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POINTWISE ESTIMATES FOR POROUS MEDIUM TYPE EQUATIONS WITH LOW ORDER TERMS AND MEASURE DATA

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Abstract. We study a Cauchy-Dirichlet problem with homogeneous boundary conditions on the parabolic boundary of a space-time cylinder for degenerate porous medium type equations with low order terms and a non-negative, finite Radon measure on the right-hand side. The central objective is to acquire linear pointwise estimates for weak solutions in terms of Riesz potentials. Our main result, Theorem 1.1, generalizes an estimate previously obtained by Bögelein, Duzaar and Gianazza [3, Theorem 1.2]), since the problem and the structure conditions considered here, are more universal.

1. Introduction and main result

In this introductory section, we determine the basic setting for our further observations, describe the treated problem, specify some notation, mention the main conclusion and unveil the proof strategies.

1.1. **Setting.** In this section, we present the covered problem and explain the occurring quantities, including some of their properties. Let $T > 0$ and $E \subset \mathbb{R}^n$ be a bounded, open domain, where $n \geq 2$. By $E_T := E \times (0, T)$, we define a space-time cylinder, and write $\partial_{\mathrm{par}} E_T := (E \times \{0\}) \cup (\partial E \times [0, T))$ for its parabolic boundary. Throughout this paper, we study a Cauchy-Dirichlet problem for porous medium type equations of the form

$$\begin{gathered} \partial_t u - \operatorname{div}\big(\mathbf{A}(x, t, u, Du)\big) - \mathbf{B}(x, t, u, Du) = \mu \quad \text{in } E_T, \\ u = 0 \quad \text{on } \partial_{\mathrm{par}} E_T, \end{gathered} \tag{1.1}$$

where μ is a non-negative Radon measure on E_T with finite total mass $\mu(E_T) < \infty$. The vector fields $\mathbf{A} : E_T \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{B} : E_T \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are assumed to be measurable with respect to $(x, t) \in E_T$ for all $(u, \xi) \in \mathbb{R} \times \mathbb{R}^n$ and continuous with respect to $(u, \xi) \in \mathbb{R} \times \mathbb{R}^n$ for a. e. $(x, t) \in E_T$. Moreover, we require them to satisfy the ellipticity condition

$$\mathbf{A}(x, t, u, \xi) \cdot \xi \geq C_0 m |u|^{m-1} |\xi|^2 - C^2 |u|^{m+1} \tag{1.2}$$

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as well as the two growth conditions

$$|\mathbf{A}(x,t,u,\xi)| \le C_1 m|u|^{m-1}|\xi| + C|u|^m, \tag{1.3}$$

$$|\mathbf{B}(x,t,u,\xi)| \le Cm|u|^{m-1}|\xi| + C^2|u|^m \tag{1.4}$$

for any $(x,t) \in E_T$, $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$, where $C_0 > 0$, $C_1 > 0$ and $C \ge 0$ are fixed constants and $m > 1$, i. e. we are concerned with the degenerate case of the equation. Finally, in order to prove the existence of very weak solutions (cf. [3, Theorem 1.4 on page 3287]), one requires the monotonicity assumption

$$\big(\mathbf{A}(x,t,u,\xi_1) - \mathbf{A}(x,t,u,\xi_2)\big)\cdot(\xi_1-\xi_2) \ge C_0|u|^{m-1}|\xi_1-\xi_2|^2$$

to hold for any $u \in \mathbb{R}$, $\xi_1, \xi_2 \in \mathbb{R}^n$ and a. e. $(x,t) \in E_T$. However, since our objective here is not an existence proof, we do not need to have any monotonicity condition in the further course of this paper. The prototype for equations treated in the sequel is given by the classical porous medium equation

$$\partial_t u - \operatorname{div}\big(Du^m\big) = \mu \quad \text{in } E_T. \tag{1.5}$$

This ends the passage on the fundamental requirements, and some comments on the porous medium equation, its fields of utilization and the history of the problem are to come up next.

1.2. **The porous medium equation.** There are lots of different applications in which one can portray the underlying process using an equation of the above form. Besides considering such an equation for the characterization of ground water problems, heat radiation in plasmas, or spread of viscous fluids, one of the most important examples is the modeling of an ideal gas flowing isoentropically in a homogeneous porous medium, e. g. soil or foam. The flow is controlled by the following three physical laws, where for each one we like to give just a sketchy idea of what the law signifies.

Since we are guided from the concept that the total amount of gas is conserved, i. e. the rate at which mass enters some region of the medium is proportional to the rate at which mass leaves that region (the constant of proportionality $\tilde\kappa \in (0,1)$ provides information on the porosity of the medium), we postulate that the mass conservation law $\tilde\kappa\partial_t\tilde\varrho + \operatorname{div}(\tilde\varrho\tilde v) = 0$ holds, where $\tilde v \equiv \tilde v(x,t)$ is the velocity vector and $\tilde\varrho \equiv \tilde\varrho(x,t)$ is the density of the gas. Next, we may demand that also Darcy's diffusion law, an empirically derived law describing the gas flow, applies to the situation, meaning that $\tilde\nu\tilde v = -\tilde\mu D\tilde p$ is satisfied. Here, $\tilde\nu \in \mathbb{R}^+$ denotes the viscosity of the gas, $\tilde\mu \in \mathbb{R}^+$ stands for the permeability of the medium, and $\tilde p \equiv \tilde p(x,t)$ is the pressure. At last, we ask the equation of state for ideal gases $\tilde p = \tilde p_0\tilde\varrho^\alpha$ to hold with constants $\tilde p_0 \in \mathbb{R}^+$ and $\alpha \in [1,\infty)$. Combining these laws, one can eliminate the quantities $\tilde p$ and $\tilde v$ from the equations, which finally leads to the porous medium equation (1.5) with $\mu \equiv 0$, where in the physical context $m = 1+\alpha \ge 2$, and u represents a scaled density. Therefore, it is completely natural to assume $u \ge 0$ for our reflections.

Although from the physical background it seems instinctive to consider $m \ge 2$, it is sufficient to impose $m > 1$ as a condition on m, because the mathematical theory makes no distinction between the exponents as long as they are larger than 1. More precisely, the modulus of ellipticity of the treated equation is $|u|^{m-1}$. For $m > 1$, it vanishes if u becomes 0, such that the equation is degenerate on the set $\{|u| = 0\}$, whereas in the case that $0 < m < 1$, the modulus of ellipticity $|u|^{m-1}$ tends to ∞ as

$|u| \to 0$, and the equation is singular on the set $\{|u| = 0\}$. Throughout the paper, we will only look at the nonlinear, degenerate case, in which $m > 1$.

Having in mind the physical intuition, we expect that the support supp $\big(\mathcal{B}_m(\cdot, t)\big)$ of the Barenblatt fundamental solution, that is the (unique, cf. [16, Theorem 1 on page 175]) very weak solution of the porous medium equation $\partial_t u - \Delta u^m = \delta_{(0,0)}$ in $\mathbb{R}^n \times [0, \infty)$,

$$\mathcal{B}_m(x,t) := \begin{cases} t^{-\frac{n}{k}} \Big[1 - b\big(|x| t^{-\frac{1}{k}}\big)^2\Big]_+^{\frac{1}{m-1}} & \text{for } t > 0, \\ 0 & \text{for } t \le 0 \end{cases}$$

is bounded for any fixed $t > 0$ (here, $b = \frac{n(m-1)}{2nmk}$ and $k = n(m-1)+2$). This means that if we suppose that the gas solely occurs in some bounded area at time $t = 0$, the gas will have propagated after some time $t > 0$ only to a certain finite region, i. e. the gas propagates with finite speed, which coincides with our imagination of $\mathcal{B}_m$ as the distribution of the density of the gas (note that this mental image is also in perfect accordance with the fact that the solution is radial in x, in other words, the process does not prefer any specific direction). However, this imagination fails in the case $m = 1$, where the equation is nondegenerate and (1.5) passes into the well-known (linear) heat equation $\partial_t u = \Delta u$, which characterizes the distribution of heat over time not taking into account any exterior heat sources, and for which a rich theory is available (cf. [14]). The finite and infinite propagation speed, respectively, is one of the most remarkable differences between the porous medium equation with $m > 1$ and the heat equation.

As regards the regularity of solutions of the porous medium type equation

$$\partial_t u - \operatorname{div}\big(\mathbf{A}(x,t,u,Du)\big) - \mathbf{B}(x,t,u,Du) = 0$$

under the structure conditions (1.2)-(1.4), the fact that locally bounded solutions are locally Hölder continuous was established in [7]. In [8], local Hölder continuity is deduced from a Harnack inequality, and [5] already contains the regularity result for the special case of (1.5) with $\mu \equiv 0$.

Unlike in large parts of the literature existing so far, we examine a fairly general version of the porous medium equation involving a Radon measure on the right-hand side. In addition to diverse applications, such as the description of explosions, Radon measures are equipped with their own mathematical charm, which is why it is worth studying the behavior of equations of the above form. In order to get a more profound overview of the considered problem and the associated results, we refer to [2], [8], [17] as well as the list of references at the end of this article. At this point, we finish our annotations concerning the classification of the treated problem. The next subsection is devoted to settle some notations that we will employ in the sequel.

1.3. **Notation.** As to the notation, for a point $z \in \mathbb{R}^{n+1} \cong \mathbb{R}^n \times \mathbb{R}$, we always write $z = (x, t)$. As is customary, we denote by $B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$ the open ball in $\mathbb{R}^n$ with center $x_0 \in \mathbb{R}^n$ and radius $r > 0$, and we define parabolic cylinders by $Q_{r,\theta}(z_0) := B_r(x_0) \times (t_0 - \theta, t_0)$, where $z_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$, $\theta > 0$ and $r \in (0, R_0]$. Here, $R_0 > 0$ is an arbitrary upper bound for the radius r, which shall be fixed for the rest of this report. What is more, for a cylinder $Q \equiv Q_{r,\theta}(z_0)$, we use the abbreviation $2Q$ for the cylinder $Q_{2r,4\theta}(z_0)$.

By $\{u > a\}$, we express the superlevel set $\{(x,t) \in E_T : u(x,t) > a\}$ where the function u exceeds the level $a > 0$, and we address the positive part of u as $u_+ := \max\{u, 0\}$. We denote the weak spatial derivative of the function u by $Du = D_x u = (D_{x_1}u, D_{x_2}u, \dots, D_{x_n}u)$, and $\partial_t = \frac{\partial}{\partial t}$ is the operator for the time derivative. Finally, $\gamma \equiv \gamma(\cdot)$ stands for a constant which may vary from line to line and depends only on the parameters presented behind. This completes our remarks on the notations, and we turn our attention towards the central statement of this paper.

1.4. **Main result.** We now provide the principal theorem containing the linear pointwise estimate (1.6) for a weak solution of the Cauchy-Dirichlet problem (1.1) in terms of the Riesz potential $\mathbf{I}_2^\mu(z_0, r, \theta)$, which will be introduced in Definition 2.2. The proof of Theorem 1.1 will be performed in Chapter 4.

Theorem 1.1. *Let u be a weak solution of the Cauchy-Dirichlet problem* (1.1) *for the inhomogeneous porous medium type equation in the sense of Definition 2.1 and $R_0 \in (0, \infty)$ be fixed. Suppose that the structure conditions* (1.2)-(1.4) *are fulfilled. Then, for any $\lambda \in (0, \frac{1}{n}]$, almost every $z_0 \in E_T$ and every parabolic cylinder $Q_{r,\theta}(z_0) \Subset E_T$ with $r \in (0, R_0]$ and $\theta > 0$, the linear potential estimate*

$$u(z_0) \le 5\Big(\frac{r^2}{\theta}\Big)^{\frac{1}{m-1}} + \gamma\Big[\frac{1}{r^{n+2}} \iint_{Q_{r,\theta}(z_0)} u^{m+\lambda}\, dz\Big]^{\frac{1}{1+\lambda}} + \gamma \mathbf{I}_2^\mu(z_0, r, \theta) \tag{1.6}$$

holds with a universal constant $\gamma \equiv \gamma(n, C_0, C_1, C, m, \lambda, R_0)$.

This estimate is optimal in the sense that the Barenblatt solution has exactly the same behavior. Note that the bound depends on the Riesz potential in the considered point z_0, hence, viewed in this light, it is very fine. Having at hand the estimate, we ought to compare it with already existing results.

First substantial moves in the history of this field were achieved in [11, Theorem 4.1 on page 608] and [12, Theorem 1.6 on page 139], where potential estimates were established for the elliptic p-Laplacian equation. Beyond that, our conclusion generalizes some previously obtained estimates for weak solutions of the porous medium equation. To begin with, if $C = 0$ in (1.2) and (1.3), respectively, and additionally $\mu \equiv 0$ and $\mathbf{B} \equiv 0$ in (1.1), then our pointwise estimate (1.6) reduces to the $L^\infty_{\rm loc}$-bound for weak solutions of the porous medium equation [1, (1.6) on page 139]. If merely $C = 0$ and $\mathbf{B} \equiv 0$, we receive the result from [3, Theorem 1.2 on page 3285]. Furthermore, for solutions of (1.5), a similar bound was derived earlier in [15, Theorem 1.1 on page 260], but the estimate is weaker than ours and the one from [3], since it comprises an extra term

$$\gamma \sup_{t \in (t_0 - \theta, t_0)} \frac{1}{\varrho^n} \int_{B_\varrho(x_0)} u(x,t)\, dx$$

on the right-hand side. Thus, the sup-bound from [1] cannot be retrieved in the case $\mu \equiv 0$. Given the preceding observations, our potential estimate (1.6) is natural, in the sense that it implies the known results from [1], [3] and [15] in the mentioned special cases.

Moreover, when $m = 1$ and $\mu \not\equiv 0$, our result becomes a bound related to the potential estimate from [9, Theorem 1.4 on page 1101], which is stronger than ours, however, the authors postulate that another continuity assumption holds. The only distinction in the outcome concerns the exponent $1+\lambda > 1$ in the integral

$$\gamma\Big[\frac{1}{r^{n+2}}\iint_{Q_{r,\theta}(z_0)} u^{1+\lambda}\,dz\Big]^{\frac{1}{1+\lambda}}.$$

Note that we are not allowed to pass to the limit $\lambda \searrow 0$, because the constant γ blows up as $\lambda \searrow 0$.

As demonstrated in [3, Theorem 1.4 on page 3287], one can expect no more than very weak solutions to exist. For such solutions, the pointwise estimate (1.6) follows for the case $\mathbf{B} \equiv 0$ by an approximation procedure (cf. [3, Theorem 1.5 on page 3287]). If actually $\mu \in L^\infty(E_T)$, one can prove the existence of weak solutions (cf. [10, Theorem 3.1 on page 2739]). In this report, we will not pick up the theory of very weak solutions, we merely speak of weak solutions instead, being conscious of the fact that the existence of such a solution is not guaranteed as long as we consider a general Radon measure μ without any further qualities.

Since, in contrast to [3], in our structure conditions (taken from [8, Chapter 5 on page 33]) there may additionally occur low order terms, we are allowed to explore even more extensive versions of the porous medium equation, for instance, equations with principal part

$$\operatorname{div}\big(\mathbf{A}(x,t,u,Du)\big) = \sum_{i,j=1}^{n} D_{x_j}\Big(|u|^{m-1}a_{ij}(x,t)D_{x_i}u\Big) + \sum_{j=1}^{n} D_{x_j}\Big(f(x,t)|u|^m\frac{D_{x_j}u}{|Du|}\Big),$$

where f is a bounded, non-negative function, and the matrix $(a_{ij})_{1\le i,j\le n}$ is supposed to be measurable and locally positive definite in E_T (cf. [8, Section 5.2 on page 35]). Next, we go a little bit into detail about the contents of the following text and outline the strategy of our argumentation.

1.5. **Contents and proof strategies.** First of all, in Section 2.1 we will declare the concept of a weak solution of the Cauchy-Dirichlet problem (1.1) for the inhomogeneous porous medium type equation. We will then define our notion of the localized parabolic Riesz potential, which we require for writing down the pointwise estimate (1.6), and quote a parabolic Sobolev embedding, including an associated Gagliardo-Nirenberg inequality (2.2). After that, we study three auxiliary functions G_λ, V_λ and W_λ, which will turn up in the proof of Theorem 1.1. Finally, we will prepare a mollification in time and on its basis develop the regularized variant (2.8) of the weak formulation (2.1).
In the third section, we will initially define parabolic cylinders and then deduce the energy estimate (3.1). To this end, we will insert a purpose-built testing function in the regularized form (2.8) and analyze all appearing terms by applying, inter alia, convergence results for the above mollification, standard estimates like Hölder's and Young's inequality, or the ellipticity and growth conditions (1.2)-(1.4), pursuing the objective of gaining an inequality which enables us to properly bound G_λ, DV_λ and DW_λ. The idea is to express these functions, which will show up in the computations of the proof of Theorem 1.1 in a natural way, by terms that one can reasonably cope with in the further course of the paper.

The fourth paragraph is designated for the proof of the pointwise estimate (1.6) for weak solutions of the Cauchy-Dirichlet problem (1.1) for the nonhomogeneous porous medium type equation in terms of a Riesz potential. For the proof, we firstly define appropriate sequences of cylinders $(Q_j)_{j\in\mathbb{N}_0}$ and parameters $(a_j)_{j\in\mathbb{N}_0}$ and $(d_j)_{j\in\mathbb{N}_0}$ and record simple but beneficial tools for our upcoming reflections.

The matter of Chapter 4.2 is to establish the recursive bound (4.10) for d_j. To achieve this, we apply, among others, the Gagliardo-Nirenberg inequality and the energy estimate (3.1) in its version (4.21) with the previously designed cylinders $2Q_j$ and the quantities a_j and d_j. Here, the presence of the low order terms from the structure conditions (1.2)-(1.4) causes extra difficulties, since in principle we have to replace $|u|$ by $|u - a_{j-1}|$. Eventually adding up (4.10) yields a convenient bound for a_j and subsequently passing to the limit $j \to \infty$ results in the asserted bound (1.6) for $u(z_0)$, which ends the proof.

2. Preliminaries

In this section, we characterize precisely the terms *weak solution* and *Riesz potential*. Moreover, we will state a parabolic Sobolev embedding, including a Gagliardo-Nirenberg inequality, and introduce some auxiliary functions, together with three lemmata concerning their properties. At last, we create a regularized version of the weak formulation of the Cauchy-Dirichlet problem for the porous medium type equation by means of a special time mollification.

2.1. Weak solutions, Riesz potentials and a Sobolev embedding.

This part deals with weak solutions, Riesz potentials, and a Sobolev embedding with a Gagliardo-Nirenberg inequality. To begin with, we declare the definition of a weak solution of the Cauchy-Dirichlet problem for the inhomogeneous porous medium type equation, remarking that our notion of a weak solution differs from the one used in [3, Definition 1.1 on page 3284], where the regularity condition on u^m is replaced by the assumption $u^{\frac{m+1}{2}} \in L^2\big((0,T); W_0^{1,2}(E)\big)$.

Definition 2.1. A non-negative function $u : \overline{E_T} \to \mathbb{R}$ satisfying

$$u \in C^0\big([0,T]; L^2(E)\big),\ u^m \in L^2\big((0,T); W_0^{1,2}(E)\big) \text{ and } u(\cdot,0) = 0 \text{ in } E$$

is termed a weak solution of the Cauchy-Dirichlet problem (1.1) for the inhomogeneous porous medium type equation if and only if the identity

$$\begin{aligned} &\int_E u\varphi\Big|_0^T dx + \iint_{E_T} [-u\partial_t\varphi + \mathbf{A}(x,t,u,Du)\cdot D\varphi - \mathbf{B}(x,t,u,Du)\varphi]\,dz \\ &= \iint_{E_T} \varphi\,d\mu \end{aligned} \tag{2.1}$$

holds for any testing function $\varphi \in C^\infty(\overline{E_T})$ vanishing on $\partial E \times (0,T)$.

At this point, we have to give a meaning to the symbol Du and become aware of the sense which it has to be understood in, because in Definition 2.1 we have imposed $Du^m \in L^2(E_T)$, among others, as a condition on u, hence, the existence of Du cannot be assured. Formally, we set

$$Du := \frac{1}{m}\chi_{\{u>0\}} u^{1-m} Du^m$$

and like to interpret Du in that way. On $\{u > \sigma\}$, where $\sigma > 0$, Du indeed is the weak derivative of u, and we have $Du \in L^2(E_T \cap \{u > \sigma\})$. In other words, whenever we will integrate over a superlevel set of the form $\{u > \sigma\}$ with $\sigma > 0$, writing Du under the integral sign is permissible and unproblematic (in the proofs of Theorem 3.2 and Theorem 1.1, the parameter $a > 0$ and the members $a_j > 0$ of the yet to be defined sequence $(a_j)_{j\in\mathbb{N}_0}$, respectively, will take on the role of σ).

After that succinct discussion about the problems associated with Du, we get to the so-called localized parabolic Riesz potential.

Definition 2.2. For $\beta \in (0, n+2]$, $z_0 \in E_T$ and $r, \theta > 0$ such that $Q_{r,\theta}(z_0) \Subset E_T$, we define the localized parabolic Riesz potential by

$$\mathbf{I}^{\mu}_{\beta}(z_0, r, \theta) := \int_0^r \frac{\mu(Q_{\varrho,\varrho^2\theta/r^2}(z_0))}{\varrho^{n+2-\beta}} \frac{d\varrho}{\varrho}.$$

Next, we cite a parabolic Sobolev embedding (cf. [6, Proposition 3.7 on page 7]), which we will employ later many a time.

Theorem 2.3. *Let $Q_{\varrho,\theta}(z_0)$ be a parabolic cylinder with $\varrho, \theta > 0$ and let $1 < p < \infty$ and $0 < r < \infty$. Then, there exists a constant $\gamma \equiv \gamma(n, p, r)$ such that for every*

$$u \in L^{\infty}\big((t_0 - \theta, t_0); L^r(B_\varrho(x_0))\big) \cap L^p\big((t_0 - \theta, t_0); W^{1,p}(B_\varrho(x_0))\big)$$

there holds the Gagliardo-Nirenberg inequality

$$\begin{aligned} &\iint_{Q_{\varrho,\theta}(z_0)} |u|^q \, dz \\ &\le \gamma\Big(\sup_{t \in (t_0-\theta, t_0)} \int_{B_\varrho(x_0)\times\{t\}} |u|^r \, dx\Big)^{p/n} \iint_{Q_{\varrho,\theta}(z_0)} \Big[\Big|\frac{u}{\varrho}\Big|^p + |Du|^p\Big] dz, \end{aligned} \tag{2.2}$$

where q is given by $q = \frac{p(n+r)}{n}$.

Having specified the terms *weak solution* and *localized parabolic Riesz potential* and displayed the helpful Gagliardo-Nirenberg inequality, we hereby finish this section.

2.2. **Auxiliary functions.** In this part, we will introduce some mappings which will occur in the third section in the energy estimate (3.1). The assertions collected in the following lemmata will turn out to be useful in the proof of Theorem 1.1. We start our reflections by announcing the auxiliary functions.

Definition 2.4. For $\lambda \in (0, 1)$ and $s \ge 0$, we define the functions G_λ, V_λ and W_λ by

$$G_\lambda(s) := \int_0^s \big[1 - (1+\sigma)^{-\lambda}\big] \, d\sigma = s - \frac{1}{1-\lambda}\big[(1+s)^{1-\lambda} - 1\big],$$

$$V_\lambda(s) := \int_0^s \sigma^{\frac{m-1}{2}} (1+\sigma)^{-\frac{1+\lambda}{2}} \, d\sigma,$$

$$W_\lambda(s) := \int_0^s (1+\sigma)^{-\frac{1+\lambda}{2}} \, d\sigma = \frac{2}{1-\lambda}\big[(1+s)^{\frac{1-\lambda}{2}} - 1\big].$$

We now mention one lemma for each of those auxiliary functions containing some characteristics which are required afterwards. The corresponding proofs can be found in [3, Section 2.3 on page 3291].

Lemma 2.5. *For any $\varepsilon \in (0, 1]$ and $s \ge 0$, there holds*

$$s \le \varepsilon + \gamma_\varepsilon G_\lambda(s) \tag{2.3}$$

for a constant $\gamma_\varepsilon \equiv \frac{\gamma(\lambda)}{\varepsilon}$.

Lemma 2.6. *For any $\varepsilon \in (0,1]$ and $s \geq 0$, there hold*

$$V_\lambda(s) \leq \frac{2}{m-\lambda} s^{\frac{m-\lambda}{2}}, \tag{2.4}$$

$$s^{m+\lambda} \leq \varepsilon^{1+\lambda} s^{m-1} + \gamma_\varepsilon V_\lambda(s)^{\frac{2(m+\lambda)}{m-\lambda}}, \tag{2.5}$$

where the constant $\gamma_\varepsilon \equiv \gamma(m, \lambda, \varepsilon)$ blows up as $\varepsilon^{-(1+\lambda)\frac{m+\lambda}{m-\lambda}}$ in the limit $\varepsilon \searrow 0$.

Lemma 2.7. *For any $\varepsilon \in (0,1]$ and $s \geq 0$, there hold*

$$W_\lambda(s) \leq \frac{2}{1-\lambda} s^{\frac{1-\lambda}{2}}, \tag{2.6}$$

$$s^{1+\lambda} \leq \varepsilon^{1+\lambda} + \gamma_\varepsilon W_\lambda(s)^{\frac{2(1+\lambda)}{1-\lambda}}, \tag{2.7}$$

where the constant $\gamma_\varepsilon \equiv \gamma(\lambda, \varepsilon)$ blows up as $\varepsilon^{-\frac{(1+\lambda)^2}{1-\lambda}}$ in the limit $\varepsilon \searrow 0$.

We conclude the segment about the auxiliary functions and their properties on this occasion and arrive at the passage that treats the time mollification.

2.3. **Regularization via time mollification.** In this subsection, we write down the weak form (2.1) in a regularized way with the aid of a particular mollification, because the weak formulation proves to be unsuitable for inserting the testing function φ as defined in the proof of Theorem 3.2. Basically, the trouble arises from the time derivative of u, which does not need to exist, but would appear when calculating $\partial_t \varphi$. Thus, the objective of this paragraph is to find a regularized version of (2.1) where choosing the desired testing function in the proof of Theorem 3.2 is no longer an issue. At first, we describe what we mean by the mollification of a function.

Definition 2.8. For $v \in L^1(E_T)$, we define the mollification in time by

$$[\![v]\!]_h(\cdot, t) := \frac{1}{h} \int_0^t e^{\frac{s-t}{h}} v(\cdot, s)\, ds$$

and its time reversed analogue by

$$[\![v]\!]_{\bar{h}}(\cdot, t) := \frac{1}{h} \int_t^T e^{\frac{t-s}{h}} v(\cdot, s)\, ds$$

for any $h \in (0, T]$ and $t \in [0, T]$.

Before establishing the regularized version (2.8) of (2.1), we like to provide in the next lemma various useful attributes of the mollification (cf. [4, Lemma B.2 on page 261], [13, Lemma 2.2 on page 417]).

Lemma 2.9. *Let $p \geq 1$ and $v \in L^1(E_T)$. Then, the mollification $[\![v]\!]_h$ as introduced in Definition 2.8 has the following properties:*

- (i) *If $v \in L^p(E_T)$, then also $[\![v]\!]_h \in L^p(E_T)$, and the convergence $[\![v]\!]_h \to v$ in $L^p(E_T)$ as $h \searrow 0$ holds.*
- (ii) *If $v \in L^p\big((0,T); W^{1,p}(E)\big)$, then also $[\![v]\!]_h \in L^p\big((0,T); W^{1,p}(E)\big)$, and the convergence $[\![v]\!]_h \to v$ in $L^p\big((0,T); W^{1,p}(E)\big)$ as $h \searrow 0$ holds. Moreover, we have the componentwise identity $D[\![v]\!]_h = [\![Dv]\!]_h$.*
- (iii) *If $v \in L^\infty\big((0,T); L^2(E)\big)$, then $\partial_t [\![v]\!]_h \in L^\infty\big((0,T); L^2(E)\big)$.*
- (iv) *With analogous proofs, these properties hold for the time reversed mollification $[\![v]\!]_{\bar{h}}$ as well.*

After this overview of the most important features of the mollification, we can go a little bit more into detail about the regularized version (2.8), which later on allows us to apply testing functions φ whose time derivative does not necessarily have to exist. This is the essential benefit of the formulation exposed in the upcoming theorem and makes the mollification argument inevitable.

Theorem 2.10. *If u is a weak solution of the Cauchy-Dirichlet problem* (1.1)*, then its time mollification $[\![u]\!]_h$ satisfies the regularized variant of the inhomogeneous porous medium type equation*

$$\begin{aligned} &\iint_{E_T} \big[\partial_t [\![u]\!]_h \varphi + [\![\mathbf{A}(x,t,u,Du)]\!]_h \cdot D\varphi - [\![\mathbf{B}(x,t,u,Du)]\!]_h \varphi\big]\, dz \\ &= \iint_{E_T} [\![\varphi]\!]_{\bar{h}}\, d\mu \end{aligned} \tag{2.8}$$

for any testing function $\varphi \in L^2\big((0,T); W^{1,2}(E)\big) \cap L^\infty(E_T)$ with compact support in E_T.

Proof. Let $\varphi \in L^2\big((0,T); W^{1,2}(E)\big) \cap L^\infty(E_T)$ be an arbitrary testing function with compact support in E_T. To prove the identity (2.8), we insert $[\![\varphi]\!]_{\bar{h}}$ as a testing function in the weak form (2.1). In this context, we have to note that $[\![\varphi]\!]_{\bar{h}}$ is a valid testing function in (2.1) by Lemma 2.9 and a standard approximation argument. Analyzing all involved terms (as performed in [3, Chapter 2.4 on page 3293]), one will easily receive the result (2.8). □

Having at hand the terms *weak solution* and *Riesz potential*, the Sobolev embedding, the auxiliary functions, and the time regularized version of the weak formulation of the porous medium type equation, we finish this part so as to reach the next segment, which revolves around another tool, i. e. an energy estimate, for the proof of the pointwise estimate (1.6).

3. Energy estimates

In this section, we deduce the energy estimate (3.1), which we require in the proof of Theorem 1.1. In view of this aim, we first of all present parabolic cylinders, which we will use in the course of the following observations. For this purpose, we recall the upper bound $R_0 > 0$ for the radius, which was determined at the beginning of Paragraph 1.3.

Definition 3.1. For $a > 0$, $\varrho \in (0, R_0]$ and $z_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$, we define parabolic cylinders by

$$Q_\varrho^{(a)}(z_0) := B_\varrho(x_0) \times \Lambda_\varrho^{(a)}(t_0) := B_\varrho(x_0) \times (t_0 - a^{1-m}\varrho^2, t_0).$$

For the sake of simplicity, we omit the (fixed) point z_0 in our notation from now on; for instance, we will write $Q_\varrho^{(a)}$ or B_ϱ instead of $Q_\varrho^{(a)}(z_0)$ and $B_\varrho(x_0)$, respectively. Next, we derive the energy estimate.

Theorem 3.2. *Let $\lambda \in (0,1)$, $d > 0$ and further suppose that $z_0 \in \mathbb{R}^{n+1}$, $a > 0$ and $\varrho \in (0, R_0]$ are such that $Q_\varrho^{(a)}(z_0) \equiv Q_\varrho^{(a)} \subset E_T$. Then, for a weak solution u*

of the Cauchy-Dirichlet problem (1.1), *the energy estimate*

$$
\begin{aligned}
&\sup_{t\in\Lambda^{(a)}_{\varrho/2}} \int_{B_{\varrho/2}\times\{t\}\cap\{u>a\}} G_\lambda\Big(\frac{u-a}{d}\Big)\,dx \\
&+ \iint_{Q^{(a)}_{\varrho/2}\cap\{u>a\}} \Big[d^{m-1}\Big|DV_\lambda\Big(\frac{u-a}{d}\Big)\Big|^2 + a^{m-1}\Big|DW_\lambda\Big(\frac{u-a}{d}\Big)\Big|^2\Big]dz \\
&\le \frac{\gamma}{\varrho^2}\iint_{Q^{(a)}_{\varrho}\cap\{u>a\}} u^{m-1}\Big(1+\frac{u-a}{d}\Big)^{1+\lambda}\,dz + \frac{\gamma}{d\varrho}\iint_{Q^{(a)}_{\varrho}\cap\{u>a\}} u^m\,dz \\
&\quad + \frac{\gamma}{d^2}\iint_{Q^{(a)}_{\varrho}\cap\{u>a\}} \frac{u^{m+1}}{(1+\frac{u-a}{d})^{1+\lambda}}\,dz + \frac{\gamma\mu\big(Q^{(a)}_{\varrho}\big)}{d}
\end{aligned}
\tag{3.1}
$$

holds with a constant $\gamma \equiv \gamma(C_0, C_1, C, m, \lambda, R_0)$.

Proof. Let u be a weak solution of (1.1) in the sense of Definition 2.1. In the regularized form (2.8), we choose the testing function $\varphi := \eta^2\zeta v$, where v is given by

$$
v := g(u) := 1 - \Big(1 + \frac{(u-a)_+}{d}\Big)^{-\lambda},
$$

$\eta \in C^1_0(B_\varrho(x_0), [0,1])$ is a function with $\eta \equiv 1$ on $B_{\varrho/2}(x_0)$ and $|D\eta| \le \frac{4}{\varrho}$, and $\zeta \in W^{1,\infty}_0(\mathbb{R}, [0,1])$ fulfills

$$
\zeta(t) := \begin{cases} 0 & \text{for } t \in (-\infty, t_0 - a^{1-m}\varrho^2) \cup [\tau, \infty), \\ \frac{4a^{m-1}}{3\varrho^2}\big(t - (t_0 - a^{1-m}\varrho^2)\big) & \text{for } t \in [t_0 - a^{1-m}\varrho^2, t_0 - a^{1-m}(\frac{\varrho}{2})^2), \\ 1 & \text{for } t \in [t_0 - a^{1-m}(\frac{\varrho}{2})^2, \tau - \varepsilon), \\ \frac{1}{\varepsilon}(\tau - t) & \text{for } t \in [\tau - \varepsilon, \tau) \end{cases}
$$

for a fixed $\tau \in \Lambda^{(a)}_{\varrho/2}$ and $\varepsilon > 0$. To avoid an overburdened notation, we employ the abbreviations

$$
\begin{gathered}
Q^+ := Q^{(a)}_\varrho(z_0) \cap \{u > a\} = \big(B_\varrho(x_0) \times (t_0 - a^{1-m}\varrho^2, t_0)\big) \cap \{u > a\}, \\
B^+(t) := B_\varrho(x_0) \cap \{u(\cdot, t) > a\}.
\end{gathered}
$$

Since u is a weak solution of (1.1), by Theorem 2.10 the identity (2.8) holds, which we now insert the above concrete testing function φ in. Using the shortcuts

$$
\mathrm{I}^{(1)} := \iint_{E_T} \partial_t [\![u]\!]_h \varphi\,dz, \tag{3.2}
$$

$$
\mathrm{II}^{(1)} := \iint_{E_T} [\![\mathbf{A}(x,t,u,Du)]\!]_h \cdot D\varphi\,dz, \tag{3.3}
$$

$$
\mathrm{III}^{(1)} := \iint_{E_T} [\![\mathbf{B}(x,t,u,Du)]\!]_h \varphi\,dz, \tag{3.4}
$$

$$
\mathrm{IV}^{(1)} := \iint_{E_T} [\![\varphi]\!]_{\overline{h}}\,d\mu, \tag{3.5}
$$

we obtain the equation

$$
\mathrm{I}^{(1)} + \mathrm{II}^{(1)} - \mathrm{III}^{(1)} - \mathrm{IV}^{(1)} = 0. \tag{3.6}
$$

In the following, we will separately estimate the terms (3.2)-(3.5), starting with (3.2). As g is increasing, the identity $\partial_t [\![u]\!]_h = -\frac{1}{h}([\![u]\!]_h - u)$ implies

$$\partial_t [\![u]\!]_h \big(g(u) - g([\![u]\!]_h)\big) = \frac{1}{h}\big([\![u]\!]_h - u\big)\big(g([\![u]\!]_h) - g(u)\big) \geq 0$$

which yields

$$\begin{aligned}
\mathrm{I}^{(1)} &= \iint_{E_T} \partial_t [\![u]\!]_h \varphi \, dz \\
&\geq \iint_{Q^+} \eta^2 \zeta \partial_t [\![u]\!]_h g([\![u]\!]_h) \, dz \\
&= \iint_{Q^+} \eta^2 \zeta \frac{\partial}{\partial t}\Big[\int_a^{[\![u]\!]_h} g(\sigma)\, d\sigma\Big] \, dz \\
&= -\iint_{Q^+} \eta^2 \partial_t \zeta \int_a^{[\![u]\!]_h} g(\sigma)\, d\sigma \, dz \\
&= -\frac{4a^{m-1}}{3\varrho^2} \int_{t_0 - a^{1-m}\varrho^2}^{t_0 - a^{1-m}(\varrho/2)^2} \int_{B^+(t)} \eta^2 \int_a^{[\![u]\!]_h} g(\sigma)\, d\sigma \, dx \, dt \\
&\quad + \frac{1}{\varepsilon} \int_{\tau-\varepsilon}^{\tau} \int_{B^+(t)} \eta^2 \int_a^{[\![u]\!]_h} g(\sigma)\, d\sigma \, dx \, dt \\
&=: \mathrm{I}^{(2)}(h) + \mathrm{II}^{(2)}(h,\varepsilon).
\end{aligned} \tag{3.7}$$

First, we consider $\mathrm{II}^{(2)}(h,\varepsilon)$. Passing to the limits $\varepsilon \searrow 0$ and $h \searrow 0$, by the Lebesgue differentiation theorem we receive

$$\begin{aligned}
\lim_{h\searrow 0} \lim_{\varepsilon \searrow 0} \mathrm{II}^{(2)}(h,\varepsilon) &= \lim_{h\searrow 0} \lim_{\varepsilon \searrow 0} \fint_{\tau-\varepsilon}^{\tau} \int_{B^+(t)} \eta^2 \int_a^{[\![u]\!]_h(x,t)} g(\sigma)\, d\sigma \, dx \, dt \\
&= \lim_{h\searrow 0} \int_{B^+(\tau)} \eta^2 \int_a^{[\![u]\!]_h(x,\tau)} \Big[1 - \Big(1 + \frac{\sigma - a}{d}\Big)^{-\lambda}\Big] d\sigma \, dx \\
&= d \int_{B^+(\tau)} \eta^2 \Big[\frac{u-a}{d} - \frac{1}{1-\lambda}\Big(\Big(1 + \frac{u-a}{d}\Big)^{1-\lambda} - 1\Big)\Big] dx \\
&= d \int_{B^+(\tau)} \eta^2 G_\lambda\Big(\frac{u-a}{d}\Big)\, dx
\end{aligned} \tag{3.8}$$

for a. e. $\tau \in \Lambda_{\varrho/2}^{(a)}$, where we have exploited the L^2-convergence $[\![u]\!]_h \to u$ as $h \searrow 0$ (cf. Lemma 2.9). Next, we let $h \searrow 0$ also in the term $\mathrm{I}^{(2)}(h)$ which results in

$$\begin{aligned}
\lim_{h\searrow 0} |\mathrm{I}^{(2)}(h)| &\leq \frac{4d}{3\varrho^2} \int_{t_0 - a^{1-m}\varrho^2}^{t_0} \int_{B^+(t)} \eta^2 a^{m-1} \frac{u-a}{d} \, dx \, dt \\
&\leq \frac{4d}{3\varrho^2} \int_{t_0 - a^{1-m}\varrho^2}^{t_0} \int_{B^+(t)} u^{m-1} \Big(1 + \frac{u-a}{d}\Big)^{1+\lambda} dx \, dt.
\end{aligned} \tag{3.9}$$

To get this, we have used the inequality $g(\sigma) \leq 1$ for $\sigma \geq a$, enlarged the domain of integration, and in the last step estimated $\eta \leq 1$, $a \leq u$ and

$$\frac{u-a}{d} \leq 1 + \frac{u-a}{d} \leq \Big(1 + \frac{u-a}{d}\Big)^{1+\lambda}$$

on the domain of integration. Inserting (3.8) and (3.9) in (3.7), we can record as an interim conclusion the lower bound

$$\begin{aligned}\lim_{h\searrow 0}\lim_{\varepsilon\searrow 0}\mathrm{I}^{(1)} &\geq d\int_{B^+(\tau)}\eta^2 G_\lambda\Big(\frac{u-a}{d}\Big)\,dx\\ &\quad-\frac{4d}{3\varrho^2}\int_{t_0-a^{1-m}\varrho^2}^{t_0}\int_{B^+(t)}u^{m-1}\Big(1+\frac{u-a}{d}\Big)^{1+\lambda}\,dx\,dt\end{aligned}\tag{3.10}$$

for $\mathrm{I}^{(1)}$, which holds for a. e. $\tau\in\Lambda^{(a)}_{\varrho/2}$. In the following, we deal with the term $\mathrm{II}^{(1)}$. Again building the limits $\varepsilon\searrow 0$ and $h\searrow 0$, we find

$$\begin{aligned}\lim_{h\searrow 0}\lim_{\varepsilon\searrow 0}\mathrm{II}^{(1)} &= \iint_{Q^+}\mathbf{A}(x,t,u,Du)\cdot D\varphi\,dz\\ &= \iint_{Q^+}\eta^2\zeta\mathbf{A}(x,t,u,Du)\cdot Dv\,dz\\ &\quad+2\iint_{Q^+}\eta\zeta v\mathbf{A}(x,t,u,Du)\cdot D\eta\,dz\\ &=: \mathrm{I}^{(3)}+\mathrm{II}^{(3)}.\end{aligned}\tag{3.11}$$

Before turning towards the term $\mathrm{II}^{(3)}$, we treat the term $\mathrm{I}^{(3)}$. Having in mind the ellipticity condition (1.2), we compute for the latter

$$\begin{aligned}\mathrm{I}^{(3)} &= \frac{\lambda}{d}\iint_{Q^+}\eta^2\zeta\Big(1+\frac{u-a}{d}\Big)^{-1-\lambda}\mathbf{A}(x,t,u,Du)\cdot Du\,dz\\ &\geq \frac{\lambda C_0 m}{d}\iint_{Q^+}\eta^2\zeta\frac{u^{m-1}|Du|^2}{(1+\frac{u-a}{d})^{1+\lambda}}\,dz-\frac{\lambda C^2}{d}\iint_{Q^+}\frac{u^{m+1}}{(1+\frac{u-a}{d})^{1+\lambda}}\,dz.\end{aligned}\tag{3.12}$$

For the other term, we exploit in turn the fact that $v\leq 1$, the growth condition (1.3), the bounds $|D\eta|\leq\frac{4}{\varrho}$ and $\eta\zeta\leq 1$, Young's inequality, and $\zeta\leq 1$ to conclude that

$$\begin{aligned}|\mathrm{II}^{(3)}| &\leq 2\iint_{Q^+}\eta\zeta v|\mathbf{A}(x,t,u,Du)||D\eta|\,dz\\ &\leq \frac{8C_1 m}{\varrho}\iint_{Q^+}\eta\zeta u^{m-1}|Du|\,dz+\frac{8C}{\varrho}\iint_{Q^+}u^m\,dz\\ &\leq \frac{\lambda C_0 m}{2d}\iint_{Q^+}\eta^2\zeta\frac{u^{m-1}|Du|^2}{(1+\frac{u-a}{d})^{1+\lambda}}\,dz\\ &\quad+\frac{64mC_1^2 d}{2\lambda C_0\varrho^2}\iint_{Q^+}u^{m-1}\Big(1+\frac{u-a}{d}\Big)^{1+\lambda}\,dz+\frac{8C}{\varrho}\iint_{Q^+}u^m\,dz.\end{aligned}\tag{3.13}$$

Combining (3.12) and (3.13) with (3.11) leads us to the estimate

$$\begin{aligned}&\lim_{h\searrow 0}\lim_{\varepsilon\searrow 0}\mathrm{II}^{(1)}\\ &\geq \frac{\lambda C_0 m}{2d}\iint_{Q^+}\eta^2\zeta\frac{u^{m-1}|Du|^2}{(1+\frac{u-a}{d})^{1+\lambda}}\,dz-\frac{\lambda C^2}{d}\iint_{Q^+}\frac{u^{m+1}}{(1+\frac{u-a}{d})^{1+\lambda}}\,dz\\ &\quad-\frac{32mC_1^2 d}{\lambda C_0\varrho^2}\iint_{Q^+}u^{m-1}\Big(1+\frac{u-a}{d}\Big)^{1+\lambda}\,dz-\frac{8C}{\varrho}\iint_{Q^+}u^m\,dz\end{aligned}\tag{3.14}$$

as an outcome of our thoughts on the term $\mathrm{II}^{(1)}$. We now give our attention to the third summand of (3.6), initially letting $\varepsilon\searrow 0$ and $h\searrow 0$ and subsequently using $v \le 1$, the growth condition (1.4), the fact that $\eta^2\zeta \le 1$, and finally Young's inequality to obtain

$$\begin{aligned}
&\Big|\lim_{h\searrow 0}\lim_{\varepsilon\searrow 0} \mathrm{III}^{(1)}\Big| \\
&\le \iint_{Q^+} \eta^2\zeta v|\mathbf{B}(x,t,u,Du)|\,dz \\
&\le Cm \iint_{Q^+} \eta^2\zeta u^{m-1}|Du|\,dz + C^2 \iint_{Q^+} u^m\,dz \\
&\le \frac{\lambda C_0 m}{4d} \iint_{Q^+} \eta^2\zeta \frac{u^{m-1}|Du|^2}{(1+\frac{u-a}{d})^{1+\lambda}}\,dz \\
&\quad + \frac{dmC^2}{\lambda C_0} \iint_{Q^+} u^{m-1}\Big(1+\frac{u-a}{d}\Big)^{1+\lambda}\,dz + C^2 \iint_{Q^+} u^m\,dz.
\end{aligned} \tag{3.15}$$

It remains to estimate the term $\mathrm{IV}^{(1)}$. Passing to the limits first and then applying Lemma 2.9 and $\varphi \le 1$, we have

$$\lim_{h\searrow 0}\lim_{\varepsilon\searrow 0} \mathrm{IV}^{(1)} = \lim_{h\searrow 0} \iint_{Q^+} [\![\varphi]\!]_{\bar{h}}\,d\mu = \iint_{Q^+} \varphi\,d\mu \le \mu(Q^+). \tag{3.16}$$

This completes the evaluations of the terms appearing in (3.6), and we can insert the results (3.10) and (3.14)-(3.16) there. Noting that the inclusions $Q^+ \supset Q_*$ (where $Q_* := B_{\varrho/2} \times (t_0 - a^{1-m}(\frac{\varrho}{2})^2, \tau) \cap \{u > a\}$) and $B^+(\tau) \supset B_{\varrho/2} \cap \{u(\cdot,\tau) > a\}$ hold, (3.6) gives

$$\begin{aligned}
&\int_{B_{\varrho/2}\times\{\tau\}\cap\{u>a\}} \eta^2 G_\lambda\Big(\frac{u-a}{d}\Big)\,dx + \frac{\lambda C_0 m}{4d^2} \iint_{Q_*} \eta^2\zeta \frac{u^{m-1}|Du|^2}{(1+\frac{u-a}{d})^{1+\lambda}}\,dz \\
&\le \Big(\frac{4}{3\varrho^2} + \frac{32mC_1^2}{\lambda C_0\varrho^2} + \frac{mC^2}{\lambda C_0}\Big) \iint_{Q^+} u^{m-1}\Big(1+\frac{u-a}{d}\Big)^{1+\lambda}\,dz \\
&\quad + \frac{1}{d}\Big(\frac{8C}{\varrho} + C^2\Big) \iint_{Q^+} u^m\,dz + \frac{\lambda C^2}{d^2} \iint_{Q^+} \frac{u^{m+1}}{(1+\frac{u-a}{d})^{1+\lambda}}\,dz + \frac{\mu(Q^+)}{d}
\end{aligned} \tag{3.17}$$

for a.e. $\tau \in \Lambda_{\varrho/2}^{(a)}$. Since $\eta \equiv 1$ on $B_{\varrho/2}$ and $\zeta \equiv 1$ on $(t_0 - a^{1-m}(\frac{\varrho}{2})^2, \tau)$, by respectively taking the supremum over all $\tau \in \Lambda_{\varrho/2}^{(a)}$ we infer from (3.17) that both

$$\sup_{t\in\Lambda_{\varrho/2}^{(a)}} \int_{B_{\varrho/2}\times\{t\}\cap\{u>a\}} G_\lambda\Big(\frac{u-a}{d}\Big)\,dx,$$

$$\frac{\lambda C_0 m}{4d^2} \iint_{Q_{\varrho/2}^{(a)}\cap\{u>a\}} \frac{u^{m-1}|Du|^2}{(1+\frac{u-a}{d})^{1+\lambda}}\,dz$$

can be bounded by the right-hand side of (3.17) which easily leads us to

$$\begin{aligned}
&\sup_{t\in\Lambda_{\varrho/2}^{(a)}} \int_{B_{\varrho/2}\times\{t\}\cap\{u>a\}} G_\lambda\Big(\frac{u-a}{d}\Big)\,dx + \frac{\lambda}{d^2} \iint_{Q_{\varrho/2}^{(a)}\cap\{u>a\}} \frac{u^{m-1}|Du|^2}{(1+\frac{u-a}{d})^{1+\lambda}}\,dz \\
&\le \frac{\gamma}{\varrho^2} \iint_{Q_\varrho^{(a)}\cap\{u>a\}} u^{m-1}\Big(1+\frac{u-a}{d}\Big)^{1+\lambda}\,dz + \frac{\gamma}{d\varrho} \iint_{Q_\varrho^{(a)}\cap\{u>a\}} u^m\,dz \\
&\quad + \frac{\gamma}{d^2} \iint_{Q_\varrho^{(a)}\cap\{u>a\}} \frac{u^{m+1}}{(1+\frac{u-a}{d})^{1+\lambda}}\,dz + \frac{\gamma\mu(Q_\varrho^{(a)})}{d}
\end{aligned} \tag{3.18}$$

with a constant $\gamma \equiv \gamma(C_0, C_1, C, m, \lambda, R_0)$. On the set $Q_{\varrho/2}^{(a)} \cap \{u > a\}$, we have

$$DV_\lambda\Big(\frac{u-a}{d}\Big) = \Big(\frac{u-a}{d}\Big)^{\frac{m-1}{2}} \Big(1 + \frac{u-a}{d}\Big)^{-\frac{1+\lambda}{2}} \frac{Du}{d},$$

on the other hand, there holds

$$DW_\lambda\Big(\frac{u-a}{d}\Big) = \Big(1 + \frac{u-a}{d}\Big)^{-\frac{1+\lambda}{2}} \frac{Du}{d}$$

which together yields

$$\begin{aligned} &d^{m-1}\Big|DV_\lambda\Big(\frac{u-a}{d}\Big)\Big|^2 + a^{m-1}\Big|DW_\lambda\Big(\frac{u-a}{d}\Big)\Big|^2 \\ &= \frac{|Du|^2}{d^2}\Big(1 + \frac{u-a}{d}\Big)^{-(1+\lambda)} \big[(u-a)^{m-1} + a^{m-1}\big] \\ &\le \frac{|Du|^2}{d^2}\Big(1 + \frac{u-a}{d}\Big)^{-(1+\lambda)} 2u^{m-1}. \end{aligned}$$

Hence, we are allowed to rewrite (3.18) in the form

$$\begin{aligned} &\sup_{t \in \Lambda_{\varrho/2}^{(a)}} \int_{B_{\varrho/2}\times\{t\}\cap\{u>a\}} G_\lambda\Big(\frac{u-a}{d}\Big)\,dx \\ &+ \frac{\lambda}{2}\iint_{Q_{\varrho/2}^{(a)}\cap\{u>a\}} \Big[d^{m-1}\Big|DV_\lambda\Big(\frac{u-a}{d}\Big)\Big|^2 + a^{m-1}\Big|DW_\lambda\Big(\frac{u-a}{d}\Big)\Big|^2\Big]dz \\ &\le \frac{\gamma}{\varrho^2}\iint_{Q_\varrho^{(a)}\cap\{u>a\}} u^{m-1}\Big(1 + \frac{u-a}{d}\Big)^{1+\lambda}\,dz + \frac{\gamma}{d\varrho}\iint_{Q_\varrho^{(a)}\cap\{u>a\}} u^m\,dz \\ &\quad + \frac{\gamma}{d^2}\iint_{Q_\varrho^{(a)}\cap\{u>a\}} \frac{u^{m+1}}{(1+\frac{u-a}{d})^{1+\lambda}}\,dz + \frac{\gamma\mu\big(Q_\varrho^{(a)}\big)}{d}. \end{aligned} \tag{3.19}$$

As $\lambda \in (0,1)$, the inequality (3.19) remains true if we multiply the term involving the supremum by $\frac{\lambda}{2}$. After that, the assertion (3.1) eventually results from dividing the whole inequality by $\frac{\lambda}{2}$. □

The energy estimate (3.1) is now at our disposal, and we end this paragraph. Moreover, we have finished the preparations for the proof of Theorem 1.1, which permits us to head for this central statement.

4. Proof of Theorem 1.1

We arrive at the core of this report. The instruments developed in the previous two sections enable us to explicitly prove the pointwise estimate (1.6) for weak solutions of the Cauchy-Dirichlet problem (1.1) for the nonhomogeneous porous medium type equation.

Proof. We will proceed as described in Section 1.5.

4.1. **Choice of parameters.** In this segment, we will provide cylinders and parameters which later on will turn out to be suitable, when inserted in the energy estimate (3.1). Therefore, we have to mention the quantities $\mathbf{K}_j$ and $\mathbf{k}_j$ that will show up in a natural way in the proof, which is why we will additionally detect some of their features in this passage. What is more, we will outline several expedient relations between functions, cylinders etc. that will emerge in the further course of the proof.

Let $\lambda \in (0, \frac{1}{n}]$ and $Q_{r,\theta}(z_0) \Subset E_T$, where $r \in (0, R_0]$ and $\theta > 0$. As before, we omit the center z_0 in our notation. For $j \in \mathbb{N}_0$, we define sequences of radii

$$r_j := \frac{r}{2^j},$$

parameters

$$\theta_j := \frac{\theta}{2^{2j}},$$

and cylinders

$$Q_j := B_j \times \Lambda_j := B_{r_j} \times (t_0 - a_j^{1-m} r_j^2, t_0),$$

where the quantities a_j will be chosen inductively below. We set

$$a_0 := \Big(\frac{r^2}{\theta}\Big)^{\frac{1}{m-1}}$$

and assume for $j \geq 0$ that $a_0, \dots, a_j$ have already been specified. For the purpose of selecting a_{j+1}, we first define

$$\mathbf{K}_j(a) := \frac{1}{r_j^{n+2}} \iint_{Q_j \cap \{u > a_j\}} u^{m-1} \Big(\frac{u - a_j}{a - a_j}\Big)^{1+\lambda} dz$$

for $a > a_j$ and observe the convergence $\mathbf{K}_j(a) \to 0$ as $a \to \infty$. Let $\kappa \in (0,1)$ be a fixed parameter which we will determine later. Then we choose

$$a_{j+1} := \big[1 + 2^{-(j+2)}\big] a_j \tag{4.1}$$

if

$$\mathbf{K}_j \Big([1 + 2^{-(j+2)}] a_j \Big) \leq \kappa \tag{4.2}$$

holds, and

$$a_{j+1} := \sup \big\{ a \in \big([1 + 2^{-(j+2)}] a_j, \infty \big) : \mathbf{K}_j(a) > \kappa \big\}, \tag{4.3}$$

provided that we have

$$\mathbf{K}_j \big([1 + 2^{-(j+2)}] a_j \big) > \kappa.$$

When a_{j+1} is defined as in (4.3), there hold

$$\mathbf{K}_j(a_{j+1}) = \kappa \tag{4.4}$$

and $a_{j+1} > \big[1 + 2^{-(j+2)}\big] a_j$, because the mapping $\mathbf{K}_j : (a_j, \infty) \to \mathbb{R}$ is continuous and decreasing. In both cases, (4.1) and (4.3), we set $d_j := a_{j+1} - a_j$ for $j \in \mathbb{N}_0$ and define

$$\mathbf{k}_j := \mathbf{K}_j(a_{j+1}),$$

which satisfies

$$\mathbf{k}_j = \frac{1}{r_j^{n+2}} \iint_{Q_j \cap \{u > a_j\}} u^{m-1} \Big(\frac{u - a_j}{d_j}\Big)^{1+\lambda} dz \leq \kappa \tag{4.5}$$

for any $j \in \mathbb{N}_0$, since we have (4.2) if a_{j+1} is defined via (4.1), and, in the case that a_{j+1} is given by (4.3), there even holds equality in (4.5) by (4.4). In order to be enabled to replace u by $u - a_{j-1}$ later in the proof, we need the estimation

$$u \leq 2^{j+2} (u - a_{j-1}) \tag{4.6}$$

for any $j \in \mathbb{N}$ on the set $\{u > a_j\}$, which we briefly establish in the following. Both if a_j is defined as in (4.1) and if a_j is stated in (4.3), there holds $a_j \geq \big[1 + 2^{-(j+1)}\big] a_{j-1}$, or equivalently, $a_j - a_{j-1} \geq 2^{-(j+1)} a_{j-1}$. This leads us to the estimate

$$\frac{a_j}{a_j - a_{j-1}} = 1 + \frac{a_{j-1}}{a_j - a_{j-1}} \leq 2^{j+2}. \tag{4.7}$$

On $\{u > a_j\}$, we compute $\big(1 - \frac{a_{j-1}}{a_j}\big)u = u - a_{j-1} + \frac{a_{j-1}}{a_j}(a_j - u) \le u - a_{j-1}$, and using the inequality (4.7), we obtain $u \le \frac{a_j}{a_j - a_{j-1}}(u - a_{j-1}) \le 2^{j+2}(u - a_{j-1})$.

We terminate this paragraph with two statements regarding the previously initiated cylinders. To begin with, due to the fact that $2r_{j+1} = r_j$ and $a_{j+1} > a_j$, we infer the inclusion

$$2Q_{j+1} \subset Q_j \tag{4.8}$$

for any $j \in \mathbb{N}_0$, and, furthermore, we have

$$Q_j \subset Q_{r_j,\theta_j} \tag{4.9}$$

for any $j \in \mathbb{N}_0$, since $a_j \ge a_0 = (r^2/\theta)^{\frac{1}{m-1}}$ and thus $a_j^{1-m} r_j^2 \le \frac{\theta}{r^2} r_j^2 = \theta_j$ holds.

4.2. **Recursive bounds for d_j.** Having prepared the sequences $(Q_j)_{j\in\mathbb{N}_0}$, $(a_j)_{j\in\mathbb{N}_0}$ and $(d_j)_{j\in\mathbb{N}_0}$, we arrive at this section whose objective is to show that the inequality

$$d_j \le \frac{1}{2} d_{j-1} + 2^{-(j+2)} a_j + \frac{\gamma \mu(2Q_{r_j,\theta_j})}{r_j^n} \tag{4.10}$$

is valid for any $j \in \mathbb{N}$, where $\gamma \equiv \gamma(n, C_0, C_1, C, m, \lambda, R_0)$ is a constant. We will roughly proceed as follows: After various introductory comments, we will apply the energy estimate (3.1) with the concrete cylinders and parameters from Chapter 4.1 and modify the outcome until we reach the assertion (4.21). Next, we estimate by the right-hand side of (4.33) the terms $\mathrm{I}^{(5)}$ and $\mathrm{II}^{(5)}$ which will occur in a natural way in (4.22). To achieve this, we will repeatedly avail ourselves to the Gagliardo-Nirenberg inequality (2.2) and the energy estimate in its version (4.21). Then, immediately after rewriting (4.33) in the more convenient form (4.35) and a simple case analysis, the conclusion (4.10) ensues.

We start our considerations by excluding certain trivial cases. According to (4.5), we have $\mathbf{k}_j \le \kappa$ for any $j \in \mathbb{N}_0$. If $\mathbf{k}_j < \kappa$ holds, a_{j+1} is defined via (4.1), meaning that we have $a_{j+1} = [1 + 2^{-(j+2)}]a_j$, which is equivalent to $d_j = 2^{-(j+2)} a_j$, so that (4.10) is obviously satisfied. Consequently, let

$$\mathbf{k}_j = \kappa \tag{4.11}$$

from now on. Moreover, we can assume without loss of generality that

$$d_j > \frac{1}{2} d_{j-1} \tag{4.12}$$

holds, since otherwise we would have $d_j \le \frac{1}{2} d_{j-1}$ which again instantly implies (4.10). Before approaching the proof of the bound (4.10), we shall establish some helpful estimates which we will frequently require later on. For one thing, we have

$$1 = \frac{a_j - a_{j-1}}{d_{j-1}} \le \frac{u - a_{j-1}}{d_{j-1}} \tag{4.13}$$

on the set $2Q_j \cap \{u > a_j\}$, for another thing, the inequality

$$\frac{u - a_j}{d_j} \le \frac{u - a_{j-1}}{d_j} \le 2 \frac{u - a_{j-1}}{d_{j-1}} \tag{4.14}$$

holds on $2Q_j \cap \{u > a_j\}$ by the fact that $a_j > a_{j-1}$ and the assumption (4.12) from above. Beyond that, we use the observation (4.13) and the identity $\frac{1}{r_j} = \frac{2}{r_{j-1}}$, extend the domain of integration (note that $a_j > a_{j-1}$ and $2Q_j \subset Q_{j-1}$ by (4.8)), and finally consult the property (4.5) of $\mathbf{k}_{j-1}$ to obtain

$$\begin{aligned}
&\frac{1}{r_j^{n+2}} \iint_{Q_j\cap\{u>a_j\}} u^{m-1}\,dz \\
&\le \frac{1}{r_j^{n+2}} \iint_{2Q_j\cap\{u>a_j\}} u^{m-1}\,dz \\
&\le \frac{1}{r_j^{n+2}} \iint_{2Q_j\cap\{u>a_j\}} u^{m-1}\Big(\frac{u-a_{j-1}}{d_{j-1}}\Big)^{1+\lambda}\,dz \\
&\le \frac{2^{n+2}}{r_{j-1}^{n+2}} \iint_{Q_{j-1}\cap\{u>a_{j-1}\}} u^{m-1}\Big(\frac{u-a_{j-1}}{d_{j-1}}\Big)^{1+\lambda}\,dz \\
&= 2^{n+2}\mathbf{k}_{j-1} \le 2^{n+2}\kappa.
\end{aligned} \tag{4.15}$$

This completes the preliminary thoughts of this section, and we now devote ourselves to the energy estimate with the concrete quantities from Section 4.1. To this end, we fix $\lambda \in (0, \frac{1}{n}]$ and apply (3.1) with the cylinder $2Q_j$ in lieu of $Q_\varrho^{(a)}$, also replacing the parameters (a, d) from Theorem 3.2 by (a_j, d_j). This yields

$$\begin{aligned}
&\sup_{t\in\Lambda_j} \int_{B_j\times\{t\}\cap\{u>a_j\}} G_\lambda\Big(\frac{u-a_j}{d_j}\Big)\,dx \\
&+ \iint_{Q_j\cap\{u>a_j\}} \Big[d_j^{m-1}\Big|DV_\lambda\Big(\frac{u-a_j}{d_j}\Big)\Big|^2 + a_j^{m-1}\Big|DW_\lambda\Big(\frac{u-a_j}{d_j}\Big)\Big|^2\Big]\,dz \\
&\le \frac{\gamma}{r_j^2} \iint_{2Q_j\cap\{u>a_j\}} u^{m-1}\Big(1+\frac{u-a_j}{d_j}\Big)^{1+\lambda}\,dz + \frac{\gamma}{d_j r_j} \iint_{2Q_j\cap\{u>a_j\}} u^m\,dz \\
&\quad + \frac{\gamma}{d_j^2} \iint_{2Q_j\cap\{u>a_j\}} \frac{u^{m+1}}{(1+\frac{u-a_j}{d_j})^{1+\lambda}}\,dz + \frac{\gamma\mu(2Q_j)}{d_j} \\
&=: \mathrm{I}^{(4)} + \mathrm{II}^{(4)} + \mathrm{III}^{(4)} + \frac{\gamma\mu(2Q_j)}{d_j}.
\end{aligned} \tag{4.16}$$

In turn, we examine the terms $\mathrm{I}^{(4)}$, $\mathrm{II}^{(4)}$ and $\mathrm{III}^{(4)}$, starting with $\mathrm{I}^{(4)}$. With the aid of the inequalities (4.13) and (4.14) in the first and (4.15) in the second step, respectively, we compute

$$\mathrm{I}^{(4)} \le \gamma\frac{1}{r_j^2} \iint_{2Q_j\cap\{u>a_j\}} u^{m-1}\Big(\frac{u-a_{j-1}}{d_{j-1}}\Big)^{1+\lambda}\,dz \le \gamma r_j^n\kappa \tag{4.17}$$

for a constant $\gamma \equiv \gamma(n, C_0, C_1, C, m, \lambda, R_0)$. Estimating u via (4.6) and successively using the assumption (4.12), the fact that $2^j = \frac{r}{r_j} \le \frac{R_0}{r_j}$, the observation (4.13), and ultimately the inequality (4.15), we find

$$\begin{aligned}
\mathrm{II}^{(4)} &= \frac{\gamma}{d_j r_j} \iint_{2Q_j\cap\{u>a_j\}} u^{m-1}u\,dz \\
&\le \gamma\frac{2^j}{r_j} \iint_{2Q_j\cap\{u>a_j\}} u^{m-1}\frac{u-a_{j-1}}{d_j}\,dz \\
&\le \gamma\frac{1}{r_j^2} \iint_{2Q_j\cap\{u>a_j\}} u^{m-1}\Big(\frac{u-a_{j-1}}{d_{j-1}}\Big)^{1+\lambda}\,dz \le \gamma r_j^n\kappa
\end{aligned} \tag{4.18}$$

for the second term. Before discussing the term $\mathrm{III}^{(4)}$, we convince ourselves that on the domain of integration, there holds

$$\begin{aligned}\frac{u}{1+\frac{u-a_j}{d_j}} &\le 2^{j+2}\frac{(u-a_{j-1})d_j}{d_j+u-a_j} = 2^{j+2}\Big[\frac{(u-a_j)d_j}{d_j+u-a_j}+\frac{(a_j-a_{j-1})d_j}{d_j+u-a_j}\Big]\\ &\le 2^{j+2}(d_j+d_{j-1}) \le 12\cdot 2^j d_j \le 12R_0\frac{d_j}{r_j},\end{aligned} \tag{4.19}$$

where we have deployed the result (4.6), the fact that $d_j \ge 0$ and $u-a_j \ge 0$ hold true, and the assumption (4.12). Initially decreasing the denominator of the fraction in $\mathrm{III}^{(4)}$ and subsequently consulting (4.19) and the bound for $\mathrm{II}^{(4)}$ from (4.18), we are enabled to establish the estimate

$$\begin{aligned}\mathrm{III}^{(4)} &\le \frac{\gamma}{d_j^2}\iint_{2Q_j\cap\{u>a_j\}} u^m \frac{u}{1+\frac{u-a_j}{d_j}}\,dz\\ &\le \frac{\gamma}{d_j r_j}\iint_{2Q_j\cap\{u>a_j\}} u^m\,dz \le \gamma r_j^n\kappa.\end{aligned} \tag{4.20}$$

We insert the outcomes (4.17), (4.18) and (4.20) in (4.16) to obtain the energy estimate

$$\begin{aligned}&\sup_{t\in\Lambda_j}\int_{B_j\times\{t\}\cap\{u>a_j\}} G_\lambda\Big(\frac{u-a_j}{d_j}\Big)\,dx\\ &+\iint_{Q_j\cap\{u>a_j\}}\Big[d_j^{m-1}\Big|DV_\lambda\Big(\frac{u-a_j}{d_j}\Big)\Big|^2 + a_j^{m-1}\Big|DW_\lambda\Big(\frac{u-a_j}{d_j}\Big)\Big|^2\Big]\,dz\\ &\le \gamma\big[r_j^n\kappa + \frac{\mu(2Q_j)}{d_j}\big]\end{aligned} \tag{4.21}$$

for a constant $\gamma \equiv \gamma(n, C_0, C_1, C, m, \lambda, R_0)$.

Since we have reduced the situation to the case in which (4.11) holds, we can now proceed as follows:

$$\begin{aligned}\kappa = \mathbf{k}_j &= \frac{1}{r_j^{n+2}}\iint_{Q_j\cap\{u>a_j\}}\big[(u-a_j)+a_j\big]^{m-1}\Big(\frac{u-a_j}{d_j}\Big)^{1+\lambda}\,dz\\ &\le \frac{\gamma d_j^{m-1}}{r_j^{n+2}}\iint_{Q_j\cap\{u>a_j\}}\Big(\frac{u-a_j}{d_j}\Big)^{m+\lambda}\,dz\\ &\quad+\frac{\gamma a_j^{m-1}}{r_j^{n+2}}\iint_{Q_j\cap\{u>a_j\}}\Big(\frac{u-a_j}{d_j}\Big)^{1+\lambda}\,dz\\ &=: \mathrm{I}^{(5)}+\mathrm{II}^{(5)}\end{aligned} \tag{4.22}$$

for a constant $\gamma \equiv \gamma(m)$. In the sequel, we first consider $\mathrm{I}^{(5)}$ and then $\mathrm{II}^{(5)}$. For the former, we have

$$\begin{aligned}\mathrm{I}^{(5)} &\le \frac{\gamma d_j^{m-1}}{r_j^{n+2}}\Big[\varepsilon^{1+\lambda}\iint_{Q_j\cap\{u>a_j\}}\Big(\frac{u-a_j}{d_j}\Big)^{m-1}\,dz\\ &\quad+\gamma_\varepsilon\iint_{Q_j\cap\{u>a_j\}}\Big(V_\lambda\Big(\frac{u-a_j}{d_j}\Big)\Big)^{\frac{2(m+\lambda)}{m-\lambda}}\,dz\Big]\\ &\le \gamma\varepsilon^{1+\lambda}\frac{1}{r_j^{n+2}}\iint_{Q_j\cap\{u>a_j\}} u^{m-1}\,dz\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\gamma_\varepsilon d_j^{m-1}}{r_j^{n+2}} \iint_{Q_j\cap\{u>a_j\}} \Big(V_\lambda\Big(\frac{u-a_j}{d_j}\Big)\Big)^{\frac{2(m+\lambda)}{m-\lambda}}\, dz\\
&\le \gamma\varepsilon^{1+\lambda}\kappa + \frac{\gamma_\varepsilon d_j^{m-1}}{r_j^{n+2}} \iint_{Q_j} \Big(V_\lambda\Big(\frac{(u-a_j)_+}{d_j}\Big)\Big)^{\frac{2(m+\lambda)}{m-\lambda}}\, dz
\end{aligned}
$$

for constants $\gamma \equiv \gamma(n,m)$ and $\gamma_\varepsilon \equiv \gamma(m,\lambda,\varepsilon)$, using the inequality (2.5) from Lemma 2.6 for some $\varepsilon \in (0,1)$ to be chosen later and (4.15) as well as noting that $V_\lambda(0) = 0$ holds. Next, we apply the Gagliardo-Nirenberg inequality with $p = 2$, $q = \frac{2(m+\lambda)}{m-\lambda}$ and $r = \frac{2\lambda n}{m-\lambda}$ to receive

$$
\begin{aligned}
\mathrm{I}^{(5)} &\le \gamma\varepsilon^{1+\lambda}\kappa + \gamma_\varepsilon\Big[\sup_{t\in\Lambda_j} \frac{1}{r_j^n}\int_{B_j\times\{t\}} \Big|V_\lambda\Big(\frac{(u-a_j)_+}{d_j}\Big)\Big|^{\frac{2\lambda n}{m-\lambda}}\, dx\Big]^{2/n}\\
&\quad\times \frac{d_j^{m-1}}{r_j^n}\iint_{Q_j}\Big[\frac{1}{r_j^2}\Big|V_\lambda\Big(\frac{(u-a_j)_+}{d_j}\Big)\Big|^2 + \Big|DV_\lambda\Big(\frac{(u-a_j)_+}{d_j}\Big)\Big|^2\Big]dz\\
&=: \gamma\varepsilon^{1+\lambda}\kappa + \gamma_\varepsilon \mathrm{I}^{(6)}\big(\mathrm{II}^{(6)} + \mathrm{III}^{(6)}\big)
\end{aligned}
\tag{4.23}
$$

for constants $\gamma \equiv \gamma(n,m)$ and $\gamma_\varepsilon \equiv \gamma(n,m,\lambda,\varepsilon)$. At this point, we shall take a brief snapshot of the progress of the proof. We have begun to work on the expression $\mathrm{I}^{(5)}$, where further terms $\mathrm{I}^{(6)}$, $\mathrm{II}^{(6)}$ and $\mathrm{III}^{(6)}$, which are to be discussed in what follows, arose in (4.23). Our next goal is to estimate these terms, before we cope with $\mathrm{II}^{(5)}$ from (4.22). We continue the proof by looking at the term $\mathrm{I}^{(6)}$. With the help of inequality (2.4), Hölder's inequality, the statement (2.3) for some fixed $\varepsilon_1 \in (0,1)$ to be chosen later, and the energy estimate (4.21), we deduce

$$
\begin{aligned}
\mathrm{I}^{(6)} &= \Big[\sup_{t\in\Lambda_j}\frac{1}{r_j^n}\int_{B_j\times\{t\}\cap\{u>a_j\}} \Big|V_\lambda\Big(\frac{u-a_j}{d_j}\Big)\Big|^{\frac{2\lambda n}{m-\lambda}}\, dx\Big]^{2/n}\\
&\le \gamma\Big[\sup_{t\in\Lambda_j}\frac{1}{r_j^n}\int_{B_j\times\{t\}\cap\{u>a_j\}} \Big(\frac{u-a_j}{d_j}\Big)^{\lambda n}\, dx\Big]^{2/n}\\
&\le \gamma\Big[\sup_{t\in\Lambda_j}\frac{1}{r_j^n}\int_{B_j\times\{t\}\cap\{u>a_j\}} \frac{u-a_j}{d_j}\, dx\Big]^{2\lambda}\\
&\le \gamma\varepsilon_1^{2\lambda} + \gamma\varepsilon_1^{-2\lambda}\Big[\sup_{t\in\Lambda_j}\frac{1}{r_j^n}\int_{B_j\times\{t\}\cap\{u>a_j\}} G_\lambda\Big(\frac{u-a_j}{d_j}\Big)\, dx\Big]^{2\lambda}\\
&\le \gamma\varepsilon_1^{2\lambda} + \gamma\varepsilon_1^{-2\lambda}\Big[\kappa + \frac{\mu(2Q_j)}{d_j r_j^n}\Big]^{2\lambda}
\end{aligned}
\tag{4.24}
$$

for a constant $\gamma \equiv \gamma(n, C_0, C_1, C, m, \lambda, R_0)$. This completes our thoughts on the term $\mathrm{I}^{(6)}$, and we now turn towards $\mathrm{II}^{(6)}$ by applying the inequality (2.4), enlarging the domain of integration, and using the fact that $u - a_j \le u$ and (4.14). Additionally enlarging the exponent from $1-\lambda$ to $1+\lambda$ (note that (4.13) holds) and exploiting (4.15), we obtain

$$\begin{aligned}
\mathrm{II}^{(6)} &= \frac{d_j^{m-1}}{r_j^{n+2}} \iint_{Q_j\cap\{u>a_j\}} \Big|V_\lambda\Big(\frac{u-a_j}{d_j}\Big)\Big|^2\,dz \\
&\le \frac{\gamma d_j^{m-1}}{r_j^{n+2}} \iint_{Q_j\cap\{u>a_j\}} \Big(\frac{u-a_j}{d_j}\Big)^{m-\lambda}\,dz \\
&\le \frac{\gamma}{r_j^{n+2}} \iint_{2Q_j\cap\{u>a_j\}} u^{m-1}\Big(\frac{u-a_j}{d_j}\Big)^{1-\lambda}\,dz \\
&\le \frac{\gamma}{r_j^{n+2}} \iint_{2Q_j\cap\{u>a_j\}} u^{m-1}\Big(\frac{u-a_{j-1}}{d_{j-1}}\Big)^{1-\lambda}\,dz \\
&\le \frac{\gamma}{r_j^{n+2}} \iint_{2Q_j\cap\{u>a_j\}} u^{m-1}\Big(\frac{u-a_{j-1}}{d_{j-1}}\Big)^{1+\lambda}\,dz \le \gamma\kappa
\end{aligned} \tag{4.25}$$

for a constant $\gamma \equiv \gamma(n,m,\lambda)$. Studying the term $\mathrm{III}^{(6)}$, we find

$$\mathrm{III}^{(6)} = \frac{d_j^{m-1}}{r_j^{n}} \iint_{Q_j\cap\{u>a_j\}} \Big|DV_\lambda\Big(\frac{u-a_j}{d_j}\Big)\Big|^2\,dz \le \gamma\Big[\kappa + \frac{\mu(2Q_j)}{d_j r_j^n}\Big] \tag{4.26}$$

for a constant $\gamma \equiv \gamma(n,C_0,C_1,C,m,\lambda,R_0)$, where we made use of (4.21). We insert the results (4.24), (4.25) and (4.26) in (4.23) and gain

$$\mathrm{I}^{(5)} \le \gamma\varepsilon^{1+\lambda}\kappa + \gamma_\varepsilon\Big[\varepsilon_1^{2\lambda} + \varepsilon_1^{-2\lambda}\Big(\kappa + \frac{\mu(2Q_j)}{d_j r_j^n}\Big)^{2\lambda}\Big]\Big[\kappa + \frac{\mu(2Q_j)}{d_j r_j^n}\Big] \tag{4.27}$$

for constants $\gamma \equiv \gamma(n,m)$ and $\gamma_\varepsilon \equiv \gamma(n,C_0,C_1,C,m,\lambda,R_0,\varepsilon)$, which qualifies us to put aside the considerations of the first summand from (4.22) to address ourselves to some illustrations of $\mathrm{II}^{(5)}$. For the term $\mathrm{II}^{(5)}$, we involve in turn the inequalities (2.7) and $a_j^{m-1} \le u^{m-1}$ (the latter holds on the domain of integration), the fact that $W_\lambda(0)=0$, and (4.15) to derive the estimate

$$\begin{aligned}
\mathrm{II}^{(5)} &\le \frac{\gamma\varepsilon^{1+\lambda}}{r_j^{n+2}} \iint_{Q_j\cap\{u>a_j\}} a_j^{m-1}\,dz \\
&\quad + \frac{\gamma_\varepsilon a_j^{m-1}}{r_j^{n+2}} \iint_{Q_j\cap\{u>a_j\}} \Big(W_\lambda\Big(\frac{u-a_j}{d_j}\Big)\Big)^{\frac{2(1+\lambda)}{1-\lambda}}\,dz \\
&\le \gamma\varepsilon^{1+\lambda}\frac{1}{r_j^{n+2}} \iint_{Q_j\cap\{u>a_j\}} u^{m-1}\,dz \\
&\quad + \frac{\gamma_\varepsilon a_j^{m-1}}{r_j^{n+2}} \iint_{Q_j} \Big(W_\lambda\Big(\frac{(u-a_j)_+}{d_j}\Big)\Big)^{\frac{2(1+\lambda)}{1-\lambda}}\,dz \\
&\le \gamma\varepsilon^{1+\lambda}\kappa + \frac{\gamma_\varepsilon a_j^{m-1}}{r_j^{n+2}} \iint_{Q_j} \Big(W_\lambda\Big(\frac{(u-a_j)_+}{d_j}\Big)\Big)^{\frac{2(1+\lambda)}{1-\lambda}}\,dz
\end{aligned}$$

for constants $\gamma \equiv \gamma(n,m)$ and $\gamma_\varepsilon \equiv \gamma(m,\lambda,\varepsilon)$. Once again applying the Gagliardo-Nirenberg inequality from Theorem 2.3, this time for the choices $p=2$, $q=\frac{2(1+\lambda)}{1-\lambda}$

and $r = \frac{2\lambda n}{1-\lambda}$, we acquire

$$\begin{aligned} \mathrm{II}^{(5)} &\le \gamma\varepsilon^{1+\lambda}\kappa + \gamma_\varepsilon\Big[\sup_{t\in\Lambda_j}\frac{1}{r_j^n}\int_{B_j\times\{t\}}\Big|W_\lambda\Big(\frac{(u-a_j)_+}{d_j}\Big)\Big|^{\frac{2\lambda n}{1-\lambda}}dx\Big]^{2/n} \\ &\quad\times\frac{a_j^{m-1}}{r_j^n}\iint_{Q_j}\Big[\frac{1}{r_j^2}\Big|W_\lambda\Big(\frac{(u-a_j)_+}{d_j}\Big)\Big|^2+\Big|DW_\lambda\Big(\frac{(u-a_j)_+}{d_j}\Big)\Big|^2\Big]dz \\ &=: \gamma\varepsilon^{1+\lambda}\kappa+\gamma_\varepsilon \mathrm{I}^{(7)}(\mathrm{II}^{(7)}+\mathrm{III}^{(7)}) \end{aligned} \tag{4.28}$$

for constants $\gamma \equiv \gamma(n, m)$ and $\gamma_\varepsilon \equiv \gamma(n, m, \lambda, \varepsilon)$. Analogous to the approach in (4.23), we have to develop in the following some appropriate bounds for the terms $\mathrm{I}^{(7)}$, $\mathrm{II}^{(7)}$ and $\mathrm{III}^{(7)}$ as well. Fortunately, the former two can by little moves be reduced to the terms $\mathrm{I}^{(6)}$ and $\mathrm{II}^{(6)}$, with the result that we are enabled to employ the inequalities (4.24) and (4.25), respectively, which we have already deduced. For the term $\mathrm{III}^{(7)}$, the same argumentation as the one used for $\mathrm{III}^{(6)}$ is operating effectively. After this synoptic view of the further proof strategy, we commence the evaluation of $\mathrm{I}^{(7)}$. Consulting (2.6) and the accomplishments for $\mathrm{I}^{(6)}$ from above (cf. (4.24)), we find

$$\begin{aligned} \mathrm{I}^{(7)} &= \Big[\sup_{t\in\Lambda_j}\frac{1}{r_j^n}\int_{B_j\times\{t\}\cap\{u>a_j\}}\Big|W_\lambda\Big(\frac{u-a_j}{d_j}\Big)\Big|^{\frac{2\lambda n}{1-\lambda}}dx\Big]^{2/n} \\ &\le \gamma\Big[\sup_{t\in\Lambda_j}\frac{1}{r_j^n}\int_{B_j\times\{t\}\cap\{u>a_j\}}\Big(\frac{u-a_j}{d_j}\Big)^{\lambda n}dx\Big]^{2/n} \\ &\le \gamma\varepsilon_1^{2\lambda}+\gamma\varepsilon_1^{-2\lambda}\Big[\kappa+\frac{\mu(2Q_j)}{d_jr_j^n}\Big]^{2\lambda} \end{aligned} \tag{4.29}$$

for a constant $\gamma \equiv \gamma(n, C_0, C_1, C, m, \lambda, R_0)$. To deal with the term $\mathrm{II}^{(7)}$, we also exploit the inequality (2.6), replace a_j by u, and exert the observations for $\mathrm{II}^{(6)}$ from (4.25) to obtain

$$\begin{aligned} \mathrm{II}^{(7)} &= \frac{a_j^{m-1}}{r_j^{n+2}}\iint_{Q_j\cap\{u>a_j\}}\Big|W_\lambda\Big(\frac{u-a_j}{d_j}\Big)\Big|^2dz \\ &\le \frac{\gamma}{r_j^{n+2}}\iint_{Q_j\cap\{u>a_j\}}a_j^{m-1}\Big(\frac{u-a_j}{d_j}\Big)^{1-\lambda}dz \\ &\le \frac{\gamma}{r_j^{n+2}}\iint_{Q_j\cap\{u>a_j\}}u^{m-1}\Big(\frac{u-a_j}{d_j}\Big)^{1-\lambda}dz \le \gamma\kappa \end{aligned} \tag{4.30}$$

for a constant $\gamma \equiv \gamma(n, \lambda)$. Eventually, working with the same arguments as in (4.26), we get the estimate

$$\mathrm{III}^{(7)} = \frac{a_j^{m-1}}{r_j^n}\iint_{Q_j\cap\{u>a_j\}}\Big|DW_\lambda\Big(\frac{u-a_j}{d_j}\Big)\Big|^2dz \le \gamma\Big[\kappa+\frac{\mu(2Q_j)}{d_jr_j^n}\Big] \tag{4.31}$$

for a constant $\gamma \equiv \gamma(n, C_0, C_1, C, m, \lambda, R_0)$. We insert (4.29)-(4.31) in (4.28) to find that

$$\mathrm{II}^{(5)} \le \gamma\varepsilon^{1+\lambda}\kappa+\gamma_\varepsilon\Big[\varepsilon_1^{2\lambda}+\varepsilon_1^{-2\lambda}\Big(\kappa+\frac{\mu(2Q_j)}{d_jr_j^n}\Big)^{2\lambda}\Big]\Big[\kappa+\frac{\mu(2Q_j)}{d_jr_j^n}\Big] \tag{4.32}$$

holds for constants $\gamma \equiv \gamma(n, m)$ and $\gamma_\varepsilon \equiv \gamma(n, C_0, C_1, C, m, \lambda, R_0, \varepsilon)$, in other words, $\mathrm{II}^{(5)}$ can be bounded by the right-hand side of (4.27) as well. This closes our calculations for the term $\mathrm{II}^{(5)}$, and we join both the estimates (4.27) and (4.32). Afterwards, we will specify the parameters ε, ε_1 and κ. By means of a case analysis, the desired recursive bound (4.10) for d_j finally becomes apparent. Embedding the estimates (4.27) and (4.32) for $\mathrm{I}^{(5)}$ and $\mathrm{II}^{(5)}$ in (4.22) yields

$$\kappa \le \gamma\varepsilon^{1+\lambda}\kappa + \gamma_\varepsilon\Big[\varepsilon_1^{2\lambda} + \varepsilon_1^{-2\lambda}\Big(\kappa + \frac{\mu(2Q_j)}{d_j r_j^n}\Big)^{2\lambda}\Big]\Big[\kappa + \frac{\mu(2Q_j)}{d_j r_j^n}\Big] \tag{4.33}$$

for constants $\gamma \equiv \gamma(n,m)$ and $\gamma_\varepsilon \equiv \gamma(n, C_0, C_1, C, m, \lambda, R_0, \varepsilon)$. Before deciding on the values of the quantities $\varepsilon, \varepsilon_1, \kappa \in (0,1)$, we shall detect an alternative representation for (4.33). If $\kappa \le \frac{\mu(2Q_j)}{d_j r_j^n}$ holds, we conclude

$$\kappa \le \gamma\varepsilon^{1+\lambda}\kappa + \gamma_\varepsilon\Big[\varepsilon_1^{2\lambda} + \varepsilon_1^{-2\lambda}\Big(2\frac{\mu(2Q_j)}{d_j r_j^n}\Big)^{2\lambda}\Big]\Big[2\frac{\mu(2Q_j)}{d_j r_j^n}\Big],$$

whereas in the case that $\kappa > \frac{\mu(2Q_j)}{d_j r_j^n}$ is valid, one can infer

$$\kappa \le \gamma\varepsilon^{1+\lambda}\kappa + \gamma_\varepsilon\big[\varepsilon_1^{2\lambda} + \varepsilon_1^{-2\lambda}(2\kappa)^{2\lambda}\big]\big[2\kappa\big].$$

We note that $\varepsilon_1^{2\lambda} \le \varepsilon_1^{-2\lambda}$ to find that in any case, by adding the right-hand sides of the last two inequalities, (4.33) implies

$$\kappa \le \big(\gamma\varepsilon^{1+\lambda} + \gamma_\varepsilon\varepsilon_1^{2\lambda} + \gamma_\varepsilon\varepsilon_1^{-2\lambda}\kappa^{2\lambda}\big)\kappa + \gamma_\varepsilon\varepsilon_1^{-2\lambda}\Big[\frac{\mu(2Q_j)}{d_j r_j^n} + \Big(\frac{\mu(2Q_j)}{d_j r_j^n}\Big)^{1+2\lambda}\Big]. \tag{4.34}$$

We now determine the still available parameters ε, ε_1 and κ as follows: First, we choose ε such that $\gamma\varepsilon^{1+\lambda} = \frac{1}{6}$, then ε_1 such that $\gamma_\varepsilon\varepsilon_1^{2\lambda} = \frac{1}{6}$, and finally κ to satisfy $\gamma_\varepsilon\varepsilon_1^{-2\lambda}\kappa^{2\lambda} = \frac{1}{6}$, where one can easily verify that all three quantities actually lie within the demanded interval $(0,1)$. Besides, ε, ε_1 and κ only depend on n, C_0, C_1, C, m, λ and R_0. That way, the preceding inequality (4.34) evolves into

$$\kappa \le \gamma\Big[\frac{\mu(2Q_j)}{d_j r_j^n} + \Big(\frac{\mu(2Q_j)}{d_j r_j^n}\Big)^{1+2\lambda}\Big] \tag{4.35}$$

for a constant $\gamma \equiv \gamma(n, C_0, C_1, C, m, \lambda, R_0)$. With the aid of a case analysis, the inequality (4.10) will relatively quickly come out of (4.35). Indeed, if there holds $\alpha := \mu(2Q_j)/(d_j r_j^n) \le 1$, we have $\alpha^{1+2\lambda} \le \alpha$, and, consequently, one can infer

$$d_j \le \gamma\frac{\mu(2Q_j)}{r_j^n} \tag{4.36}$$

from (4.35), since κ, just like γ, solely depends on n, C_0, C_1, C, m, λ and R_0. If otherwise $\alpha > 1$ holds, we have $\alpha \le \alpha^{1+2\lambda}$, and (4.35) likewise yields (4.36). In both cases, this leads to

$$d_j \le 2\gamma\frac{\mu(2Q_j)}{r_j^n} \le \frac{\gamma\mu(2Q_{r_j,\theta_j})}{r_j^n},$$

where we have used (4.9). Eventually, the claim (4.10) ensues from this estimate. We hereby terminate this passage on the recursive bound for d_j and move on to the last subsection to establish the proposition (1.6).

4.3. **Potential estimates.** In this segment, we resort to the property (4.10) of d_j to deduce the alleged inequality (1.6). More precisely, we will add up (4.10) which gives us the bound (4.38) for the members of the sequence $(a_j)_{j\in\mathbb{N}_0}$. Appropriately estimating in (4.38) both a_1 and the involved sum (the latter in terms of a Riesz potential), subsequently passing to the limit $j\to\infty$, and additionally employing a short argument which allows to associate $u(z_0)$ with the limit a_∞, we obtain the assertion of Theorem 1.1.

Summing up the result (4.10) proved in Section 4.2, we receive

$$\begin{aligned} a_\ell - a_1 = \sum_{j=1}^{\ell-1} d_j &\le \frac{1}{2}\sum_{j=1}^{\ell-1} d_{j-1} + \sum_{j=1}^{\ell-1} 2^{-(j+2)} a_j + \gamma \sum_{j=1}^{\ell-1} \frac{\mu(2Q_{r_j,\theta_j})}{r_j^n} \\ &\le \frac{3}{4} a_\ell + \gamma \sum_{j=1}^{\ell} \frac{\mu(2Q_{r_j,\theta_j})}{r_j^n} \end{aligned} \tag{4.37}$$

for any $\ell \ge 2$, where we have worked with the estimates (note that $(a_j)_{j\in\mathbb{N}_0}$ is an increasing sequence)

$$\begin{gathered} \sum_{j=1}^{\ell-1} d_{j-1} = a_{\ell-1} - a_0 \le a_{\ell-1} \le a_\ell, \\ \sum_{j=1}^{\ell-1} 2^{-(j+2)} a_j \le \frac{a_\ell}{4} \sum_{j=1}^{\ell-1} 2^{-j} \le \frac{a_\ell}{4}. \end{gathered}$$

The inequality (4.37) connotes

$$a_\ell \le 4a_1 + \gamma \sum_{j=1}^{\ell} \frac{\mu(2Q_{r_j,\theta_j})}{r_j^n} \tag{4.38}$$

for any $\ell \ge 2$. In the following, we are interested in a bound for the parameter a_1, which is why we recall its definition. If $\mathbf{K}_0(\frac{5}{4}a_0) \le \kappa$, we have set $a_1 = \frac{5}{4}a_0$, hence, (4.38) gives

$$a_\ell \le 5\Big(\frac{r^2}{\theta}\Big)^{\frac{1}{m-1}} + \gamma \sum_{j=1}^{\ell} \frac{\mu(2Q_{r_j,\theta_j})}{r_j^n}, \tag{4.39}$$

whereas in the case that $\mathbf{K}_0(\frac{5}{4}a_0) > \kappa$ holds, the equation (4.4) for $j=0$ reads as

$$\frac{1}{r^{n+2}} \iint_{Q_0\cap\{u>a_0\}} u^{m-1}\Big(\frac{u-a_0}{a_1-a_0}\Big)^{1+\lambda}\,dz = \kappa.$$

We multiply both sides by $\frac{(a_1-a_0)^{1+\lambda}}{\kappa}$ and subsequently raise them to the power $\frac{1}{1+\lambda}$ to acquire

$$\begin{aligned} a_1 &= a_0 + \Big[\frac{1}{\kappa r^{n+2}} \iint_{Q_0\cap\{u>a_0\}} u^{m-1}(u-a_0)^{1+\lambda}\,dz\Big]^{\frac{1}{1+\lambda}} \\ &\le \Big(\frac{r^2}{\theta}\Big)^{\frac{1}{m-1}} + \Big[\frac{1}{\kappa r^{n+2}} \iint_{Q_{r,\theta}} u^{m+\lambda}\,dz\Big]^{\frac{1}{1+\lambda}}, \end{aligned}$$

where in the second step we have replaced $u-a_0$ by u and enlarged the domain of integration via (4.9). Inserting this in (4.38), we find

$$a_\ell \le 4\Big(\frac{r^2}{\theta}\Big)^{\frac{1}{m-1}} + 4\Big[\frac{1}{\kappa r^{n+2}}\iint_{Q_{r,\theta}} u^{m+\lambda}\,dz\Big]^{\frac{1}{1+\lambda}} + \gamma\sum_{j=1}^{\ell}\frac{\mu(2Q_{r_j,\theta_j})}{r_j^n}. \tag{4.40}$$

Thus, regardless of whether $\mathbf{K}_0(\frac{5}{4}a_0)\le\kappa$ or not, we derive

$$a_\ell \le 5\Big(\frac{r^2}{\theta}\Big)^{\frac{1}{m-1}} + \gamma\Big[\frac{1}{r^{n+2}}\iint_{Q_{r,\theta}} u^{m+\lambda}\,dz\Big]^{\frac{1}{1+\lambda}} + \gamma\sum_{j=1}^{\infty}\frac{\mu(2Q_{r_j,\theta_j})}{r_j^n} \tag{4.41}$$

for a constant $\gamma\equiv\gamma(n,C_0,C_1,C,m,\lambda,R_0)$ from (4.39) and (4.40). Next, we estimate the series in (4.41) by a Riesz potential. For this purpose, we set $r_{-1}:=2r$ and compute

$$\begin{aligned}
\sum_{j=1}^{\infty}\frac{\mu(2Q_{r_j,\theta_j})}{r_j^n} &= \sum_{j=1}^{\infty}\frac{1}{r_{j-2}-r_{j-1}}\int_{r_{j-1}}^{r_{j-2}}\frac{\mu(Q_{r_{j-1},r_{j-1}^2\theta/r^2})}{r_j^n}\,d\varrho\\
&\le 2^{2n}\sum_{j=1}^{\infty}\int_{r_{j-1}}^{r_{j-2}}\frac{\mu(Q_{\varrho,\varrho^2\theta/r^2})}{r_{j-2}^n(r_{j-2}-r_{j-1})}\,d\varrho\\
&\le 2^{2n+1}\sum_{j=1}^{\infty}\int_{r_{j-1}}^{r_{j-2}}\frac{\mu(Q_{\varrho,\varrho^2\theta/r^2})}{\varrho^n\varrho}\,d\varrho\\
&= 2^{2n+1}\int_0^{2r}\frac{\mu(Q_{\varrho,\varrho^2\theta/r^2})}{\varrho^n}\frac{d\varrho}{\varrho}\\
&= 2^{2n+1}\mathbf{I}_2^{\mu}(z_0,2r,4\theta)\,,
\end{aligned}$$

which, inserted in (4.41), yields the inequality

$$\begin{aligned}
a_\ell &\le 5\Big(\frac{r^2}{\theta}\Big)^{\frac{1}{m-1}} + \gamma\Big[\frac{1}{r^{n+2}}\iint_{Q_{r,\theta}} u^{m+\lambda}\,dz\Big]^{\frac{1}{1+\lambda}} + \gamma\mathbf{I}_2^{\mu}(z_0,2r,4\theta)\\
&\le 5\Big(\frac{(2r)^2}{4\theta}\Big)^{\frac{1}{m-1}} + \gamma\Big[\frac{1}{(2r)^{n+2}}\iint_{Q_{2r,4\theta}} u^{m+\lambda}\,dz\Big]^{\frac{1}{1+\lambda}} + \gamma\mathbf{I}_2^{\mu}(z_0,2r,4\theta)\,.
\end{aligned}$$

Substituting $2r$ by r and 4θ by θ, this implies in particular that

$$\begin{aligned}
a_\infty := \lim_{j\to\infty} a_j \le 5\Big(\frac{r^2}{\theta}\Big)^{\frac{1}{m-1}} + \gamma\Big[\frac{1}{r^{n+2}}\iint_{Q_{r,\theta}} u^{m+\lambda}\,dz\Big]^{\frac{1}{1+\lambda}}\\
+\gamma\mathbf{I}_2^{\mu}(z_0,r,\theta) < \infty
\end{aligned} \tag{4.42}$$

for a constant $\gamma\equiv\gamma(n,C_0,C_1,C,m,\lambda,R_0)$. By the definition of d_j $(=a_j-a_{j-1})$, we infer the convergence $d_j\to 0$ as $j\to\infty$. Now, let z_0 be a Lebesgue point of u. Defining for short $\omega_n:=\mathcal{H}^{n-1}(\mathcal{S}^{n-1})$, we then have

$$\begin{aligned}
0 &\le \Big(\frac{u(z_0)}{a_\infty}\Big)^{m-1}\big(u(z_0)-a_\infty\big)_+^{1+\lambda}\\
&= \lim_{j\to\infty}⨌_{Q_j}\Big(\frac{u}{a_j}\Big)^{m-1}\big(u-a_j\big)_+^{1+\lambda}\,dz\\
&= \lim_{j\to\infty}\frac{nd_j^{1+\lambda}}{\omega_n}\frac{1}{r_j^{n+2}}\iint_{Q_j\cap\{u>a_j\}} u^{m-1}\Big(\frac{u-a_j}{d_j}\Big)^{1+\lambda}\,dz
\end{aligned}$$

$$\le \frac{n\kappa}{\omega_n} \lim_{j\to\infty} d_j^{1+\lambda} = 0$$

by the inequality (4.5) and the limit $d_j \to 0$ as $j \to \infty$, which we have just established above. Hence, $u(z_0) - a_\infty \le 0$ necessarily holds. Taking into account the estimate (4.42), this leads us to

$$u(z_0) \le a_\infty \le 5\Big(\frac{r^2}{\theta}\Big)^{\frac{1}{m-1}} + \gamma\Big[\frac{1}{r^{n+2}} \iint_{Q_{r,\theta}} u^{m+\lambda}\, dz\Big]^{\frac{1}{1+\lambda}} + \gamma \mathbf{I}_2^{\mu}(z_0, r, \theta)$$

for any Lebesgue point z_0 of u with a constant $\gamma \equiv \gamma(n, C_0, C_1, C, m, \lambda, R_0)$ which proves the assertion of Theorem 1.1. □

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