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POSITIVE SOLUTIONS FOR PARAMETRIC NONLINEAR PERIODIC PROBLEMS WITH COMPETING NONLINEARITIES

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Abstract. We consider a nonlinear periodic problem driven by a nonhomogeneous differential operator plus an indefinite potential and a reaction having the competing effects of concave and convex terms. For the superlinear (concave) term we do not employ the usual in such cases Ambrosetti-Rabinowitz condition. Using variational methods together with truncation, perturbation and comparison techniques, we prove a bifurcation-type theorem describing the set of positive solutions as the parameter varies.

1. Introduction

In this article we study the nonlinear periodic problem (P_λ),

$$\begin{gathered} -(a(|u'(t)|)u'(t))' + \beta(t)u(t)^{p-1} = \lambda u(t)^{q-1} + f(t,u(t)) \quad \text{a.e. on } T := [0,b] \\ u(0) = u(b), \quad u'(0) = u'(b), \quad u > 0,\ 1 < q < p < \infty. \end{gathered} \tag{1.1}$$

The function $a(|x|)x$ involved in the definition of the differential operator is a continuous increasing function which satisfies certain other regularity conditions listed in hypothesis (H1) below.

These hypotheses are general enough and incorporate as special cases many differential operators of interest such as the scalar p-Laplacian. The potential $\beta \in L^\infty(T)$ may change sign (indefinite potential). Also $\lambda > 0$ is a parameter, and the term λx^{q-1} (for $x \geq 0$) is a "concave" (that is, $(p-1)$-sublinear) contribution to the reaction of problem (1.1). The perturbation $f(t,x)$ is a Carathéodory function (i.e., for all $x \in \mathbb{R}$, $t \to f(t,x)$ is measurable and for almost all $t \in T$, $x \to f(t,x)$ is continuous), which exhibits $(p-1)$-superlinear growth near $+\infty$, but without satisfying the usual in such cases Ambrosetti-Rabinowitz condition (AR-condition for short).

So, in the reaction of (1.1) we have the competing effects of concave and convex nonlinearities.

Our aim is to describe the dependence on the parameter $\lambda > 0$ of the set of positive solutions of problem (1.1). We prove a bifurcation-type theorem asserting the existence of a critical parameter value $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$, problem (1.1) admits at least two positive solutions, for $\lambda = \lambda^*$, problem (1.1) has

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at least one positive solution, and for $\lambda > \lambda^*$, there are no positive solutions for problem (1.1).

Recently, such a result was proved by Aizicovici-Papageorgiou-Staicu [2] for equations driven by the scalar p-Laplacian (that is, $a(|x|)x = |x|^{p-2}x$ with $1 < p < \infty$) and with $\beta \equiv 0$.

In the present work, the differential operator is nonhomogeneous and this is a source of serious difficulties in the analysis of problem (1.1), and many techniques used in [2] are not applicable here. We mention that recently, Aizicovici-Papageorgiou-Staicu [3] proved a bifurcation-type theorem for a class of nonlinear periodic problems driven by a nonhomogeneous differential operator, but with a positive potential function $\beta \in L^\infty(T)_+\backslash\{0\}$ and a reaction $\lambda f(t,x)$ $(\lambda > 0)$ which is strictly $(p-1)$-sublinear near $+\infty$. So, they deal with a coercive problem with no competition of different nonlinearities in the reaction. Nonlinear, nonhomogeneous periodic problems with a positive potential function $\beta \in L^\infty(T)_+\backslash\{0\}$ and with competing nonlinearities in the reaction (concave-convex terms) were studied by Aizicovici-Papageorgiou-Staicu [4]. In that paper the emphasis is on the existence of nodal (that is, sign changing) solutions.

Our investigation is motivated by applications of physical interest. For instance, in the work of Brézis-Mawhin [8] some concrete quasilinear inertia terms arise in the context of the study of the relativistic motion of particles. The corresponding differential operator is different, however it seems possible to adapt our results to the framework of [8]. Also, our problem is related to the stationary version of the parabolic equations studied by Badii-Diaz [6] in the context of some catalysis and chemical reaction models.

Finally, we mention that multiplicity results for positive solutions of equations driven by the scalar p-Laplacian with Dirichlet and Sturm-Liouville boundary conditions, were proved by Ben Naoum-De Coster [7], De Coster [9], Manasevich-Njoku-Zanolin [13], Njoku-Zanolin [14]. For periodic problems driven by the scalar p-Laplacian, we mention the works of Hu-Papageorgiou [12] and Wang [16].

Our approach is variational, based on the critical point theory, combined with suitable truncations and comparison techniques. In the next section, for easy reference, we recall the main mathematical tools that we will use in the sequel. We also introduce the hypotheses on the map $x \to a(|x|)x$ and state some useful consequences of these conditions. Our main result is stated and proved in Section 3.

2. Preliminaries

Let $(X, \|\cdot\|$ be a Banach space and $(X^*, \|\cdot\|_*)$ its topological dual. By $\langle .,. \rangle$ we denote the duality brackets for the pair (X^*, X) and by $\xrightarrow{w}$ the weak convergence in X. A map $A : X \to X^*$ is said to be *of type* $(S)_+$, if for every sequence $\{x_n\}_{n\ge1} \subseteq X$ such that $x_n \xrightarrow{w} x$ in X and

$$\limsup_{n\to\infty} \langle A(x_n), x_n - x\rangle \le 0,$$

one has

$$x_n \to x \quad \text{in } X \text{ as } n \to \infty.$$

Let $\varphi \in C^1(X)$. We say that $x^* \in X$ is a critical point of φ if $\varphi'(x^*) = 0$. If $x^* \in X$ is a critical point of φ, then $c = \varphi(x^*)$ is called a critical value of φ. The set of all critical points of φ will be denoted by K_φ.

Given $\varphi \in C^1(X)$, we say that φ satisfies the *Cerami condition* (the C-condition for short), if the following is true:

> every sequence $\{u_n\}_{n\geq 1} \subset X$ such that $\{\varphi(u_n)\}_{n\geq 1}$ is bounded in $\mathbb{R}$ and
> $$(1+\|u_n\|\varphi'(u_n) \to 0 \quad \text{in } X^* \text{ as } n\to\infty$$
> admits a strongly convergent subsequence.

This is a compactness-type condition on the functional φ, which compensates for the fact that the ambient space X needs not be locally compact (in general, X is infinite dimensional). It leads to a deformation theorem from which we can derive the minimax theory for critical values of φ. Prominent in that theory, is the so-called *mountain pass theorem,* due to Ambrosetti-Rabinowitz [5]. Here we state the result in a slightly more general form (see [11]).

Theorem 2.1. *If $(X, \|\cdot\|$ is a Banach space, $\varphi \in C^1(X)$ satisfies the $C-$condition, $u_0,\ u_1 \in X$, $\|u_1 - u_0\| > r > 0$,*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : \|u-u_0\| = r\} =: m_r,$$
$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi(\gamma(t)) \quad \textit{with}$$
$$\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = u_0,\ \gamma(1) = u_1\}$$

then $c \geq m_r$ and c is a critical value of φ.

In the study of problem (1.1) we will use the following spaces:

$$W_p := W^{1,p}_{per}(0,b) = \{u \in W^{1,p}(0,b) : u(0) = u(b)\},$$
$$\widehat{C^1}(T) := C^1(T) \cap W_p.$$

Recall that the Sobolev $W^{1,p}(0,b)$ is embedded continuously (in fact compactly) in $C(T)$. Hence the evaluations at $t=0$ and $t=b$ in the definition of W_p make sense. The Banach space $\widehat{C^1}(T)$ is an ordered Banach space with positive cone

$$\widehat{C_+} = \{u \in C^1(T) : u(t) \geq 0 \text{ for all } t \in T\}.$$

This cone has nonempty interior given by

$$\operatorname{int} \widehat{C_+} = \{u \in C^1(T) : u(t) > 0 \text{ for all } t \in T\}.$$

Now we introduce the following hypotheses on the map $x \to a(|x|)x$:

(H1) $a : (0,\infty) \to (0,\infty)$ is a C^1-function such that:

(i)] $x \to a(x)x$ is strictly increasing on $(0,\infty)$, with
$$a(x)x \to 0 \text{ and } \frac{a'(x)x}{a(x)} \to c > -1 \text{ as } x \to 0^+;$$

(ii) there exists $\widehat{c} > 0$ such that
$$|a(|x|)x| \leq \widehat{c}\big(1 + |x|^{p-1}\big) \quad \text{for all } x \in \mathbb{R};$$

(iii) there exists $C_0 > 0$ such that $a(|x|)x^2 \geq C_0|x|^p$ for all $x \in \mathbb{R}$;

(iv) if $G_0(t) := \int_0^t a(s)sds$ for all $t \geq 0$, then there exists $\xi_0 > 0$ such that
$$pG_0(t) - a(|t|)t^2 \geq -\xi_0 \quad \text{for all } t \geq 0;$$

(v) there exists $\tau \in (q,p)$ such that $t \to G(|x|^{\frac{1}{\tau}})$ is convex on $(0,\infty)$ and

$$\lim_{t\to 0^+} \frac{G_0(t)}{t^\tau} = 0.$$

Remarks: From the above hypotheses, it is clear that $G_0(\cdot)$ is strictly convex and strictly increasing on $(0,\infty)$. We set $G(x) = G_0(|x|)$ for all $x \in R$. Then $G(\cdot)$ is convex, too, and for all $x \neq 0$, we have

$$G'(x) = G_0'(|x|) = a(|x|)|x|\frac{x}{|x|} = a(|x|)x.$$

So, $G(\cdot)$ is the primitive of the function $x \to a(|x|)x$ involved in the definition of the differential operator. The convexity of $G(\cdot)$ and $G(0) = 0$ imply

$$G(x) \le a(|x|)x^2 \quad \text{for all } x \in \mathbb{R}. \tag{2.1}$$

Then, using (2.1) and hypotheses (H1)(ii), (H1)(iii), we have the following growth estimates for the primitive $G(\cdot)$:

$$\frac{C_0}{p}|x|^p \le G(x) \le C_1(1+|x|^p) \quad \text{for some } C_1 > 0, \text{ all } x \in \mathbb{R}. \tag{2.2}$$

Examples: The following functions satisfy hypotheses (H1):

$$a_1(x) = |x|^{p-2}x \quad \text{with } 1 < p < \infty;$$
$$a_2(x) = |x|^{p-2}x + \mu|x|^{q-2}x \quad \text{with } 1 < q < p < \infty \text{ and } \mu > 0;$$
$$a_3(x) = (1+|x|^2)^{\frac{p-2}{2}}x \quad \text{with } 1 < p < \infty;$$
$$a_4(x) = |x|^{p-2}x(1+\frac{1}{1+|x|^p}) \quad \text{with } 1 < p < \infty.$$

The first function corresponds to the scalar p-Laplacian, the second corresponds to the scalar (p,q)-differential operator (that is, the sum of a p-Laplacian and of a q-Laplacian) and the third function is the generalized scalar p-mean curvature differential operator.

Let $A : W_p \to W_p^*$ be the nonlinear map defined by

$$\langle A(u), y\rangle = \int_0^b a(|u'|)u'y'dt \quad \text{for all } u, y \in W_p. \tag{2.3}$$

The following result is well known; see, e.g., [4].

Proposition 2.2. *The nonlinear map $A : W_p \to W_p^*$ defined by* (2.3) *is bounded (that is, it maps bounded sets to bounded sets), demicontinuous, monotone (hence maximal monotone, too) and of type $(S)_+$.*

Let $f_0 : T \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that

$$|f_0(t,x)| \le \alpha_0(t)\big(1+|x|^{r-1}\big) \quad \text{for a.a. } t \in T, \text{ all } x \in \mathbb{R}$$

with $\alpha_0 \in L^1(T)_+$, $1 < r < \infty$. We set

$$F_0(t,x) = \int_0^x f_0(t,s)\,ds$$

and consider the C^1-functional $\varphi_0 : W_p \to \mathbb{R}$ defined by

$$\varphi_0(u) = \int_0^b G(u'(t))dt - \int_0^b F_0(t,u(t))dt \quad \text{for all } u \in W_p.$$

From Aizicovici-Papageorgiou-Staicu [4], we have:

Proposition 2.3. *If hypotheses* (H1) (i)-(iii) *hold and $u_0 \in W_p$ is a local $\widehat{C^1}(T)$-minimizer of φ_0; that is, there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \le \varphi_0(u_0 + h) \quad \text{for all } h \in \widehat{C^1}(T) \text{ with } \|h\|_{\widehat{C^1}(T)} \le \rho_0,$$

then $u_0 \in \widehat{C^1}(T)$ and u_0 is also a local W_p-minimizer of φ_0; that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \le \varphi_0(u_0 + h) \quad \text{for all } h \in W_p \text{ with } \|h\| \le \rho_1.$$

In the above result and in the sequel, by $\| \cdot \|$ we denote the norm of W_p defined by

$$\|u\| = (\|u\|_p^p + \|u'\|_p^p)^{1/p} \quad \text{for all } u \in W_p,$$

where $\| \cdot \|_p$ stands for the norm in $L^p(T)$. Also, if $x \in \mathbb{R}$, then $x^{\pm} = \max\{\pm x, 0\}$. Then given $u \in W_p$, we set $u^{\pm}(.) = u(.)^{\pm}$. We have

$$u^+, u^- \in W_p, \quad |u| = u^+ + u^-, \quad u = u^+ - u^-.$$

Finally, for any Carathéodory function $g : T \times \mathbb{R} \to \mathbb{R}$, we denote by N_g the Nemytskii operator corresponding to g, defined by

$$N_g(u)(\cdot) = g(\cdot, u(\cdot)) \quad \text{for all } u \in W_p.$$

Note that $t \to N_g(u)(t) = g(t, u(t))$ is measurable.

3. A bifurcation-type theorem

In this section, we prove a bifurcation-type theorem describing the set of positive solutions of problem (1.1), as the parameter $\lambda > 0$ varies. The following hypotheses will be needed:

- (H2) $\beta \in L^\infty(T)_+$.
- (H3) $f : T \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(t, 0) = 0$ for a.a. $t \in T$ and
  - (i) $|f(t, x)| \le \alpha(t)(1 + |x|^{r-1})$ for a.a. $t \in T$, all $x \ge 0$, with $\alpha \in L^\infty(T)_+$, $p < r < \infty$;
  - (ii) if $F(t, x) = \int_0^x f(t, s) ds$, then $\lim_{x \to +\infty} \frac{F(t,x)}{x^p} = +\infty$ uniformly for a.a. $t \in T$;
  - (iii) there exist $\mu > \max\{r - p, q\}$ and $\eta_0 > 0$ such that
    $$0 < \eta_0 \le \liminf_{x \to +\infty} \frac{f(t, x)x - pF(t, x)}{x^\mu} \quad \text{uniformly for a.a. } t \in T;$$
  - (iv) $\lim_{x \to 0^+} \frac{f(t,x)}{x^{p-1}} = 0$ uniformly for a.a. $t \in T$;
  - (v) for every $\rho > 0$, there exists $\xi_\rho > 0$ such that for a.a. $t \in T$ the function $x \to f(t, x) + \xi_\rho x^{p-1}$ is nondecreasing on $[0, \rho]$.

Remarks: Since we are looking for positive solutions and all the above hypotheses concern the positive half-axis, the values of $f(t, \cdot)$ on $(-\infty, 0)$ are irrelevant and so, without any loss of generality, we may assume that $f(t, x) = 0$ for a.a. $t \in T$, all $x < 0$. Note that hypotheses (H3) (ii), (iii) imply that

$$\lim_{x \to +\infty} \frac{f(t, x)}{x^{p-1}} = +\infty \quad \text{uniformly for a. a. } t \in T.$$

So, the perturbation $f(t,\cdot)$ is $(p-1)-$superlinear ("convex" nonlinearity). Hence in the reaction of problem (1.1) we have the competing effects of concave and convex terms.

However, for the $(p-1)$-superlinear (convex) term, we do not assume the usual in such cases AR-condition, unilateral version. This condition says that there exist $r_0 > p$ and $M > 0$ such that

$$0 < r_0 F(t,x) \le f(t,x)x \quad \text{for a.a. } t \in T, \text{ all } x \ge M, \tag{3.1}$$

$$\operatorname{essinf} F(.,M) > 0. \tag{3.2}$$

Integrating (3.1) and using (3.2) we obtain the following weaker condition

$$C_2 x^{r_0} \le F(t,x) \quad \text{for a.a. } t \in T, \text{ all } x \ge M \text{ with } C_2 > 0. \tag{3.3}$$

Evidently (3.3) implies the much weaker hypothesis in (H3) (ii). Here we use this superlinearity condition together with (H3) (iii), and the two together are weaker than the AR-condition and incorporate in our framework $(p-1)$-superlinear perturbations with slower growth near $+\infty$.

Examples: The following functions satisfy (H3) (For the sake of simplicity we drop the t-dependence):

$$f_1(x) = x^{r-1} \quad \text{for all } x \ge 0 \text{ with } p < r < \infty,$$

$$f_2(x) = x^{p-1}(\ln(x+1) + \frac{1}{p}) \quad \text{for all } x \ge 0.$$

Note that f_2 does not satisfy the AR-condition.

Our main result reads as follows.

Theorem 3.1. *If hypotheses* (H1)–(H3) *hold, then there exists* $\lambda^* > 0$ *such that:*

(i) *for* $\lambda \in (0,\lambda^*)$*, problem* (1.1) *admits at least two positive solutions* $u_0, \widehat{u} \in \operatorname{int} \widehat{C}_+$ *with* $\widehat{u} - u_0 \in \operatorname{int} \widehat{C}_+$*;*

(ii) *for* $\lambda = \lambda^*$*, problem* (1.1) *has at least one positive solution* $u_* \in \operatorname{int} \widehat{C}_+$*;*

(iii) *for* $\lambda > \lambda^*$*, problem* (1.1) *has no positive solution.*

Moreover, for every $\lambda \in (0,\lambda^*]$ *problem* (1.1) *has a smallest positive solution* u_λ^* *and the curve* $\lambda \to u_\lambda^*$ *is nondecreasing from* $(0,\lambda^*]$ *into* $\widehat{C^1}(T)$.

The proof of Theorem 3.1 is based on several propositions of independent interest. Let

$$\mathcal{P} = \{\lambda > 0 : \text{problem (1.1) admits a positive solution}\}$$

and for every $\lambda \in \mathcal{P}$, let $\mathcal{S}(\lambda)$ be the set of positive solutions of problem (1.1).

First we establish the nonemptiness of the set $\mathcal{P}$ of admissible parameters. To this end, let $\gamma > \|\beta\|_\infty$ (see hypothesis (H2)) and consider the following truncation-perturbation of the reaction of (1.1):

$$h_\lambda(t,x) = \begin{cases} 0 & \text{if } x \le 0, \\ \lambda x^{q-1} + f(t,x) + \gamma x^{p-1} & \text{if } x > 0. \end{cases} \tag{3.4}$$

This is a Carathéodory function. We set $H_\lambda(t,x) := \int_0^x h_\lambda(t,s)ds$ and introduce the C^1-functional $\widehat{\varphi}_\lambda : W_p \to \mathbb{R}$ defined by

$$\widehat{\varphi}_\lambda(u) = \int_0^b G(u'(t))dt + \frac{1}{p}\int_0^b (\beta(t)+\gamma)|u(t)|^p dt - \int_0^b H_\lambda(t,u(t))dt$$

for all $u \in W_p$. Next we show that $\widehat{\varphi}_\lambda$ satisfies the C-condition.

Proposition 3.2. *If hypotheses* (H1)–(H3) *hold and* $\lambda > 0$, *then the functional* $\widehat{\varphi}_\lambda$ *satisfies the* C*-condition.*

Proof. Let $\{u_n\}_{n\geq 1}$ be a sequence in W_p such that

$$|\widehat{\varphi}_\lambda(u_n)| \leq M_1 \quad \text{for some } M_1 > 0 \text{ and all } n \geq 1, \tag{3.5}$$

and

$$(1 + \|u_n\|\widehat{\varphi}'_\lambda(u_n) \to 0 \quad \text{in } W_p^* \text{ as } n \to \infty. \tag{3.6}$$

From (3.6) we have

$$|\langle \widehat{\varphi}'_\lambda(u_n), v\rangle| \leq \frac{\varepsilon_n\|v\|}{1 + \|u_n\|} \quad \text{for all } v \in W_p,\ \text{all } n \geq 1,\ \text{with } \varepsilon_n \to 0^+,$$

hence

$$\big|\langle A(u_n), v\rangle + \int_0^b (\beta(t) + \gamma)|u_n|^{p-2}u_n v dt - \int_0^b h_\lambda(t, u_n)v dt\big| \leq \frac{\varepsilon_n\|v\|}{1 + \|u_n\|} \tag{3.7}$$

for all $n \geq 1$. In (3.7) we choose $v = -u_n^- \in W_p$. Then we have

$$\int_0^b a(|(u_n^-)'|)[(u_n^-)']^2 dt + \int_0^b (\beta(t) + \gamma)(u_n^-)^p dt \leq \varepsilon_n \text{ for all } n \geq 1$$

(see (3.4)), hence

$$C_0\|(u_n^-)'\|_p^p + (\gamma - \|\beta\|_\infty)\|u_n^-\|_p^p \leq \varepsilon_n \quad \text{for all } n \geq 1$$

(with $\gamma > \|\beta\|_\infty$), therefore

$$u_n^- \to 0 \quad \text{in } W_p \text{ as } n \to \infty. \tag{3.8}$$

Next, in (3.7) we choose $v = u_n^+ \in W_p$. Then

$$\begin{aligned}
&\int_0^b a(|(u_n^+)'|)[(u_n^+)']^2 dt - \int_0^b (\beta(t))(u_n^+)^p dt + \lambda\|u_n^+\|_q^q + \int_0^b f(t, u_n^+)u_n^+ dt \\
&\leq \varepsilon_n \quad \text{for all } n \geq 1 \text{(see (3.4))}.
\end{aligned} \tag{3.9}$$

From (3.5) and (3.8), we have

$$\int_0^b pG((u_n^+)')dt + \int_0^b \beta(t)(u_n^+)^p dt - \frac{\lambda p}{q}\|u_n^+\|_q^q - \int_0^b pF(t, u_n^+)dt \leq M_2 \tag{3.10}$$

for some $M_2 > 0$, all $n \geq 1$. We add (3.9) and (3.10) and obtain

$$\begin{aligned}
&\int_0^b [pG\big((u_n^+)'\big) - a(|u_n^+|')\big((u_n^+)'\big)^2]dt + \int_0^b [f(t, u_n^+)u_n^+ - pF(t, u_n^+)]dt \\
&\leq M_3 + \lambda(\frac{p}{q} - 1)\|u_n^+\|_q^q \quad \text{for some } M_3 > 0 \text{ all } n \geq 1,
\end{aligned}$$

hence

$$\int_0^b [f(t, u_n^+)u_n^+ - pF(t, u_n^+)]dt \leq M_4 + \lambda(\frac{p}{q} - 1)\|u_n^+\|_q^q \tag{3.11}$$

for some $M_4 > 0$ and all $n \geq 1$ (see (H1) (iv)). Hypotheses (H3) (i), (iii) imply that we can find $\eta_1 \in (0, \eta_0)$ and $\alpha_1 \in L^1(T)_+$ such that

$$\beta_1|x|^\mu - \alpha_1(t) \leq f(t, x)x - pF(t, x) \quad \text{for a.a. } t \in T \text{ and all } x \geq 0. \tag{3.12}$$

Returning to (3.11) and using (3.12), we obtain

$$\eta_1\|u_n^+\|_\mu^\mu \leq C_4(1 + \|u_n^+\|_\mu^q) \quad \text{for some } C_4 > 0,\ \text{all } n \geq 1$$

(since $q < \mu$ and $q < p$), therefore,

$$\{u_n^+\}_{n\geq 1} \text{ is bounded in } L^\mu(T). \tag{3.13}$$

It is clear from hypothesis (H3) (iii), that we may assume that $\mu < r$. Then we can find $t \in (0,1)$ such that

$$\frac{1}{r} = \frac{t}{\mu}. \tag{3.14}$$

Invoking the interpolation inequality (see, for example, Gasinski-Papageorgiou [11, p. 905]), we have

$$\|u_n^+\|_r \leq \|u_n^+\|_\mu^t \|u_n^+\|_\infty^{1-t}$$

hence

$$\|u_n^+\|_r^r \leq C_5 \|u_n^+\|^{(1-t)r} \quad \text{for some } C_5 > 0, \text{ all } n \geq 1 \tag{3.15}$$

(see (3.13) and use the Sobolev embedding theorem).

Hypotheses (H3) (i), (iv) imply that we can find $C_6 = C_6(\lambda) > 0$ such that

$$\lambda x^q + f(t,x)x \leq C_6(1 + x^r) \quad \text{for a.a. } t \in T, \text{ all } x \geq 0. \tag{3.16}$$

In (3.7) we choose $v = u_n^+ \in W_p$. Then

$$\begin{aligned} &\int_0^b a(|u_n^+|')\big((u_n^+)'\big)^2 dt + \int_0^b \beta(t)(u_n^+)^p dt - \lambda\|u_n^+\|_q^q - \int_0^b f(t,u_n^+)u_n^+ dt \\ &\leq \varepsilon_n \quad \text{for all } n \geq 1. \end{aligned}$$

(see (3.4)), hence

$$C_0 \|(u_n^+)'\|_p^p \leq C_7(1 + \|u_n^+\|_r^r) \quad \text{for some } C_7 > 0, \text{ all } n \geq 1 \tag{3.17}$$

(see (H1) (iii) , (H2), (3.16) and recall $p < r$). We know that $y \to \|y'\|_p + \|y\|_\mu$ is an equivalent norm on W_p (see, for example, Gasinski-Papageorgiou [11], p.227). So, from (3.13), (3.15) and (3.17) we have

$$\begin{aligned} \|u_n^+\|^p &\leq C_8(1 + \|u_n^+\|_r^r) \quad \text{for some } C_8 > 0, \text{ all } n \geq 1 \\ &\leq C_9(1 + \|u_n^+\|^{(1-t)r}) \quad \text{for some } C_9 > 0, \text{ all } n \geq 1. \end{aligned} \tag{3.18}$$

From (3.14) we have $\mu = tr$. Hence $(1-t)r = r - \mu < p$; see hypothesis (H1) (iii). So, from (3.18) it follows that $\{u_n^+\}_{n\geq 1}$ is bounded in W_p; consequently

$$\{u_n\}_{n\geq 1} \quad \text{is bounded in } W_p \text{ (see 3.8)}. \tag{3.19}$$

By (3.19) and passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_p \text{ and } u_n \to u \text{ in } C(T) \text{ as } n \to \infty. \tag{3.20}$$

In (3.7) we choose $v = u_n - u \in W_p$, pass to the limit as $n \to \infty$ and use (3.20). Then

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0,$$

hence $u_n \to u$ in W_p (see Proposition 2.2). Therefore $\widehat{\varphi}_\lambda$ satisfies the C-condition. □

To establish the nonemptiness of the set $\mathcal{P}$, we use Theorem 2.1. To this end, we need to satisfy the mountain pass geometry for the functional $\widehat{\varphi}_\lambda$. The next proposition is a crucial step in this direction.

Proposition 3.3. *If hypotheses* (H1)–(H3) *hold, then there exists* $\lambda_+ > 0$ *such that for all* $\lambda \in (0, \lambda_+)$ *we can find* $\rho_\lambda > 0$ *for which we have*

$$\inf\{\widehat{\varphi}_\lambda(u) : \|u\| = \rho_\lambda\} = \widehat{m}_\lambda > 0.$$

Proof. Hypotheses (H3) (i), (iv) imply that given $\varepsilon > 0$, there exists $C_{10} = C_{10}(\varepsilon) > 0$ such that

$$\lambda x^{q-1} + f(t,x) \le (\lambda + \varepsilon)x^{q-1} + C_{10}x^{r-1} \quad \text{for a.a. } t \in T, \text{ all } x \ge 0. \tag{3.21}$$

Then, for every $u \in W_p$, we have

$$\begin{aligned} \widehat{\varphi}_\lambda(u) &= \int_0^b G(u'(t))dt + \frac{1}{p}\int_0^b (\beta(t)+\gamma)|u(t)|^p dt - \int_0^b H_\lambda(t,u(t))dt \\ &\ge \frac{C_0}{p}\|u'\|_p^p + \frac{1}{p}\int_0^b (\beta(t)+\gamma)|u(t)|^p dt - \frac{\lambda+\varepsilon}{q}\|u^+\|_q^q \\ &\quad - \frac{C_{10}}{r}\|u^+\|_r^r - \frac{\gamma}{p}\|u^+\|_p^p \quad \text{(see (2.2), (3.4) and (3.21))} \\ &\ge C_{11}\|u\|^p - \frac{\gamma}{p}\|u\|^p - C_{12}(\frac{\lambda+\varepsilon}{q}\|u\|^q + \|u\|^r) \end{aligned} \tag{3.22}$$

for some $C_{11}, C_{12} > 0$ (recall that $\gamma > \|\beta\|_\infty$). Since $q < p < r$, given $\varepsilon > 0$, one can find $C_\varepsilon > 0$ such that

$$\|u\|^p \le \frac{\varepsilon p}{\gamma q}\|u\|^q + C_\varepsilon\|u\|^r \quad \text{for all } u \in W_p.$$

So, from (3.22) we have

$$\begin{aligned} \widehat{\varphi}_\lambda(u) &\ge C_{11}\|u\|^p - C_{13}(\frac{\lambda+2\varepsilon}{q}\|u\|^q + \|u\|^r) \quad \text{for some } C_{13} > 0 \\ &= [C_{11} - C_{13}(\frac{\lambda+2\varepsilon}{q}\|u\|^{q-p} + \|u\|^{r-p})]\|u\|^p. \end{aligned} \tag{3.23}$$

Consider the function

$$\theta_\lambda(t) = \frac{\lambda+2\varepsilon}{q}t^{q-p} + t^{r-p} \quad \text{for all } t > 0.$$

Evidently, $\theta_\lambda \in C^1(0,\infty)$, and because $q < p < r$, we have

$$\theta_\lambda(t) \to +\infty \quad \text{as } t \to 0^+ \text{ and } t \to +\infty.$$

So, we can find $t_0 \in (0,\infty)$ such that

$$\theta_\lambda(t_0) = \inf\{\theta_\lambda(t) : t > 0\},$$

hence $\theta_\lambda'(t_0) = 0$; therefore

$$\frac{\lambda+2\varepsilon}{q}(p-q)t_0^{q-p-1} = (r-p)t_0^{r-p-1}.$$

We obtain

$$t_0 = t_0(\lambda,\varepsilon) = \big[\frac{(\lambda+2\varepsilon)(p-q)}{q(r-p)}\big]^{\frac{1}{r-q}}.$$

Note that $\theta_\lambda(t_0(\lambda,\varepsilon)) \to 0^+$ as $\lambda, \varepsilon \to 0^+$. Therefore, we can find λ_+, $\varepsilon_+ > 0$ small such that

$$\theta_\lambda(t_0(\lambda,\varepsilon)) < \frac{C_{11}}{C_{13}} \quad \text{for all } \lambda \in (0,\lambda_+),\ \varepsilon \in (0,\varepsilon_+). \tag{3.24}$$

So, fixing $\varepsilon \in (0, \varepsilon_+)$, from (3.23) and (3.24) we have $\widehat{\varphi}_\lambda(u) \geq \widehat{m}_\lambda > 0$ for all $u \in W_p$ with $\|u\| = \rho_\lambda = t_0(\lambda, \varepsilon)$ and all $\lambda \in (0, \lambda_+)$. $\square$

By adapting the proof of Proposition 6 in [4], we arrive at the following result, which completes the mountain pass geometry for the functional $\widehat{\varphi}_\lambda$.

Proposition 3.4. *If hypotheses* (H1)–(H3) *hold,* $\lambda > 0$ *and* $u \in \operatorname{int} \widehat{C}_+$*, then* $\widehat{\varphi}_\lambda(tu) \to -\infty$ *as* $t \to +\infty$.

Now we establish the nonemptiness of the set $\mathcal{P}$ and also determine the nature of the solution set $\mathcal{S}(\lambda)$ when $\lambda \in \mathcal{P}$.

Proposition 3.5. *If hypotheses* (H1)–(H3) *hold, then* $\mathcal{P} \neq \varnothing$ *and for all* $\lambda \in \mathcal{P}$ *we have* $\mathcal{S}(\lambda) \subseteq \operatorname{int} \widehat{C}_+$.

Proof. Propositions 3.2, 3.3 and 3.4 permit the use of Theorem 2.1 (the mountain pass theorem) on $\widehat{\varphi}_\lambda$ when $\lambda \in (0, \lambda_+)$. So, we can find $u_\lambda \in W_p$ such that

$$u_\lambda \in K_{\widehat{\varphi}_\lambda} \text{ and } 0 = \widehat{\varphi}_\lambda(0) < \widehat{m}_\lambda \leq \widehat{\varphi}_\lambda(u_\lambda). \tag{3.25}$$

Evidently $u_\lambda \neq 0$. Also, since $u_\lambda \in K_{\widehat{\varphi}_\lambda}$ we have $\widehat{\varphi}'_\lambda(u_\lambda) = 0$, hence

$$A(u_\lambda) + (\beta(t) + \gamma)|u_\lambda|^{p-2} u_\lambda = N_{h_\lambda}(u_\lambda). \tag{3.26}$$

On (3.26) we act with $-u_\lambda^- \in W_p$. Then

$$\int_0^b a(|(u_\lambda^-)'|)((u_\lambda^-)')^2 dt + \int_0^b (\beta(t) + \gamma)(u_\lambda^-)^p dt = 0 \quad \text{(see (3.4))},$$

hence

$$C_0 \|(u_\lambda^-)'\|_p^p + C_{14} \|u_\lambda^-\|_p^p \leq 0 \quad \text{for some } C_{14} > 0,$$

(see (H1) (iii) and recall that $\gamma > \|\beta\|_\infty$), therefore $u_\lambda \geq 0$, $u_\lambda \neq 0$. Then, because of (3.4), equation (3.26) becomes

$$A(u_\lambda) + \beta(t)|u_\lambda|^{p-1} = \lambda u_\lambda^{q-1} + N_f(u_\lambda),$$

therefore $u_\lambda \in \mathcal{S}(\lambda)$ and $u_\lambda \in \widehat{C}_+ \backslash \{0\}$ (see (3.4)).

Let $\rho = \|u_\lambda\|_\infty$ and let $\xi_\rho > 0$ be as postulated by hypothesis (h3) (v). Then

$$\begin{aligned}
&-(a(|u'_\lambda(t)|)u'_\lambda(t))' + (\beta(t) + \xi_\rho)[u_\lambda(t)]^{p-1} \\
&= \lambda [u_\lambda(t)]^{q-1} + f(t, u_\lambda(t)) + \xi_\rho [u_\lambda(t)]^{p-1} \geq 0 \quad \text{a.e. on } T,
\end{aligned}$$

hence

$$-(a(|u'_\lambda(t)|)u'_\lambda(t))' \leq (\|\beta\|_\infty + \xi_\rho)[u_\lambda(t)]^{p-1} \quad \text{a.e. on } T,$$

and we infer that $u_\lambda \in \operatorname{int} \widehat{C}_+$ (see Pucci-Serrin [15, pp. 111, 120]). Therefore we conclude that

$$(0, \lambda_+) \subseteq \mathcal{P} \text{ and } \mathcal{S}(\lambda) \subseteq \operatorname{int} \widehat{C}_+ \text{ for all } \lambda \in \mathcal{P}.$$

$\square$

Proposition 3.6. *If hypotheses* (H1)–(H3) *hold and* $\lambda \in \mathcal{P}$*, then* $(0, \lambda] \subseteq \mathcal{P}$.

Proof. Let $\mu \in (0, \lambda)$ and let $u_\lambda \in \mathcal{S}(\lambda)$. We introduce the following truncation-perturbation of the reaction of (1.1) with μ instead of λ, (P_μ):

$$e_\mu(t, x) = \begin{cases} 0 & \text{if } x < 0 \\ \mu x^{q-1} + f(t, x) + \gamma x^{p-1} & \text{if } 0 \leq x \leq u_\lambda(t) \\ \mu u_\lambda(t)^{q-1} + f(t, u_\lambda(t)) + \gamma u_\lambda(t)^{p-1} & \text{if } u_\lambda(t) < x. \end{cases} \tag{3.27}$$

This is a Carathéodory function. We set $E_\mu(t,x) = \int_0^x e_\mu(t,s)ds$ and consider the C^1-functional $\psi_\mu : W_p \to \mathbb{R}$ defined by

$$\psi_\mu(u) = \int_0^b G(u'(t))dt + \frac{1}{p}\int_0^b (\beta(t)+\gamma)|u(t)|^p dt - \int_0^b E_\mu(t,u(t))dt$$

for all $u \in W_p$. By (3.27) and since $\gamma > \|\beta\|_\infty$, it is clear that ψ_μ is coercive. Also, it is sequentially weakly lower semicontinuous (just use the Sobolev embedding theorem and the fact that since $G(\cdot)$ is convex, the integral functional $y \to \int_0^b G(y'(t))dt$ is sequentially weakly lower semicontinuous). So, by the Weierstrass theorem, we can find $u_\mu \in W_p$ such that

$$\psi_\mu(u_\mu) = \inf\{\psi_\mu(u) : u \in W_p\}. \tag{3.28}$$

By hypothesis (H3) (iv) , given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) \in (0,1)$ such that

$$F(t,x) \geq -\frac{\varepsilon}{p}x^p \text{ for a.a. } t \in T, \text{ all } x \in [0,\delta]. \tag{3.29}$$

Let $\xi \in (0, \min\{\delta, \min_T u_\lambda\})$ (recall that $u_\lambda \in \operatorname{int}\widehat{C}_+$). Then

$$\begin{aligned}
\psi_\mu(\xi) &\leq \frac{\xi^p}{p}\|\beta\|_\infty b - \frac{\mu\xi^q}{q}b - \int_0^b F(t,\xi)dt \text{ (see (3.27))}\\
&\leq \frac{\xi^p}{p}[\|\beta\|_\infty + \varepsilon]b - \frac{\mu\xi^q}{q}b.
\end{aligned}$$

Since $q < p$, choosing $\xi \in (0,1)$ even smaller if necessary, we have $\psi_\mu(\xi) < 0$. Then

$$\psi_\mu(u_\mu) < 0 \quad \text{(see (3.28))}.$$

hence $u_\mu \neq 0$. From (3.28), we have $\psi_\mu'(u_\mu) = 0$, hence

$$A(u_\mu) + (\beta(t)+\gamma)|u_\mu|^{p-2}u_\mu = N_{e_\mu}(u_\mu). \tag{3.30}$$

On (3.30), first we act with $-u_\mu^- \in W_p$ and then with $(u_\mu - u_\lambda)^+ \in W_p$ and obtain

$$u_\mu \in [0, u_\lambda] := \{u \in W_p : 0 \leq u(t) \leq u_\lambda(t) \text{ for all } t \in T\}.$$

From (3.27) it follows that $u_\mu \in \mathcal{S}(\mu) \subseteq \operatorname{int}\widehat{C}_+$ and so, $(0,\lambda] \subseteq \mathcal{P}$. ☐

Let $\lambda^* := \sup\mathcal{P}$.

Proposition 3.7. *If hypotheses* (H1)–(H3) *hold, then* $\lambda^* < \infty$.

Proof. Fix $\gamma_0 \geq \|\beta\|_\infty$. Hypotheses (H3) (i)–(iv) imply that there exists $\widetilde{\lambda} > 0$ such that

$$\widetilde{\lambda}x^{q-1} + f(t,x) \geq \gamma_0 x^{p-1} \quad \text{for a.a. } t \in T, \text{ all } x \geq 0 \tag{3.31}$$

(recall that $q < p$). Let $\lambda > \widetilde{\lambda}$ and suppose that $\lambda \in \mathcal{P}$. Then we can find $u_\lambda \in \mathcal{S}(\lambda) \subseteq \operatorname{int}\widehat{C}_+$ (see Proposition 3.5). Let

$$m = \min_T u_\lambda > 0.$$

For $\delta > 0$, let $m_\delta = m + \delta \in \operatorname{int}\widehat{C}_+$. For $\rho = \|u_\lambda\|_\infty$, let $\xi_\rho > 0$ be as postulated by hypothesis (H3) (v). Evidently, we may assume that $\xi_\rho > \|\beta\|_\infty$. Then

$$\begin{aligned}
&-(a(|m_\delta'|)m_\delta')' + (\beta(t)+\xi_\rho)m_\delta^{p-1}\\
&\leq (\gamma_0 + \xi_\rho)m^{p-1} + \chi(\delta) \text{ with } \chi(\delta) \to 0^+ \quad \text{as } \delta \to 0^+\\
&\leq \widetilde{\lambda}m^{q-1} + f(t,m) + \xi_\rho m^{p-1} + \chi(\delta) \quad \text{(see (3.31))}
\end{aligned}$$

$$
\begin{aligned}
&\le \widetilde{\lambda} u_\lambda(t)^{q-1} + f(t,u_\lambda(t)) + \xi_\rho u_\lambda(t)^{p-1} + \chi(\delta) \quad \text{(see (H3) (v))}\\
&= \lambda u_\lambda(t)^{q-1} + f(t,u_\lambda(t)) + \xi_\rho u_\lambda(t)^{p-1} - (\lambda-\widetilde{\lambda})u_\lambda(t)^{q-1} + \chi(\delta)\\
&\le \lambda u_\lambda(t)^{q-1} + f(t,u_\lambda(t)) + \xi_\rho u_\lambda(t)^{p-1} - (\lambda-\widetilde{\lambda})m^{q-1} + \chi(\delta) \quad (\text{since } \lambda > \widetilde{\lambda})\\
&\le \lambda u_\lambda(t)^{q-1} + f(t,u_\lambda(t)) + \xi_\rho u_\lambda(t)^{p-1} \quad \text{for } \delta > 0\\
&= -(a(|u_\lambda'(t)|)u_\lambda'(t))' + (\beta(t)+\xi_\rho)|u_\lambda(t)|^{p-1} \text{ a.e. on } T, \text{ for } \delta > 0 \text{ small,}
\end{aligned}
$$

hence $m_\delta \le u_\lambda(t)$ for all $t \in T$; therefore $m_\delta \le m$ for $\delta > 0$ small, which is a contradiction. So, $\lambda \notin \mathcal{P}$ and we have $\lambda^* \le \widetilde{\lambda} < \infty$. ☐

From Proposition 3.6, we see that $(0,\lambda^*) \subseteq \mathcal{P}$.

Proposition 3.8. *If hypotheses* (H1)–(H3) *hold and* $\lambda \in (0,\lambda^*)$*, then problem* (1.1) *admits at least two positive solutions*

$$
u_0, \widehat{u} \in \operatorname{int} \widehat{C}_+, \quad \widehat{u} - u_0 \in \operatorname{int} \widehat{C}_+,
$$

and u_0 *is a local minimizer of the functional* $\widehat{\varphi}_\lambda$*.*

Proof. Let $\theta \in (\lambda,\lambda^*)$ and let $u_\theta \in \mathcal{S}(\theta) \subseteq \operatorname{int} \widehat{C}_+$. As in the proof of Proposition 3.6, we truncate the reaction of problem (1.1) at $u_\theta(t)$ and use the direct method to obtain

$$
u_0 \in [0,u_\theta] \cap \mathcal{S}(\lambda).
$$

For $\delta > 0$, let $u_0^\delta = u_0 + \delta \in \operatorname{int} \widehat{C}_+$. Let $\rho = \|u_\theta\|_\infty$ and let $\xi_\rho > 0$ be as postulated by hypothesis (H3) (v). We can always assume that $\xi_\rho > \|\beta\|_\infty$. We have

$$
\begin{aligned}
&-(a(|(u_0^\delta)'|)(u_0^\delta)')' + (\beta(t)+\xi_\rho)(u_0^\delta)^{p-1}\\
&\le -(a(|u_0'|)u_0')' + (\beta(t)+\xi_\rho)u_0^{p-1} + \chi(\delta) \quad \text{with } \chi(\delta) \to 0^+ \text{ as } \delta \to 0^+\\
&= \lambda u_0^{q-1} + f(t,u_0) + \xi_\rho {u_0}^{p-1} + \chi(\delta) \ (\text{since } u_0 \in \mathcal{S}(\lambda))\\
&= \theta u_0^{q-1} + f(t,u_0) + \xi_\rho {u_0}^{p-1} - (\theta-\lambda)u_0^{q-1} + \chi(\delta)\\
&\le \theta u_\theta^{q-1} + f(t,u_\theta) + \xi_\rho {u_\theta}^{p-1} - (\theta-\lambda)m_0^{q-1} + \chi(\delta)\\
&\quad (\text{recall that } u_0 \le u_\theta, u_0 \in \operatorname{int} \widehat{C}_+ \text{ see (H3) (v)})\\
&\le \theta u_\theta^{q-1} + f(t,u_\theta) + \xi_\rho u_\theta^{p-1} \text{ for } \delta > 0 \text{ small}\\
&= -(a(|u_\theta'(t)|)u_\theta'(t))' + (\beta(t)+\xi_\rho)u_\theta(t)^{p-1} \text{ a.e. on } T.
\end{aligned}
$$

Then $u_0^\delta \le u_\theta$ for $\delta > 0$ small, hence $u_\theta - u_0 \in \operatorname{int} \widehat{C}_+$. So, we have that

$$
u_0 \in \operatorname{int}_{\widehat{C^1}(T)}[0,u_\theta]. \tag{3.32}
$$

Let ψ_λ be the C^1-functional corresponding to the truncation-perturbation of the reaction of (1.1) at $u_\theta(t)$ (see the proof of Proposition 3.6 and in particular (3.27) with μ replaced by λ and λ replaced by θ). We know that u_0 is a minimizer of ψ_λ and

$$
\psi_\lambda \mid_{[0,u_\theta]} = \widehat{\varphi}_\lambda \mid_{[0,u_\theta]} . \tag{3.33}
$$

From (3.32) and (3.33) it follows that u_0 is a local $\widehat{C^1}(T)$-minimizer of $\widehat{\varphi}_\lambda$. Invoking Proposition 2.3, we infer that u_0 is a local W_p-minimizer of $\widehat{\varphi}_\lambda$. We consider the following Carathéodory function

$$
\eta_\lambda(t,x) = \begin{cases} \lambda u_0(t)^{q-1} + f(t,u_0(t)) + \gamma u_0(t)^{p-1} & \text{if } x \le u_0(t)\\ \lambda x^{q-1} + f(t,x) + \gamma x^{p-1} & \text{if } u_0(t) < x. \end{cases} \tag{3.34}
$$

As before, $\gamma > \|\beta\|_\infty$. Let $H_\lambda(t,x) = \int_0^x \eta_\lambda(t,s)ds$ and consider the C^1-functional $\sigma_\lambda : W_p \to \mathbb{R}$ defined by

$$\sigma_\lambda(u) = \int_0^b G(u'(t))dt + \frac{1}{p}\int_0^b (\beta(t)+\gamma)|u(t)|^p dt - \int_0^b H_\lambda(t,u(t))dt \text{ for all } u \in W_p.$$

From (3.34) it is clear that $\sigma_\lambda = \widehat{\varphi}_\lambda + \widehat{\zeta}_\lambda$ for some $\widehat{\zeta}_\lambda \in \mathbb{R}$, hence

$$\sigma_\lambda \text{ satisfies the } C-\text{condition.} \tag{3.35}$$

Also, u_0 is a local W_p-minimizer of σ_λ (since it is a local W_p-minimizer of $\widehat{\varphi}_\lambda$). So, we can find $\rho \in (0,1)$ small such that

$$\sigma_\lambda(u_0) < \inf\{\sigma_\lambda(u) : \|u\| = \rho\} = m_\rho \tag{3.36}$$

(see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 29).

Hypothesis (H3) (iii) and (3.34)) imply that

$$\sigma_\lambda(\xi) \to -\infty \quad \text{as } \xi \to +\infty,\ \xi \in \mathbb{R}. \tag{3.37}$$

Then (3.35), (3.36), (3.37) enable us to use Theorem 2.1 (the mountain pass theorem). So, we can find $\widehat{u} \in W_p$ such that

$$\widehat{u} \in K_{\sigma_\lambda} \text{ and } m_\rho \le \sigma_\lambda(\widehat{u}). \tag{3.38}$$

From (3.36), (3.38) it follows that $\widehat{u} \neq u_0$. Also $\sigma_\lambda'(\widehat{u}) = 0$, hence

$$A(\widehat{u}) + (\beta(t)+\gamma)|\widehat{u}|^{p-2}\widehat{u} = N_{\eta_\lambda}(\widehat{u}). \tag{3.39}$$

Acting on (3.39) with $(u_0 - \widehat{u})^+ \in W_p$ and using (3.34), we show that $u_0 \le \widehat{u}$, and so,

$$\widehat{u} \in \mathcal{S}(\lambda) \subseteq \operatorname{int} \widehat{C}_+.$$

Moreover, reasoning as in the first part of the proof, we conclude that

$$\widehat{u} - u_0 \in \operatorname{int} \widehat{C}_+$$

□

Next we deal with the critical case $\lambda = \lambda^*$. To treat this case, we first need some auxiliary results.

Hypotheses (H2) and (H3) (i), (iv) imply that given $\lambda > 0$ and $\varepsilon \in (0,\lambda)$, there is a $C_{15} > 0$ such that

$$\lambda x^{q-1} + f(t,x) - \beta(t)x^{p-1} \ge (\lambda-\varepsilon)x^{q-1} - C_{15}x^{r-1} \quad \text{for a.a. } t \in T, \text{ all } x \ge 0. \tag{3.40}$$

This unilateral growth condition on the reaction of (1.1) leads to the auxiliary periodic problem (AP_λ),

$$\begin{gathered} -(a(|u'(t)|)u'(t))' = (\lambda-\varepsilon)u(t)^{q-1} - C_{15}u(t)^{r-1} \quad \text{a.e. on } T, \\ u(0) = u(b), \quad u'(0) = u'(b), \quad u > 0,\ \varepsilon \in (0,\lambda). \end{gathered} \tag{3.41}$$

Proposition 3.9. *If hypotheses* (H1)–(H3) *hold and* $\lambda > 0$*, then problem* (3.41) *has a unique positive solution* $\overline{u}_\lambda \in \operatorname{int} \widehat{C}_+$ *and the map* $\lambda \to \overline{u}_\lambda$ *is nondecreasing from* $(0,\infty)$ *into* $\widehat{C^1}(T)$ *(that is, if* $\lambda < \mu$*, then* $\overline{u}_\lambda \le \overline{u}_\mu$*)*

Proof. First we show the existence of a positive solution for the problem (3.41). For this purpose, we introduce the C^1-functional $\epsilon_\lambda : W_p \to \mathbb{R}$ defined by

$$\epsilon_\lambda(u) = \int_0^b G(u'(t))dt + \frac{1}{p}\|u\|_p^p - \frac{\lambda-\varepsilon}{q}\|u^+\|_q^q + \frac{C_{15}}{r}\|u^+\|_r^r - \frac{1}{p}\|u^+\|_p^p$$

for all $u \in W_p$. We observe that since by hypothesis $q < p < r$, ϵ_λ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\overline{u}_\lambda \in W_p$ such that

$$\epsilon_\lambda(\overline{u}_\lambda) = \inf\{\epsilon_\lambda(u) : u \in W_p\}. \tag{3.42}$$

Since $q < p < r$, for $\xi \in (0,1)$ small, we have $\epsilon_\lambda(\xi) < 0$. Then

$$\epsilon_\lambda(\overline{u}_\lambda) < 0 = \epsilon_\lambda(0),$$

hence $\overline{u}_\lambda \neq 0$. From (3.42) we have

$$\epsilon'_\lambda(\overline{u}_\lambda) = 0,$$

hence

$$A(\overline{u}_\lambda) + |\overline{u}_\lambda|^{p-2}\overline{u}_\lambda = (\lambda-\varepsilon)(\overline{u}_\lambda^+)^{q-1} - C_{15}(\overline{u}_\lambda^+)^{r-1} + (\overline{u}_\lambda^+)^{p-1}. \tag{3.43}$$

On (3.43) we act with $-\overline{u}_\lambda^- \in W_p$ and obtain

$$\overline{u}_\lambda \geq 0,\ \overline{u}_\lambda \neq 0.$$

Then (3.43) becomes

$$A(\overline{u}_\lambda) = (\lambda-\varepsilon)(\overline{u}_\lambda)^{q-1} - C_{15}(\overline{u}_\lambda)^{r-1},$$

hence $\overline{u}_\lambda \in C_+\backslash\{0\}$ is a solution of (3.41). Moreover, we have

$$(a(|\overline{u}'_\lambda|)\overline{u}'_\lambda)' \leq C_{15}\|\overline{u}_\lambda\|_\infty^{r-p}\overline{u}_\lambda^{p-1},$$

hence $\overline{u}_\lambda \in \operatorname{int}\widehat{C}_+$ (see Pucci-Serrin [15, pp. 111, 120],).

Next we show the uniqueness of this solution. To do this, we introduce the integral functional $\chi : L^1(T) \to \overline{\mathbb{R}} = \mathbb{R}\cup\{+\infty\}$ defined by

$$\chi(u) = \begin{cases} \int_0^b G((u^{\frac{1}{\tau}})')dt & \text{if } u \geq 0, u^{\frac{1}{\tau}} \in W_p \\ +\infty & \text{otherwise,} \end{cases}$$

where τ is as in (H1) (v). Let

$$\operatorname{dom}\chi = \{u \in L^1(T) : \chi(u) < +\infty\}$$

be the effective domain of χ and consider $u_1,\ u_2 \in \operatorname{dom}\chi$. Let $t \in [0,1]$ and set

$$y = [tu_1 + (1-t)u_2]^{1/\tau},\ v_1 = u_1^{1/\tau},\ v_2 = u_2^{1/\tau}.$$

From Diaz-Saa [10] (Lemma 1), we have

$$|y'(t)| = [t|v_1'(t)|^\tau + (1-t)|v_2'(t)|^\tau]^{1/\tau} \quad \text{for a.a. } t \in T,$$

hence

$$\begin{aligned} G_0(|y'(t)|) &\leq G_0([t|v_1'(t)|^\tau + (1-t)|v_2'(t)|^\tau]^{1/\tau}) \quad (G_0(\cdot)s \text{ is increasing}) \\ &\leq tG_0(|v_1'(t)|) + (1-t)G_0(|v_2'(t)|) \quad \text{for a.a. } t \in T \text{ (see (H1) (v))} \end{aligned}$$

therefore

$$G(y'(t)) \leq tG((u_1(t)^{1/\tau})') + (1-t)G((u_2(t)^{\frac{1}{\tau}})') \quad \text{for a.a. } t \in T,$$

hence $\chi(\cdot)$ is convex. Moreover, via Fatou's lemma we can see that $\chi(\cdot)$ is also lower semicontinuous.

Suppose that u, $v \in W_p$ are two solutions of (3.41). From the first part of the proof, we have that u, $v \in \operatorname{int} \widehat{C}_+$. Therefore, if $h \in \widehat{C^1}(T)$ and $|\mu| < 1$ is small, then

$$u^\tau + \mu h \in \operatorname{dom}\chi \text{ and } v^\tau + \mu h \in \operatorname{dom}\chi.$$

It follows that χ is Gâteaux differentiable at u^τ and v^τ in the direction h. Moreover, using the chain rule and the density of $\widehat{C^1}(T)$ in W_p, we have for all $h \in W_p$,

$$\begin{aligned}
\chi'(u^\tau)(h) &= \frac{1}{\tau}\int_0^b \frac{-(a(|u'|)u')'}{u^{\tau-1}} h\,dt,\\
\chi'(v^\tau)(h) &= \frac{1}{\tau}\int_0^b \frac{-(a(|v'|)v')'}{v^{\tau-1}} h\,dt.
\end{aligned}$$

The convexity of $\chi(\cdot)$ implies the monotonicity of $\chi(\cdot)$. Therefore

$$\begin{aligned}
0 &\le \frac{1}{\tau}\int_0^b \Big(\frac{-(a(|u'|)u')'}{u^{\tau-1}} + \frac{(a(|v'|)v')'}{v^{\tau-1}}\Big)(u^\tau - v^\tau)dt\\
&= \frac{1}{\tau}\int_0^b [(\lambda-\varepsilon)\Big(\frac{1}{u^{\tau-q}} - \frac{1}{v^{\tau-q}}\Big) - C_{15}(u^{r-\tau} - v^{r-\tau})](u^\tau - v^\tau)dt\\
&\le 0,
\end{aligned}$$

and we conclude that $u = v$. This proves the uniqueness of the solution $\overline{u}_\lambda \in \operatorname{int}\widehat{C}_+$.

Next we show that $\lambda \to \overline{u}_\lambda$ is nondecreasing from $(0,\infty)$ into $\widehat{C^1}(T)$. Indeed, let $\lambda < \mu$ and let $\overline{u}_\mu \in \operatorname{int}\widehat{C}_+$ be the unique positive solution of problem (AP_μ). We consider the following truncation-perturbation of the reaction in problem (3.41):

$$\theta_\lambda(t,x) = \begin{cases} 0 & \text{if } x < 0\\ (\lambda-\varepsilon)x^{q-1} - C_{15}x^{r-1} + x^{p-1} & \text{if } 0 \le x \le \overline{u}_\mu(t)\\ (\lambda-\varepsilon)\overline{u}_\mu(t)^{q-1} - C_{15}\overline{u}_\mu(t)^{r-1} + \overline{u}_\mu(t)^{p-1} & \text{if } \overline{u}_\mu(t) < x. \end{cases}$$

This is a Carathéodory function. We set $\Theta_\lambda(t,x) = \int_0^x \theta_\lambda(t,s)ds$ and consider the C^1-functional $\psi_\lambda : W_p \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = \int_0^b G(u'(t))dt + \frac{1}{p}\|u\|_p^p - \int_0^b \Theta_\lambda(t,u(t))dt \quad \text{for all } u \in W_p.$$

It is clear that $\psi_\lambda(\cdot)$ is coercive. Also, $\psi_\lambda(\cdot)$ is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_\lambda \in W_p$ such that

$$\psi_\lambda(\widetilde{u}_\lambda) = \inf\{\psi_\lambda(u) : u \in W_p\}. \tag{3.44}$$

As in the proof of Proposition 3.6, since $q < p < r$, for $\xi \in (0, \min\{1, \min_T \overline{u}_\mu\})$ small (recall that $\overline{u}_\mu \in \operatorname{int}\widehat{C}_+$), we have $\psi_\lambda(\xi) < 0$. Then

$$\psi_\lambda(\widetilde{u}_\lambda) < 0 = \psi_\lambda(0) \quad \text{(see (3.44))},$$

hence $\widetilde{u}_\lambda \ne 0$. From (3.44), we have

$$\psi_\lambda'(\widetilde{u}_\lambda) = 0,$$

which implies

$$A(\widetilde{u}_\lambda) + |\widetilde{u}_\lambda|^{p-2}\widetilde{u}_\lambda = N_{\theta_\lambda}(\widetilde{u}_\lambda). \tag{3.45}$$

On (3.45) we first act with $-\widetilde{u}_\lambda^- \in W_p$ and obtain

$$\widetilde{u}_\lambda \ge 0, \quad \widetilde{u}_\lambda \ne 0$$

(see hypothesis (H1) (iii). Also we act with $(\widetilde{u}_\lambda - \overline{u}_\mu)^+ \in W_p$. Then

$$\begin{aligned}
&\langle A(\widetilde{u}_\lambda), (\widetilde{u}_\lambda - \overline{u}_\mu)^+\rangle + \int_0^b \widetilde{u}_\lambda^{p-1}(\widetilde{u}_\lambda - \overline{u}_\mu)^+ dt \\
&= \int_0^b \theta_\lambda(t, \widetilde{u}_\lambda)(\widetilde{u}_\lambda - \overline{u}_\mu)^+ dt \\
&= \int_0^b [(\lambda-\varepsilon)\overline{u}_\mu^{q-1} - C_{15}\overline{u}_\mu^{r-1} + \overline{u}_\mu^{p-1}](\widetilde{u}_\lambda - \overline{u}_\mu)^+ dt \\
&= \langle A(\overline{u}_\mu), (\widetilde{u}_\lambda - \overline{u}_\mu)^+\rangle + \int_0^b \widetilde{u}_\mu^{p-1}(\widetilde{u}_\lambda - \overline{u}_\mu)^+ dt
\end{aligned}$$

(since $\overline{u}_\mu$ is a solution of (AP_μ), hence $\widetilde{u}_\lambda \leq \overline{u}_\mu$, therefore

$$\widetilde{u}_\lambda \in [0, \overline{u}_\mu]\backslash\{0\}.$$

Consequently (3.45) becomes

$$A(\widetilde{u}_\lambda) = (\lambda-\varepsilon)(\widetilde{u}_\lambda)^{q-1} - C_{15}(\widetilde{u}_\lambda)^{r-1},$$

hence $\widetilde{u}_\lambda$ is a positive solution of (3.41). Invoking the uniqueness of solutions to (3.41) we get $\widetilde{u}_\lambda = \overline{u}_\lambda$, therefore

$$\overline{u}_\lambda \leq \overline{u}_\mu.$$

This proves that the map $\lambda \to \overline{u}_\lambda$ is nondecreasing from $(0, \infty)$ into $\widehat{C^1}(T)$. □

Proposition 3.10. *If hypotheses* (H1)–(H3) *hold and* $\lambda \in (0, \lambda^*)$, *then* $\overline{u}_\lambda \leq u$ *for all* $u \in \mathcal{S}(\lambda)$.

Proof. Let $u \in \mathcal{S}(\lambda) \subseteq \operatorname{int} \widehat{C}_+$ and consider the following Carathéodory function

$$\gamma_\lambda(t,x) = \begin{cases} 0 & \text{if } x < 0 \\ (\lambda-\varepsilon)x^{q-1} - C_{15}x^{r-1} + x^{p-1} & \text{if } 0 \leq x \leq u(t) \\ (\lambda-\varepsilon)u(t)^{q-1} - C_{15}u(t)^{r-1} + u(t)^{p-1} & \text{if } u(t) < x. \end{cases}$$

Let $\Gamma_\lambda(t,x) = \int_0^x \gamma_\lambda(t,s)ds$ and consider the C^1−functional $\widehat{\sigma}_\lambda : W_p \to \mathbb{R}$ defined by

$$\widehat{\sigma}_\lambda(u) = \int_0^b G(u'(t))dt + \frac{1}{p}\|u\|_p^p - \int_0^b \Gamma_\lambda(t, u(t))dt \quad \text{for all } u \in W_p.$$

As in the proof of Proposition 3.9, using the direct method, we can find $\widehat{u}_\lambda \in W_p$ such that

$$\widehat{\sigma}_\lambda(\widehat{u}_\lambda) = \inf\{\widehat{\sigma}_\lambda(u) : u \in W_p\} < 0 = \widehat{\sigma}_\lambda(0),$$

hence $\widehat{u}_\lambda \neq 0$. In fact, we can show that $\widehat{u}_\lambda \in [0, u]\backslash\{0\}$ (see the proof of Proposition 3.9 and (3.40)). Then we have

$$A(\widehat{u}_\lambda) = (\lambda-\varepsilon)\widehat{u}_\lambda^{q-1} - C_{15}\widehat{u}_\lambda^{r-1},$$

hence $\widehat{u}_\lambda = \overline{u}_\lambda$ (see Proposition 3.9), therefore $\overline{u}_\lambda \leq u$ for all $u \in \mathcal{S}(\lambda)$. □

Now we can deal with the critical case $\lambda = \lambda^*$.

Proposition 3.11. *If hypotheses* (H1)–(H3) *hold, then* $\lambda^* \in \mathcal{P}$ *and so* $\mathcal{P} = (0, \lambda^*]$.

Proof. Let $\{\lambda_n\}_{n\geq 1} \subseteq \mathcal{P}$ be such that $\lambda_n \to (\lambda^*)_-$ and let $u_n \in \mathcal{S}(\lambda_n) \subseteq \operatorname{int} \widehat{C}_+$ for all $n \geq 1$. From the proof of Proposition 3.6, we see that

$$\varphi_{\lambda_n}(u_n) < 0 \quad \text{for all } n \geq 1. \tag{3.46}$$

Also, we have

$$A(u_n) + \beta(t) u_n^{p-1} = \lambda_n u_n^{q-1} + N_f(u_n) \text{ for all } n \geq 1, \tag{3.47}$$

hence

$$\int_0^b a(|u_n'|)(u_n')^2 dt + \int_0^b \beta(t)|u_n|^p dt = \lambda_n \|u_n\|_q^q + \int_0^b f(t, u_n) u_n dt \tag{3.48}$$

for all $n \geq 1$. From (3.46) it follows that

$$\int_0^b pG(u_n') dt + \int_0^b \beta(t)|u_n|^p dt - \frac{\lambda_n p}{q}\|u_n\|_q^q - \int_0^b pF(t, u_n) dt < 0 \tag{3.49}$$

for all $n \geq 1$. Using (3.48), (3.49), hypothesis (H1) (iv), and recalling that $\lambda_n < \lambda^*$ for all $n \geq 1$, we have

$$\int_0^b [f(t, u_n)u_n - pF(t, u_n)] dt \leq \lambda^* (\frac{p}{q} - 1)\|u_n\|_q^q + C_{16} \text{ for some } C_{16} > 0. \tag{3.50}$$

From (3.50), reasoning as in the proof of Proposition 3.2, and using hypothesis (H3) (iii), we infer that $\{u_n\}_{n\geq 1} \subseteq W_p$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u_* \text{ in } W_p \quad \text{and} \quad u_n \to u_* \text{ in } C(T) \quad \text{as } n \to \infty. \tag{3.51}$$

On (3.47) we act with $u_n - u_* \in W_p$, pass to the limit as $n \to \infty$ and use (3.51). Then

$$\lim_{n\to\infty} \langle A(u_n), u_n - u_* \rangle = 0,$$

hence $u_n \to u_*$ in W_p (see Proposition 2.2); therefore

$$A(u_*) + \beta(t) u_*^{p-1} = \lambda^* u_*^{q-1} + N_f(u_*) \text{ for all } n \geq 1. \tag{3.52}$$

From Propositions 3.9 and 3.10, we have

$$\overline{u}_{\lambda_1} \leq \overline{u}_{\lambda_n} \leq u_n \quad \text{for all } n \geq 1.$$

Then $\overline{u}_{\lambda_1} \leq u_*$; therefore $u_* \in \mathcal{S}(\lambda^*)$ (see (3.52)). Hence $\lambda^* \in \mathcal{P}$ and $\mathcal{P} = (0, \lambda^*]$. ☐

Proof of Theorem 3.1 . We just observe that the conclusions of Theorem 3.1 follow directly from Propositions 3.5, 3.7, 3.9, 3.11.

The existence of the smallest positive solution follows as in [4] using the lower bound for the elements of $\mathcal{S}(\lambda)$ established in Proposition 3.10. The monotonicity of the curve $\lambda \to u_\lambda^*$ is established as the corresponding result for $\lambda \to \overline{u}_\lambda$ in the proof of Proposition 3.9, using hypothesis (H3) (v). ☐

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