,\mathrm{d}\tau|\\ &\le CM^2(t)(1+t)^{-\frac{2+l-7}{2}}B_1(x-bt,t).\end{aligned}$$

When $|\alpha| = l-7$, we also have

$$|I_{211}| \le CM^2(t)(1+t)^{-\frac{2+l-7}{2}}B_1(x-bt,t).$$

When $|\alpha| = l-6$,

$$\begin{aligned}|I_{211}| &= |\int_0^t D^{\alpha}\partial_{x_2}(G-F_\alpha)(t-\tau) * h_1(\tau)\,\mathrm{d}\tau|\\ &= |\int_0^t D^{\alpha''}(D^{\beta'}\partial_{x_2}(G-F_\alpha)(t-\tau)) * h_1(\tau)\,\mathrm{d}\tau|,\end{aligned}$$

where $|\alpha''| = l-8$, $|\alpha''|+|\beta| = l-6$. Besides, we have

$$\begin{aligned}|D^{\alpha''}(D^{\beta'}\partial_{x_2}G(t-\tau))| &\le C(1+t)^{-\frac{2+l-6}{2}}B_1(x-bt,t),\\ |D^{\alpha''}h_1(x)| &\le CM^2(t)(1+t)^{-2-\frac{l-6}{2}}B_2(x-bt,t).\end{aligned}$$

By Lemma 5.5, we obtain the estimate

$$\begin{aligned}&|\int_0^t D^{\alpha''}(D^{\beta'}\partial_{x_2}(G-F_\alpha)(t-\tau)) * h_1(\tau)\,\mathrm{d}\tau|\\ &\le CM^2(t)(1+t)^{-\frac{2+l-6}{2}}B_1(x-bt,t).\end{aligned}$$

Therefore, when $|\alpha| = l-6$,

$$|I_{211}| \le CM^2(t)(1+t)^{-\frac{2+l-2}{2}}B_1(x-bt,t).$$

When $|\alpha| < l-6$,

$$\begin{aligned}|I_{211}| &= |\int_0^t D^{\alpha}\partial_{x_2}(G-F_\alpha)(t-\tau) * h_1(\tau)\,\mathrm{d}\tau|\\ &= |\int_0^t D^{\gamma}(D^{\eta}\partial_{x_2}(G-F_\alpha)(t-\tau)) * h_1(\tau)\,\mathrm{d}\tau|,\end{aligned}$$

where $|\gamma| = l-8$, $|\gamma|+|\eta| = |\alpha|$.

Notice that we also have

$$|D^{\gamma}(D^{\eta}\partial_{x_2}(G-F_\alpha)(t-\tau))| \le C(1+t)^{-\frac{2+|\alpha|+1}{2}}B_1(x-bt,t).$$

and

$$\begin{aligned}|D^\gamma h_1(x)| &\le CM^2(t)(1+t)^{-2-\frac{|\gamma|+3}{2}}B_2(x-bt,t)\\ &\le CM^2(t)(1+t)^{-2-\frac{|\alpha|+1}{2}}B_2(x-bt,t)\end{aligned}$$

By Lemma 5.5, we have

$$\begin{aligned}|I_{211}| &= |\int_0^t D^\alpha\partial_{x_2}(G-F_\alpha)(t-\tau)*h_1(\tau)\,\mathrm{d}\tau|\\ &= |\int_0^t D^\gamma(D^\eta\partial_{x_2}(G-F_\alpha)(t-\tau))*h_1(\tau)\,\mathrm{d}\tau|\\ &\le CM^2(t)(1+t)^{-\frac{2+|\alpha|}{2}}B_1(x-bt,t).\end{aligned}$$

When $l-6<|\alpha|\le l-3$, we have $\nu(|\alpha|)\le 2+|\alpha|$. Then we have the following conclusion when $|\alpha|\le l-3$,

$$|I_{211}|\le CM^2(t)(1+t)^{-\frac{2+\nu(|\alpha|)}{2}}B_1(x-bt,t).$$

Now, we aim to deal with I_{212}. Firstly, we can get the following conclusion. When $|\xi|$ is large enough, there exist a positive constant b, such that

$$\begin{aligned}\hat{F}_\alpha(\xi,t) &= \chi_3(\xi)\mathrm{e}^{-\lambda|\xi|^2t}\big(\sum_{j=1}^m p_i(t)q_j(\xi)+p_{m+1}(t)O(|\xi|^{-(2m+1)})\big)\\ &= \mathrm{e}^{-bt}\sum_{j=1}^{m+1}p_j(t)\chi_3(\xi)\mathrm{e}^{-\frac{-\lambda|\xi|^2t}{2}}q_j(\xi)\\ &= \mathrm{e}^{-bt}\sum_{j=1}^{m+1}p_j(t)\hat{f}_j(\xi)\end{aligned}$$

According to the definition of $f(\xi)$ and Lemma 5.4, there exists f_1, f_2 and C_0 such that

$$f(x)=f_1(x)+f_2(x)+C_0\delta(x).$$

For any positive $2m>2+|\alpha|$ and ε_0, we have

$$|D_x^\alpha f_1(x)|\le C(1+|x|^2)^{-m},$$
$$\|f_2\|_{L^1}\le C,\quad \operatorname{supp} f_2(x)\subset\{x||x|<2\varepsilon_0\}.$$

Therefore,

$$\begin{aligned}|I_{212}| &= |\int_0^t F_\alpha(t-\tau)*D_x^\alpha\partial_{x_2}h_1(\tau)\,\mathrm{d}\tau|\\ &\le C\sum_{j=1}^{m+1}|\int_0^t p_j(t)(f_{1j}+f_{2j}+C_0\delta)\mathrm{e}^{-b(t-\tau)}*D_x^\alpha\partial_{x_2}h_1(\tau)\,\mathrm{d}\tau|\end{aligned}$$

Choose $m=\max\{N',1+l\}$, then $|D_x^\alpha f_{1j}(x)|\le(1+|x|)^{-m}$, where N' is the constant N from Lemma 5.6. We have already obtain that when $l\le 9$ and $|\alpha|\le l-3$,

$$|D_x^\alpha\partial_{x_2}h_1(\tau)|\le C(M^2(t)+EM(t))(1+\tau)^{-1-\nu(|\alpha|)}B_1(x-b\tau,\tau).$$

By Lemma 5.6, we have

$$|\int_0^t f_{1j}\mathrm{e}^{-b(t-s)}*D_x^\alpha\partial_{x_2}h_1(s)\,\mathrm{d}s|$$

$$\begin{aligned}
&\le C(M^2(t)+EM(t))\Big|\int_0^t\int \mathrm{e}^{-b(t-s)}(1+|x-y|^2)^{-m}(1+s)^{-1-\nu(|\alpha|)}\\
&\quad\times B_1(x-bs,s)\,\mathrm{d}y\,\mathrm{d}s\Big|\\
&\le C(M^2(t)+EM(t))(1+t)^{-1-\nu(|\alpha|)}B_1(x-bt,t).
\end{aligned}$$

When $|x-y|\le 2\varepsilon_0$ and ε_0 is small enough, by Lemma 5.6 we obtain the estimate

$$\begin{aligned}
&\Big|\int_0^t f_{2j}\mathrm{e}^{-b(t-s)} * D_x^\alpha\partial_{x_2}h_{1j}(s)\,\mathrm{d}s\Big|\\
&\le \|f_{2j}\|_{L^1}\|\int_0^t D_x^\alpha\partial_{x_2}h_{1j}(s)(\cdot,s)\mathrm{e}^{-b(t-s)}\,\mathrm{d}s\|_{L^\infty}(|\cdot-x|\le 2\varepsilon_0)\\
&\le C(M^2(t)+EM(t))(1+t)^{-1-\nu(|\alpha|)}B_1(x-bt,t).
\end{aligned}$$

Notice that

$$\int_{\mathbb{R}^2}\delta(x-y)D_x^\alpha\partial_{x_2}h_1(t)\,\mathrm{d}y = D_x^\alpha\partial_{x_2}h_1(x).$$

To conclude, when $|\alpha|\le l-3$, $l<9$, and t is large enough, we have

$$|I_{212}|\le C(M^2(t)+EM(t))(1+t)^{-\frac{2+\nu(|\alpha|)}{2}}B_1(x-bt,t).$$

By the above two conclusions we obtain the estimates for I_{21}, i.e. when $|\alpha|\le l-3$ and $l<9$,

$$|I_{21}|\le C(M^2(t)+EM(t))(1+t)^{-\frac{2+\nu(|\alpha|)}{2}}B_1(x-bt,t).$$

Similarly, we can get the decay estimates of I_{22}. Then we obtain the estimates of I_2 in the following.

Lemma 4.2. *When $|\alpha|\le l-3$, $l<9$, t is large enough, we have*

$$|I_2|\le C(M^2(t)+EM(t))(1+t)^{-\frac{2+\nu(|\alpha|)}{2}}B_1(x-bt,t).$$

Combining Lemma 4.1 and Lemma 4.2, we have

$$|D^\alpha u(x)|\le C(E+M^2(t))(1+t)^{-\frac{2+\nu(|\alpha|)}{2}}B_1(x-bt,t).$$

From the definition of $M(t)$, it is easy to get

$$M(t)\le C(E+M^2(t)).$$

Taking E small enough and using the continuity of $M(t)$ and induction, we conclude that

$$M(t)\le CE.$$

Finally, Theorem 1.1 is proved.

5. Appendix

We list some known facts and lemmas which used in this paper. Lemma 5.1 can be found in [17], which is a vital tool to get the pointwise estimates of the Green function $G(x,t)$. Lemma 5.4 gives us a useful tool to deal with the singular part of the higher frequency part in $G(x,t)$, which can be found in [11].

Lemma 5.1. *If $\hat{f}(\xi,t)$ satisfies,*

$$|D^{2\beta}\partial_t^l(\xi^\alpha \hat{f}(\xi,t))| \le C|\xi|^{|\alpha|+k-2|\beta|+2l}(1+(t|\xi|^2))^m \mathrm{e}^{-\nu|\xi|^2 t/2},$$

for any positive integers l and m, and multi-indexes α,β with $|\beta|\le N$, then

$$|D^\alpha f(x,t)| \le C_N t^{-\frac{n+|\alpha|+k}{2}+l} B_N(x,t),$$

where N is any fixed integer, and

$$B_N(x,t) = (1+\frac{|x|^2}{1+t})^{-N}.$$

Lemma 5.2. *For any positive integer N, there exists a constant $C>0$, such that*

$$\mathrm{e}^{-t} B_N(x+bt,t) \le C\mathrm{e}^{-t/2} B_N(x,t), \tag{5.1}$$

where $x=(x_1,x_2)$ and $b=(b_1,b_2)$.

Remark 5.3. By a similar proof to that for Lemma 5.2, we can also get

$$\mathrm{e}^{-t} B_N(x,t) \le C\mathrm{e}^{-t/2} B_N(x-bt,t).$$

Lemma 5.4. *If $\operatorname{supp}\hat{f}(\xi) \subset O_R = \{\xi : |\xi| > R\}$, $\hat{f}(\xi) \in L^\infty \cap C^{n+1}(O_R)$, and $\hat{f}$ satisfies*

$$|\hat{f}(\xi)| \le C_0, \quad |D_\xi^\beta \hat{f}(\xi)| \le C_0|\xi|^{-|\beta|-1}, (|\beta|\ge 1),$$

then there exist distributions $f_1(x), f_2(x)$ and a constant $C>0$ depending on n, such that

$$f(x) = f_1(x) + f_2(x) + C\delta(x),$$

Furthermore, for any positive constant $2m > 2+|\alpha|$, we have

$$|D_x^\alpha f_1(x)| \le C(1+|x|^2)^{-m}, \quad \|f_2\|_{L^1} \le C, \quad \operatorname{supp} f_2(x) \subset \{x : |x| < 2\varepsilon_0\},$$

with ε_0 small enough.

Lemma 5.5 ([17, Lemma 5.2])**.** *If functions $F(x,t)$ and $S(x,t)$ satisfy*

$$|D_x^\alpha F(x,t)| \le C(1+t)^{-\frac{n+|\beta|}{2}} B_{n_1}(x-bt,t),$$
$$|D_x^\alpha S(x,t)| \le C(1+t)^{-\frac{2n+|\beta|}{2}} B_{n_2}(x-bt,t),$$

where α and β are multi-indexes. Then,

$$|D_x^\alpha\Big(\int_0^t F(t-s) * S(s)\,\mathrm{d}s\Big)| \le C(1+t)^{-\frac{n+|\beta|}{2}} B_{n_3}(x-bt,t),$$

where $n_1,n_2 > n/2$ and $n_3 = \min(n_1,n_2)$.

Lemma 5.6 ([16, Proposition 3.2])**.** *For any positive integers $\alpha, N' > 4$, suppose $b>0$, then for the large t, we have*

$$\Big|\int_0^t \int_{\mathbb{R}^2} \mathrm{e}^{-b(t-s)}(1+s)^{-\frac{2+|\alpha|}{2}}(1+|x-y|^2)^{-N'} B_1(y,s)\,\mathrm{d}y\,\mathrm{d}s\Big|$$
$$\le C(1+t)^{-\frac{2+|\alpha|}{2}} B_1(x,t).$$

Lemma 5.7. *Suppose $n_1,n_2 > n/2$, and let $n_3 = \min(n_1,n_2)$, then we have*

$$\int_{\mathbb{R}^n} (1+\frac{|x-y|^2}{1+t})^{-n_1}(1+|y|^2)^{-n_2}\,\mathrm{d}y \le C(1+\frac{|x|^2}{1+t})^{-n_3}.$$

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