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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR NONHOMOGENEOUS KLEIN-GORDON-MAXWELL EQUATIONS

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Abstract. This article concerns the nonhomogeneous Klein-Gordon-Maxwell equation

$$\begin{gathered} -\Delta u + u - (2\omega + \phi)\phi u = |u|^{p-1}u + h(x), \quad \text{in } \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi)u^2, \quad \text{in } \mathbb{R}^3, \end{gathered}$$

where $\omega > 0$ is constant, $p \in (1, 5)$. Under appropriate assumptions on $h(x)$, the existence of at least two solutions is obtained by applying the Ekeland's variational principle and the Mountain Pass Theorem in critical point theory.

1. Introduction

In this article, we consider the existence of multiple solutions for the nonhomogeneous Klein-Gordon-Maxwell equation

$$\begin{gathered} -\Delta u + u - (2\omega + \phi)\phi u = |u|^{p-1}u + h(x), \quad \text{in } \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi)u^2, \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{1.1}$$

where $\omega > 0$ is constant, $1 < p < 5$. We assume that the function $h(x)$ satisfies the following hypotheses.

- (H1) $0 \le h(x) \in L^2(\mathbb{R}^3) \bigcap C^1(\mathbb{R}^3)$ and $h(x) = h(|x|) \not\equiv 0$.
- (H2) $\|h(x)\|_{L^2} < m_p$, where $m_p = \frac{p-1}{2p}\big(\frac{p+1}{2p\eta_p^{p+1}}\big)^{\frac{1}{p-1}}$, $\eta_p > 0$ is the Sobolev embedding constant.
- (H3) $\langle \nabla h(x), x\rangle \in L^2(\mathbb{R}^3)$.

Such system was first introduced in [2] as a model which describes the nonlinear Klein-Gordon field interacting with the electromagnetic field in the electrostatic case. The unknowns of the system are the field u associated to the particle and the electric potential ϕ, while ω denotes the phase. The presence of the nonlinear term simulates the interaction between many particles or external nonlinear perturbations.

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When $h(x) = 0$, the homogeneous case, a several works have been devoted to the Klein-Gordon-Maxwell:

$$\begin{gathered} -\Delta u + [m^2 - (\omega + \phi)^2]u = |u|^{p-1}u, \quad \text{in } \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi)u^2, \quad \text{in } \mathbb{R}^3. \end{gathered} \tag{1.2}$$

The first result is due to Benci and Fortunato. In [2], they proved the existence of infinitely many radially symmetric solutions for (1.2) under the assumption $3 < p < 5$. D'Aprile and Mugnai [5] covered the case $1 < p < 3$ and the case $p = 3$. Under the assumption $1 < p < 5$, Azzollini and Pomponio proved the existence of a ground state solution for (1.2) in [1]. In [6], some nonexistence results of nontrivial solutions for (1.2) were obtained when $p \geq 5$ or $p \leq 1$.

Recently, by combining the minimization of the corresponding Euler-Lagrange functional on the Nehari manifold with the Brezis and Nirenberg technique, Carrião, Cunha and Miyagaki proved the existence of positive ground state solutions of system (1.1) with $h(x) = 0$ when the nonlinearity exhibits critical growth, see [3].

The nonhomogeneous case, that is $h(x) \neq 0$. The authors [4] considered the following nonhomogeneous Klein-Gordon-Maxwell equations:

$$\begin{gathered} -\Delta u + [m^2 - (\omega + \phi)^2]u = |u|^{p-2}u + h(x) \quad \text{in } \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi)u^2, \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{1.3}$$

where $m > \omega > 0$ and $2 < p < 6$. This is the first paper dealing with the nonhomogeneous Klein-Gordon-Maxwell equations. However, since [4, equality (9)] is in error, the authors could not obtain the boundedness of $\{u_n\}$ under the assumption $2 < p < 6$. Then [4, Lemma 3.6 and Theorem 1.3] could not be obtained.

Motivated by the works described above, in the present paper, we establish the existence of multiple solution results for system (1.1). The method is inspired by [9].

By Ekeland's variational principle, it is not difficult to get a solution u_0 of (1.1) for all $\omega > 0$, $1 < p < 5$ and $\|h\|_{L^2}$ suitably small. Moreover, u_0 is a local minimizer of I_ω and $I_\omega(u_0) < 0$, where I_ω is defined by (2.2). However, under our assumptions it seems difficult to get a second solution(different from u_0) of (1.1) by applying the Mountain Pass Theorem. So we have to study problem (1.1) in the following two cases: $p \in (1, 2]$ and $p \in (2, 5)$, respectively.

For $p \in [3, 5)$, we can directly prove the boundedness of $\{u_n\}$ and the $(PS)_c$ condition. But for $p \in (2, 3)$, it is difficult to show if the $(PS)_c$ condition satisfies. To overcome the difficulty, by introducing a suitable approximation problem, we use an indirect method to obtain the boundedness of $\{u_n\}$ sequence for I_ω based on the weak solutions of the approximation problem, and then show that this special (PS) sequence converges to a solution of problem (1.1). However, when $p \in (1, 2]$, it is more delicate. For this case, we note that (1.1) has no positive energy solution for $\omega > 0$ large enough (see Theorem 5.1). Based on this observation, by using the cut-off technique as in [7], we finally get a positive energy solution for problem (1.1) with $\omega > 0$ small enough.

Our main results read as follows.

Theorem 1.1. *Let $p \in (2, 5)$ and* (H1)–(H3) *hold. Then, for all $\omega > 0$, problem* (1.1) *has at least two nontrivial solutions u_0 and u_1 such that $I_\omega(u_0) < 0 < I_\omega(u_1)$.*

Theorem 1.2. *Assume that $p \in (1, 2]$ and* (H1)–(H2) *hold. Then, if $\omega > 0$ small, problem* (1.1) *possesses two nontrivial solutions u_0 and $\tilde{u}_1$ such that $I_\omega(u_0) < 0 <$*

$I_\omega(\tilde{u_1})$. *However, if* $\omega > 0$ *large enough, problem* (1.1) *has no solution with positive energy.*

Remark 1.3. According to our results, for any $\omega > 0$, problem (1.1) has always a solution with negative energy.

Throughout this article mC denotes various positive constants.

2. Variational setting

In this section, we introduce some preliminary results concerning the variational structure for (1.1). Our working space is $E := H^1(\mathbb{R}^3)$ equipped with the inner product and norm

$$\langle u, v\rangle := \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv)dx, \quad \|u\| := \langle u, u\rangle^{1/2}.$$

Let $D^{1,2}(\mathbb{R}^3)$ be the completion of $C_0^\infty(\mathbb{R}^3, R)$ with respect to the norm

$$\|u\|_{D^{1,2}} = (\int_{\mathbb{R}^3} |\nabla u|^2 dx)^{\frac{1}{2}}.$$

And for any $1 \le s < \infty$, $\|u\|_{L^s} := (\int_{\mathbb{R}^3} |u|^s dx)^{\frac{1}{s}}$ denotes the usual norm of the Lebesgue space $L^s(\mathbb{R}^3)$.

Due to the variational nature of problem (1.1), its weak solutions $(u, \phi) \in E \times D^{1,2}(\mathbb{R}^3)$ are critical points of the functional $J : E \times D^{1,2}(\mathbb{R}^3) \to R$ defined by

$$\begin{aligned} J(u, \phi) = &\frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi|^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} (2\omega + \phi)\phi u^2 dx \\ &- \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1} dx - \int_{\mathbb{R}^3} h(x)u\, dx. \end{aligned}$$

Obviously, the action functional J belongs to $C^1(E \times D^{1,2}(\mathbb{R}^3), R)$ and exhibits a strong indefiniteness. To avoid the indefiniteness we apply a reduction method, as has been done by the aforementioned authors.

Lemma 2.1 ([5,6])**.** *For every* $u \in E$ *there exists a unique* $\phi = \phi_u \in D^{1,2}(\mathbb{R}^3)$ *which solves* $\Delta\phi = (w + \phi)u^2$. *Furthermore*

(i) *in the set* $\{x : u(x) \neq 0\}$ *we have* $-\omega \le \phi_u \le 0$ *for* $\omega > 0$*;*
(ii) *if* u *is radially symmetric,* ϕ_u *is radial too.*

According to Lemma 2.1, we can consider the functional $I_\omega : E \to R$ *defined by* $I_\omega(u) = J(u, \phi_u)$. *After multiplying both members of the second equation in equations* (1.1) *by* ϕ_u *and integrating by parts, we obtain*

$$\int_{\mathbb{R}^3} |\nabla\phi_u|^2 dx = -\int_{\mathbb{R}^3} \omega\phi_u u^2 dx - \int_{\mathbb{R}^3} \phi_u^2 u^2 dx. \tag{2.1}$$

Then, the reduced functional takes the form

$$I_\omega(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2 - \omega\phi_u u^2)dx - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1} dx - \int_{\mathbb{R}^3} h(x)u\, dx. \tag{2.2}$$

Furthermore I *is* C^1 *and we have for any* $u, v \in E$*,*

$$\begin{aligned} \langle I_\omega'(u), v\rangle = &\int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv - (2\omega + \phi_u)\phi_u uv)dx \\ &- \int_{\mathbb{R}^3} |u|^{p-1} uv dx - \int_{\mathbb{R}^3} h(x)v\, dx. \end{aligned} \tag{2.3}$$

Remark 2.2. By (2.1), we can note that

$$\|\phi_u\|^2_{D^{1,2}(\mathbb{R}^3)} \le \int_{\mathbb{R}^3} \omega|\phi_u|u^2 dx \le \omega\|\phi_u\|_{L^6}\|u\|^2_{L^{12/5}},$$

then

$$\|\phi_u\|_{D^{1,2}(\mathbb{R}^3)} \le C_1\omega\|u\|^2_{L^{12/5}}, \quad \int_{\mathbb{R}^3} \omega|\phi_u|u^2 dx \le \omega C_1\|u\|^4.$$

Now, we can apply [6, Lemma 2.2] to our functional I_ω and obtain the following result.

Lemma 2.3. *The following statements are equivalent:*

1. $(u, \phi) \in E \times D^{1,2}(\mathbb{R}^3)$ *is a critical point of* J *(i.e.*(u, ϕ) *is a solution of* (1.1).
2. u *is a critical point of* I_ω *and* $\phi = \phi_u$.

Set

$$H^1_r(\mathbb{R}^3) := \{u \in H^1(\mathbb{R}^3) : u = u(r),\ r = |x|\}.$$

We shall consider the functional I_ω on $H^1_r(\mathbb{R}^3)$. Then any critical point $u \in H^1_r(\mathbb{R}^3)$ of $I_\omega|_{H^1_r(\mathbb{R}^3)}$ is also a critical point of I_ω since $H^1_r(\mathbb{R}^3)$ is a natural constraint for I_ω. Thus we are reduced to look for critical points of $I_\omega|_{H^1_r(\mathbb{R}^3)}$. In the following, we still denote $I_\omega|_{H^1_r(\mathbb{R}^3)}$ by I_ω. It follows from [2] that for $2 < s < 6$, $H^1_r(\mathbb{R}^3)$ is compactly embedded into $L^s(\mathbb{R}^3)$. Therefore, there exists a positive constant $\eta_s > 0$ such that

$$\|u\|_{L^s} \le \eta_s\|u\|, \quad \forall u \in H^1_r(\mathbb{R}^3).$$

To obtain our results, the following theorem will be needed in our argument.

Theorem 2.4 ([8])**.** $(X, \|\cdot\|)$ *is a Banach space and* $S \subset R_+$ *an interval. Let us consider the family of* C^1 *functionals on* X

$$I_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in S,$$

with B *nonnegative and either* $A(u) \to +\infty$ *or* $B(u) \to +\infty$ *as* $\|u\| \to \infty$ *and such that* $I_\lambda(0) = 0$. *Set*

$$\Gamma_\lambda = \{\gamma \in C([0,1], X) : \gamma(0) = 0, I_\lambda(\gamma(1)) < 0\}, \quad \text{for any } \lambda \in S.$$

If for every $\lambda \in S$ *the set* Γ_λ *is nonempty and* $c_\lambda = \inf_{\gamma\in\Gamma_\lambda} \max_{t\in[0,1]} I_\lambda(\gamma(t)) > 0$, *then for almost every* $\lambda \in S$, *there exists a sequence* $\{u_n\} \subset X$ *satisfying*

(i) $\{u_n\}$ *is bounded;*
(ii) $I_\lambda(u_n) \to c_\lambda$*;*
(iii) $I'_\lambda(u_n) \to 0$ *in the dual* X^{-1} *of* X.

3. A weak solution with negative energy

In this section, we prove that (1.1) has a weak solution with negative energy for any $\omega > 0$ and $p \in (1, 5)$. With the aid of Ekeland's variational principle, this weak solution is obtained by seeking a local minimum of the energy functional I_ω.

Lemma 3.1. *Suppose that* $p \in (1, 5)$ *and* (H1)–(H2) *hold. Then there exist* ρ, α, *and* m_p *positive such that* $I_\omega(u)|_{\|u\|=\rho} \ge \alpha > 0$ *for all* h *satisfying* $\|h\|_{L^2} < m_p$, *where* $m_p = \frac{p-1}{2p}\big(\frac{p+1}{2p\eta_P^{p+1}}\big)^{\frac{1}{p-1}}$.

Proof. For all $\omega > 0$ and $u \in H^1(\mathbb{R}^3)$, by Lemma 2.1, the Hölder inequality and Sobolev's embedding theorem, we have

$$\begin{aligned} I_\omega(u) &\geq \frac{1}{2}\|u\|^2 - \frac{1}{p+1}\|u\|_{L^p}^{p+1} - \|h\|_{L^2}\|u\| \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\eta_p^{p+1}}{p+1}\|u\|^{p+1} - \|h\|_{L^2}\|u\| \\ &= \|u\|(\frac{1}{2}\|u\| - \frac{\eta_p^{p+1}}{p+1}\|u\|^p - \|h\|_{L^2}). \end{aligned} \tag{3.1}$$

Set

$$g(t) = \frac{1}{2}t - \frac{\eta_p^{p+1}}{p+1}t^p \quad \text{for } t \geq 0.$$

By direct calculations, we see that $\max_{t\geq 0} g(t) = g(\rho) = \frac{p-1}{2p}(\frac{p+1}{2p\eta_p^{p+1}})^{\frac{1}{p-1}} := m_p$, where $\rho = (\frac{p+1}{2p\eta_p^{p+1}})^{\frac{1}{p-1}}$. Then it follows from (3.1) that, if $\|h\|_{L^2} < m_p$, there exists $\alpha = \rho(g(\rho) - \|h\|_{L^2}) > 0$ such that $I_\omega(u)|_{\|u\|=\rho} \geq \alpha > 0$ for all $\omega > 0$. □

Lemma 3.2. *If $p \in (1,5)$ and (H1)–(H2) hold. Then, for any $\omega > 0$, there exists $u_0 \in H_r^1(\mathbb{R}^3)$ such that*

$$I_\omega(u_0) = \inf\{I_\omega(u) : u \in H_r^1(\mathbb{R}^3) \text{ and } \|u\| \leq \rho\} < 0.$$

where ρ is given by Lemma 3.1. Moreover, u_0 is a solution of problem (1.1).

Proof. By (H1), we can choose a function $\varphi \in H_r^1(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} h(x)\varphi dx > 0$. Hence, for $t > 0$ small enough, we obtain

$$\begin{aligned} I_\omega(t\varphi) &= \frac{t^2}{2}\int_{\mathbb{R}^3}(|\nabla\varphi|^2 + \varphi^2)dx - \frac{1}{2}\int_{\mathbb{R}^3}\omega\phi_{t\varphi}(t\varphi)^2 dx \\ &\quad - \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^3}|\varphi|^{p+1}dx - t\int_{\mathbb{R}^3}h(x)\varphi dx \\ &\leq \frac{t^2}{2}\|\varphi\|^2 + \frac{t^4 C_1\omega}{2}\|\varphi\|^4 - \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^3}|\varphi|^{p+1}dx - t\int_{\mathbb{R}^3}h(x)\varphi dx < 0, \end{aligned}$$

which shows that $c_0 = \inf\{I_\omega(u) : u \in \bar{B}_\rho\} < 0$, where

$$\bar{B}_\rho = \{u \in H_r^1(\mathbb{R}^3) \text{ and } \|u\| \leq \rho\}.$$

By the Ekeland's variational principle, there exists a sequence $\{u_n\} \subset \bar{B}_\rho$ such that

$$c_0 \leq I_\omega(u_n) \leq c_0 + \frac{1}{n}, \quad I_\omega(\vartheta) \geq I_\omega(u_n) - \frac{1}{n}\|\vartheta - u_n\| \quad \forall \vartheta \in \bar{B}_\rho.$$

By a standard procedure, see, for example [12], we can show that $\{u_n\}$ is bounded (PS) sequence of I_ω. Then, by the compactness of the embedding $H_r^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)(2 < s < 6)$, there exists $u_0 \in H_r^1(\mathbb{R}^3)$ such that $\{u_n\} \to u_0$ strongly in $H_r^1(\mathbb{R}^3)$. Hence $I_\omega(u_0) = c_0 < 0$, $I'_\omega(u_0) = 0$. □

4. Positive energy solution for $p \in (2,5)$

In this section, we aim to prove that problem (1.1) has a positive energy solution for any $\omega > 0$, $p \in (2,5)$. It is well-known that, for $p \in [3,5)$, we can directly prove the boundedness of $\{u_n\}$ of the functional I_ω. But for $p \in (1,3)$, it is not easy to do this. Particularly, $p \in (1,2)$ is the hardest case. To show the boundedness of a

(PS) sequence of I_ω when $p \in (2,5)$ is also nontrivial. Here we have to use Theorem 2.4. Consider the approximation problem

$$\begin{gathered} -\Delta u + u - (2\omega+\phi)\phi u = \lambda|u|^{p-1}u + h(x), \quad \text{in } \mathbb{R}^3, \\ \Delta\phi = (\omega+\phi)u^2, \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{4.1}$$

where $p \in (2,5)$ and $\lambda \in [1/2,1]$. Set $X = H_r^1(\mathbb{R}^3)$,

$$A(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{R}^3} \omega\phi_u u^2 dx - \int_{\mathbb{R}^3} h(x)u\,dx$$

and $B(u) = \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}dx$. Thus we study the perturbed functional

$$I_{\omega,\lambda}(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2 - \omega\phi_u u^2)dx - \int_{\mathbb{R}^3} h(x)u\,dx - \frac{\lambda}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}dx.$$

Then, $I_{\omega,\lambda}$ is a family of C^1-functionals on X, $B(u) \ge 0$ and $A(u) \ge \frac{1}{2}\|u\|^2 - \|h\|_{L^2}\|u\| \to +\infty$ as $\|u\| \to \infty$.

Lemma 4.1. *Assume $p \in (1,5)$ and* (H1)–(H2) *satisfy. Then, the following hold.*

(i) *$\Gamma_\lambda \neq \emptyset$, for any $\lambda \in [1/2,1]$;*
(ii) *There exists a constant $\tilde{c}$ such that $c_\lambda \ge \tilde{c} > 0$ for all $\lambda \in [1/2,1]$.*

Proof. (i) For any $\lambda \in [1/2,1]$, we choose a function $\psi \in X \ge (\not\equiv 0)$. Then, by Lemma 2.1, we obtain

$$I_{\omega,\lambda}(t\psi) \le \frac{t^2}{2}\|\psi\|^2 + \frac{t^2}{2}\omega^2\int_{\mathbb{R}^3} \psi^2 dx - \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^3} |\psi|^{p+1}dx.$$

Since $p \in (1,5)$, there exists t_0 large enough such that $I_{\omega,\lambda}(t_0\psi) < 0$. Hence (i) holds.

(ii) By Lemma 2.1, for any $u \in X$ and $\lambda \in [1/2,1]$, we have

$$I_{\omega,\lambda}(u) \ge \frac{1}{2}\|u\|^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}dx$$

Since $p > 1$, we conclude that there exists $\rho > 0$ such that $I_{\omega,\lambda}(u) > 0$ for any $u \in X$ and $\lambda \in [1/2,1]$ with $\|u\| \le \rho$. In particular, for any $\|u\| = \rho$, we have $I_{\omega,\lambda}(u) > \tilde{c} > 0$. Now fix $\lambda \in [1/2,1]$ and $\gamma \in \Gamma_\lambda$, by the definition of Γ_λ, certainly $\|\gamma(1)\| > \rho$. By continuity, we deduce that there exists $t_\gamma \in (0,1)$ such that $\|\gamma(t_\gamma)\| = \rho$. Therefore, for any $\lambda \in [1/2,1]$, we have

$$c_\lambda \ge \inf_{\gamma\in\Gamma_\lambda} I_{\omega,\lambda}(\gamma(t_\gamma)) \ge \tilde{c} > 0.$$

Thus, (ii) holds. □

Since $I_{\omega,\lambda}(0) = 0$, then by Lemma 4.1 and Theorem 2.4, there exist (i) $\{\lambda_j\} \subset [1/2,1]$ such that $\lambda_j \to 1$ as $j \to \infty$ and (ii) a bounded sequence $\{v_n^j\}$ of the functional I_{ω,λ_j}. By the compactness of the embedding $H_r^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)(2 < s < 6)$ and [11, Lemma 2.1], we can show that for each $j \in \mathrm{N}$ there exists $v_j \in H_r^1(\mathbb{R}^3)$ such that $v_n^j \to v_j$ strongly in $H_r^1(\mathbb{R}^3)$. Moreover, for all $j \in \mathrm{N}$, we have

$$0 < \tilde{c} \le I_{\omega,\lambda_j}(v_j) = c_{\omega,\lambda_j} \le c_{\omega,\frac{1}{2}}, \quad I'_{\omega,\lambda_j}(v_j) = 0. \tag{4.2}$$

Lemma 4.2. *If $v_j \in X$ solves the problem $(*)$, then the following Pohožaev type identity*

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3}|\nabla v_j|^2dx+\frac{3}{2}\int_{\mathbb{R}^3}v_j^2dx-\int_{\mathbb{R}^3}(\frac{5}{2}\omega+\phi_{v_j})\phi_{v_j}v_j^2dx\\
&=\int_{\mathbb{R}^3}[\frac{3\lambda}{p+1}|v_j|^{p+1}+(3h(x)+\langle x,\nabla h(x)\rangle)v_j]dx.
\end{aligned}
\tag{4.3}
$$

holds.

The proof can be done as in [6, Lemma 3.1] and details are omitted here. In what follows, we turn to showing that $\{v_j\}$ converges to a solution of problem (1.1). For this purpose, we have to prove $\{v_j\}$ is the bounded in $H_r^1(\mathbb{R}^3)$.

Lemma 4.3. *Under the conditions of Theorem 1.1, if $p \in (2,5)$, then $\{v_j\}$ is bounded in $H_r^1(\mathbb{R}^3)$.*

Proof. The proof of this theorem is divided into two steps.

Step 1: $\{\|v_j\|_{L^2}\}$ is bounded. By contradiction, we assume that $\|v_j\|_{L^2}\to\infty$ as $j\to\infty$. Set $u_j=\frac{v_j}{\|v_j\|_{L^2}}$, $X_j=\int_{\mathbb{R}^3}|\nabla u_j|^2dx$, $Y_j=\int_{\mathbb{R}^3}\omega\phi_{v_j}u_j^2dx$, $Z_j=\int_{\mathbb{R}^3}\phi_{v_j}^2u_j^2dx$, and $T_j=\lambda_j\|u_j\|_{L^{p+1}}^{p+1}\|v_j\|_{L^2}^{p-1}$. By (4.2), we have

$$
\begin{gathered}
\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla v_j|^2+v_j^2-\omega\phi_{v_j}v_j^2)dx-\int_{\mathbb{R}^3}h(x)v_jdx-\frac{\lambda_j}{p+1}\int_{\mathbb{R}^3}|v_j|^{p+1}dx=c_{\omega,\lambda_j},\\
\int_{\mathbb{R}^3}|\nabla v_j|^2+v_j^2-(2\omega+\phi_{v_j})\phi_{v_j}v_j^2)dx-\int_{\mathbb{R}^3}h(x)v_jdx=\lambda_j\int_{\mathbb{R}^3}|v_j|^{p+1}dx,
\end{gathered}
\tag{4.4}
$$

and $\{c_{\omega,\lambda_j}\}$ is bounded. Note that $h(x),\langle x,h(x)\rangle\in L^2(\mathbb{R}^3)$. Multiplying (4.3) and (4.4) by $\frac{1}{\|v_j\|_{L^2}}$, we obtain

$$
\begin{gathered}
\frac{1}{2}X_j-\frac{5}{2}Y_j-Z_j-\frac{3}{p+1}T_j=o(1)-\frac{3}{2},\\
\frac{1}{2}X_j-\frac{1}{2}Y_j-\frac{1}{p+1}T_j=o(1)-\frac{1}{2},\\
X_j-2Y_j-Z_j-T_j=o(1)-1,
\end{gathered}
\tag{4.5}
$$

where $o(1)$ denotes that the quantity tends to zero as $j\to\infty$. Solving (4.5), we have

$$
X_j=\frac{(1-p)(1+Z_j)}{2(p-2)}+o(1),\quad \text{for } p\in(2,5).
$$

Since $Z_j\geq 0$ and $X_j\geq 0$ for all $j\in \mathrm{N}$, (4.5) is a contradiction for j large enough. Thus, $\{\|v_j\|_{L^2}\}$ is bounded for $p\in(2,5)$.

Step 2: $\|\nabla v_j\|_{L^2}$ is bounded. Similarly, by contradiction, we can assume that $\|\nabla v_j\|_{L^2}\to\infty$ as $j\to\infty$. Set $w_j=\frac{v_j}{\|\nabla v_j\|_{L^2}}$, $M_j=\int_{\mathbb{R}^3}\omega\phi_{v_j}w_j^2dx$, $N_j=\int_{\mathbb{R}^3}\phi_{v_j}^2w_j^2dx$, $S_j=\lambda_j\|w_j\|_{L^{p+1}}^{p+1}\|\nabla v_j\|_{L^2}^{p-1}$. Then, multiplying (4.3) and (4.4) by $\frac{1}{\|\nabla v_j\|_{L^2}^2}$, and noting that $\|v_j\|_{L^2}$ is bounded, we obtain

$$
\begin{gathered}
-\frac{5}{2}M_j-N_j-\frac{3}{p+1}S_j=o(1)-\frac{1}{2},\\
-\frac{1}{2}M_j-\frac{1}{p+1}S_j=o(1)-\frac{1}{2},\\
-2M_j-N_j-S_j=o(1)-1.
\end{gathered}
\tag{4.6}
$$

For $p \in (2,5)$, solving (4.6), we obtain

$$N_j = \frac{2(2-p)}{(p-1)} + o(1), \quad \text{for } p \in (2,5),$$

which implies a contradiction for j large enough since $N_j \geq 0$ for all $j \in \mathrm{N}$. Thus, $\{\|\nabla v_j\|_{L^2}\}$ is bounded for $p \in (2,5)$. The proof is complete. $\square$

Proof of Theorem 1.1. Lemma 4.3 implies that $\{v_j\}$ is a bounded sequence of I_ω. Then, by the compactness of the embedding $H_r^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)(2 < s < 5)$, for any $\omega > 0$, we show that problem (1.1) has a solution u_1 satisfying $I_\omega(u_1) > 0$. Combining with Lemma 3.2, we complete the proof. $\square$

5. Positive energy solution for $p \in (1,2]$

In this section, we first prove that (1.1) with $1 < p \leq 2$ has no solution with positive energy for $\omega > 0$ large enough.

Theorem 5.1. *Assume that $p \in (1,2]$ and* (H1)–(H2) *hold (in fact, $h(x)$ may not be radially symmetric). Then* (1.1) *has no solution with positive energy if $\omega > 0$ is large enough.*

Proof. Let $u \in H^1(\mathbb{R}^3)$ be a solution of (1.1). Then $\langle I'_\omega(u), u\rangle = 0$. By (2.2) and (2.3), we have

$$\begin{aligned} I_\omega(u) = &-(\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{3}{2}\int_{\mathbb{R}^3} \omega\phi_u u^2 dx - \int_{\mathbb{R}^3} \phi_u^2 u^2 dx) \\ &- \frac{1}{2}\int_{\mathbb{R}^3} u^2 dx + \frac{p}{p+1}\int_{\mathbb{R}^3} |u|^{p+1} dx. \end{aligned} \tag{5.1}$$

Similar to [11, (20)], we obtain

$$\sqrt{\frac{3}{4}}\int_{\mathbb{R}^3} (\omega + \phi_u)|u|^3 \leq \frac{1}{4}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{3}{4}\int_{\mathbb{R}^3} |\nabla \phi|^2 dx. \tag{5.2}$$

Then, by Lemma 2.1, one has

$$\begin{aligned} \sqrt{3}\int_{\mathbb{R}^3} (\omega + \phi_u)|u|^3 &\leq \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{3}{2}\int_{\mathbb{R}^3} |\nabla \phi|^2 dx \\ &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{3}{2}\int_{\mathbb{R}^3} \omega\phi_u u^2 dx - \frac{3}{2}\int_{\mathbb{R}^3} \phi_u^2 u^2 dx \\ &\leq \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{3}{2}\int_{\mathbb{R}^3} \omega\phi_u u^2 dx - \int_{\mathbb{R}^3} \phi_u^2 u^2 dx. \end{aligned} \tag{5.3}$$

For $p \in (1,2]$ and $\omega > 0$ large enough such that $\omega + \phi_u > 0$, it follows from (5.1) and (5.3) that

$$I_\omega(u) \leq -\{\sqrt{3}\int_{\mathbb{R}^3} [(\omega + \phi_u)|u|^3 + \frac{1}{2}u^2 - \frac{p}{p+1}|u|^{p+1}] dx\} < 0.$$

Hence, problem (1.1) must have no solution with positive energy if $\omega > 0$ is large enough. $\square$

Obviously, when $p \in (1,2]$, Theorem 5.1 implies that we may find a solution with positive energy to problem (1.1) only for $\omega > 0$ small. To overcome the difficulty in finding bounded $(PS)_c (c > 0)$ sequence for the associated functional I_ω, following [10], we introduce the cut-off function $\eta \in C^\infty(\mathbb{R}^+, \mathbb{R}^+)$ satisfying

$$\begin{gathered} \eta(t) = 1, \quad \text{for } t \in [0,1], \\ 0 \le \eta(t) \le 1, \quad \text{for } t \in (1,2), \\ \eta(t) = 0, \quad \text{for } t \in [2,+\infty), \\ |\eta'|_\infty \le 2, \end{gathered}$$

and consider the modified functional

$$\begin{aligned} I_{\omega,T}(u) &= \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx - \frac{\omega}{2}\int_{\mathbb{R}^3} K_T(u)\phi_u u^2 dx \\ &\quad - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1} dx - \int_{\mathbb{R}^3} h(x) u \, dx. \end{aligned} \tag{5.4}$$

where, for $T > 0$, $K_T(u) = \eta(\frac{\|u\|^2}{T^2})$. If $h(x) = h(|x|) \in L^2(\mathbb{R}^3)$ and $p \in (1,5]$, then $I_{\omega,T}$ is a C^1 functional, and

$$\begin{aligned} \langle I'_{\omega,T}(u), v\rangle &= \int_{\mathbb{R}^3} (\nabla u \nabla v + uv) dx - \int_{\mathbb{R}^3} K_T(u)(2\omega + \phi_u)\phi_u uv \, dx \\ &\quad - \frac{\omega}{T^2}\eta'(\frac{\|u\|^2}{T^2})\int_{\mathbb{R}^3} \phi_u u^2 dx \int_{\mathbb{R}^3} (\nabla u \nabla v + uv) dx \\ &\quad - \int_{\mathbb{R}^3} |u|^{p-1} uv \, dx - \int_{\mathbb{R}^3} h(x) v \, dx, \end{aligned} \tag{5.5}$$

for every $u, v \in E$.

Lemma 5.2. *Assume that $p \in (1,5)$ and* (H1)–(H2). *Then the functional $I_{\omega,T}$ satisfies the following:*

- (i) *$I_{\omega,T}|_{\|u\|=\rho} > \alpha > 0$ for all $\omega, T > 0$.*
- (ii) *For each $T > 0$, there exists a function $e_T \in H^1_r(\mathbb{R}^3)$ with $\|e_T\| > \rho$ such that $I_{\omega,T}(e_T) < 0$, where ρ, α is given by Lemma 3.1.*

Proof. The proof of (i) is similar to that of Lemma 3.1.

(ii) we choose $\varphi \in E$ with $\varphi \ge 0$, $\|\varphi\| = 1$. By (5.4) and the definition of η, there exists $t_T \ge 2T > 0$ large enough such that $K_T(t_T\varphi) = 0$ and $I_{\omega,T}(t_T\varphi) < 0$. Hence, (ii) holds by taking $e_T = t_T\varphi$. Set

$$c_{\omega,T} = \inf_{\gamma \in \Gamma_{\omega,T}} \max_{t \in [0,1]} I_{\omega,T}(\gamma(t)),$$

where $\Gamma_{\omega,T} := \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e_T\}$. Then, by Lemma 5.2, we have

$$c_{\omega,T} \ge \alpha > 0, \quad \text{for all } \omega,\ T > 0. \tag{5.6}$$

Applying the Mountain Pass Theorem, there exists $\{u^n_{\omega,T}\} \in H^1_r(\mathbb{R}^3)$ (denoted by $\{u_n\}$ for simplicity) such that

$$I_{\omega,T}(u_n) \to c_{\omega,T}, \ (1 + \|u_n\|)\|I'_{\omega,T}(u_n)\|_{H_r^{-1}} \to 0 \tag{5.7}$$

as $n \to \infty$, where H_r^{-1} denotes the dual space of $H^1_r(\mathbb{R}^3)$. □

Lemma 5.3. *Suppose that $p \in (1,5)$ and* (H1)–(H2) *hold. Let $\{u_n\}$ be given by* (5.7). *Then there exists $T_0 > 0$ such that*

$$\lim_{n\to\infty} \sup \|u_n\| \le \frac{T_0}{2}, \quad \forall 0 < \omega < T_0^{-3},$$

which implies $\{u_n\}$ being a bounded (PS) sequence of I_ω in $H_r^1(\mathbb{R}^3)$.

Proof. Motivated by [10], we will argue by contradiction. Assume that, for every $T > 0$ there exists $0 < \omega_T < T^{-3}$ such that $\lim_{n\to\infty} \sup \|u_n\| > \frac{T}{2}$. So, up to a subsequence, we obtain $\|u_n\| \geq \frac{T}{2}$ for all $n \in \mathrm{N}$. On the one hand, by (5.4), (5.5) and Lemma 2.1, we have

$$
\begin{aligned}
&(p+1)I_{\omega,T}(u_n) - \langle I'_{\omega,T}(u_n), u_n\rangle \\
&= \frac{p-1}{2}\|u_n\|^2 - \frac{\omega(p-3)}{2}\int_{\mathbb{R}^3} K_T(u_n)\phi_{u_n}u_n^2dx \\
&\quad + \int_{\mathbb{R}^3} K_T(u_n)\phi_{u_n}^2u_n^2dx + \frac{\omega}{T^2}\eta'(\frac{\|u_n\|^2}{T^2})\|u_n\|^2\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx - p\int_{\mathbb{R}^3} h(x)u_ndx.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\frac{p-1}{2}\|u_n\|^2 - \|I'_{\omega,T}(u_n)\|\|u_n\| \\
&\leq \frac{p-1}{2}\|u_n\|^2 + \langle I'_{\omega,T}(u_n), u_n\rangle \\
&\leq (p+1)I_{\omega,T}(u_n) + \frac{\omega(p-3)}{2}\int_{\mathbb{R}^3} K_T(u_n)\phi_{u_n}u_n^2dx \\
&\quad + \int_{\mathbb{R}^3} K_T(u_n)\phi_{u_n}^2u_n^2dx - \frac{\omega}{T^2}\eta'(\frac{\|u_n\|^2}{T^2})\|u_n\|^2\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx + p\int_{\mathbb{R}^3} h(x)u\,dx \\
&\leq (p+1)I_{\omega,T}(u_n) + \frac{\omega(p-3)}{2}\int_{\mathbb{R}^3} K_T(u_n)\phi_{u_n}u_n^2dx \\
&- \omega\int_{\mathbb{R}^3} K_T(u_n)\phi_{u_n}u_n^2dx - \frac{\omega}{T^2}\eta'(\frac{\|u_n\|^2}{T^2})\|u_n\|^2\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx + p\int_{\mathbb{R}^3} h(x)u\,dx \\
&= (p+1)I_{\omega,T}(u_n) + \frac{\omega(5-p)}{2}\int_{\mathbb{R}^3} K_T(u_n)(-\phi_{u_n})u_n^2dx \\
&\quad + \frac{\omega}{T^2}\eta'(\frac{\|u_n\|^2}{T^2})\|u_n\|^2\int_{\mathbb{R}^3}(-\phi_{u_n})u_n^2dx + p\int_{\mathbb{R}^3} h(x)u\,dx.
\end{aligned} \tag{5.8}
$$

On the other hand, we claim that there exist $T_1, C, M_1 > 0$ such that

$$
c_{\omega,T} \leq C\omega T^4 + M_1, \quad \forall T \geq T_1. \tag{5.9}
$$

Let φ be the function taken in the proof of (ii) of Lemma 5.2. By (5.4), we have

$$
I_{\omega,T}(2T\varphi) \leq 2T^2 - \frac{2^{p+1}}{p+1}T^{p+1}\|\varphi\|_{L^{p+1}}^{p+1}. \tag{5.10}
$$

Then there exists $T_1 > 0$ such that $I_{\omega,T}(2T\varphi) < 0$ for all $T > T_1$. Thus

$$
c_{\omega,T} \leq \max_{t\in[0,1]} I_{\omega,T}(2tT\varphi), \quad \forall T \geq T_1. \tag{5.11}
$$

By (5.4) and Remark 2.2, we have

$$
\begin{aligned}
&\max_{t\in[0,1]} I_{\omega,T}(2tT\varphi)\\
&\le \max_{t\in[0,1]}\{2(tT)^2-\frac{2^{p+1}}{p+1}(tT)^{p+1}\|\varphi\|_{L^{p+1}}^{p+1}\}+\max_{t\in[0,1]}\{-\frac{\omega}{2}\int_{\mathbb{R}^3}\phi_{2tT\varphi}(2tT\varphi)^2dx\}\\
&\le \max_{m\ge 0}\{2(m)^2-\frac{2^{p+1}}{p+1}(m)^{p+1}\|\varphi\|_{L^{p+1}}^{p+1}\}+C\omega T^4\\
&= M_1+C\omega T^4.
\end{aligned}
\tag{5.12}
$$

It follows from (5.11) and (5.12) that (5.9) holds. By Remark 2.2, and noting that $K_T(u_n)=0$ for $\|u_n\|^2\ge 2T^2$, we obtain

$$\int_{\mathbb{R}^3} K_T(u_n)(-\phi_{u_n})u_n^2dx\le CT^4, \tag{5.13}$$

$$\eta'(\frac{\|u_n\|^2}{T^2})\frac{\|u_n\|^2}{T^2}\int_{\mathbb{R}^3}(-\phi_{u_n})u_n^2dx\le CT^4. \tag{5.14}$$

Combining (5.7), (5.8), (5.9), (5.13) with (5.14), one has, for all $T>T_1$,

$$\frac{p-1}{2}\|u_n\|^2\le C_2\omega T^4+M_2+p\int_{\mathbb{R}^3}h(x)u\,dx, \tag{5.15}$$

where $C_2,M_2>0$ independent of T. Then, for any $\varepsilon>0$, by the inequality $\int_{\mathbb{R}^3}h(x)u_n\le\varepsilon\|u_n\|^2+C(\varepsilon,\|h\|_{L^2})$ and (5.15), there exist $C,M>0$ independent of T such that, for all $T>T_1$,

$$\|u_n\|^2\le C\omega T^4+M. \tag{5.16}$$

Since $0<\omega<T_0^{-3}$ and $\|u_n\|\ge\frac{T}{2}$, (5.16) is impossible for $T>0$ large enough. Thus we complete the proof. □

Proof of Theorem 1.2. By Lemma 5.3, we obtain that $\{u_n\}$ is given by (5.7) is bounded sequence of I_ω in $H_r^1(\mathbb{R}^3)$ for all $0<\omega<T_0^{-3}$. Moreover, by using (5.6) and (5.7), we see that

$$I_\omega(u_n)\to c_{\omega,T_0}\ge\alpha>0,\quad \text{as } n\to\infty.$$

Then, by the compactness of the embedding $H_r^1(\mathbb{R}^3)\hookrightarrow L^{s+1}(\mathbb{R}^3)(1<s<5)$, for any $0<\omega<T_0^{-3}$, problem (1.1) has a solution $\tilde{u_1}$ satisfying $I_\omega(\tilde{u_1})>0$. Then, by Theorem 5.1 and Lemma 3.2, we easily complete the proof. □

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