Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 05, pp. 1–17.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

COMPARISON RESULTS FOR ELLIPTIC VARIATIONAL INEQUALITIES RELATED TO GAUSS MEASURE

YUJUAN TIAN, CHAO MA

Abstract. In this article, we study linear elliptic variational inequalities that are defined on a possibly unbounded domain and whose ellipticity condition is given in terms of the density of Gauss measure. Using the notion of rearrangement with respect to the Gauss measure, we prove a comparison result with a problem of the same type defined in a half space, with data depending only on the first variable.

1. Introduction

This article concerns the problem

$$\begin{gathered} a(u,\psi-u)\geq\int_\Omega f(\psi-u)\varphi\,dx,\quad \forall\psi\in H_0^1(\varphi,\Omega),\ \psi\geq 0,\\ u\in H_0^1(\varphi,\Omega),\quad u\geq 0, \end{gathered}\tag{1.1}$$

where

$$\begin{aligned} a(u,\psi-u)=&\int_\Omega\sum_{i,j=1}^n a_{ij}D_iuD_j(\psi-u)\,dx-\int_\Omega\sum_{i=1}^n b_iuD_i(\psi-u)\,dx\\ &+\int_\Omega\sum_{i=1}^n d_iD_iu(\psi-u)\,dx+\int_\Omega cu(\psi-u)\,dx, \end{aligned}$$

$\varphi(x)=(2\pi)^{-\frac{n}{2}}e^{-\frac{|x|^2}{2}}$ is the density of Gauss measure, Ω is an open subset of $\mathbb{R}^n (n\geq 2)$ with Gauss measure less than one, a_{ij}, b_i, d_i, c and f are measurable functions on Ω that satisfy the following assumptions:

(A1) $a_{ij}/\varphi, c/\varphi\in L^\infty(\Omega)$, $f\in L^2(\varphi,\Omega)$;
(A2) $\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j\geq\varphi(x)|\xi|^2$, a.e. $x\in\Omega,\forall\xi\in\mathbb{R}^n$;
(A3) $\big(\sum_{i=1}^n|b_i(x)+d_i(x)|^2\big)^{1/2}\leq B\varphi(x)$, a.e. $x\in\Omega$, $B>0$;
(A4) $\sum_{i=1}^n D_ib_i(x)+c(x)\geq c_0(x)\varphi(x)$ in $\mathcal{D}'(\Omega)$, $c_0\in L^\infty(\Omega)$;

We obtain a priori estimates for the solutions of (1.1) using rearrangement techniques. As Ω is bounded, the operator is uniformly elliptic. This issue has been

---

2000 *Mathematics Subject Classification.* 35J86, 35J70, 35B45.
Key words and phrases. Comparison results; rearrangements; gauss measure;
Elliptic variational inequality.

Submitted November 19, 2014. Published Janaury 5, 2015.

studied by many authors, firstly by Weinberger [28] and Talenti [24]. Actually it is well known that, one can use Schwarz symmetrization to estimate the solutions of elliptic and parabolic equations in terms of the solutions of (one dimentional) radially symmetric problems (for a comprehensive bibliography see the paper [10] and [27]). For variational inequalities, similar comparison results in terms of linear elliptic variational inequalities can be found, for example, in [1, 3, 18, 19]; while nonlinear elliptic variational inequalities are discussed, for example, in [5, 20]. In the same case, comparison results for parabolic variational inequalities can be found in [11, 14].

In the elliptic variational inequalities (1.1), since Ω maybe unbounded, the degeneracy of the operator does not allow to use the classical approach via Schwarz symmetrization. Based on the structure of the problem, it is more appropriate to use the Gauss symmetrization as has been done for elliptic and parabolic equations in [7, 8, 12, 13]. Our aim is to compare the solutions of problem (1.1) with the symmetric solutions of a problem in which the data depend only on the first variable and the domain is a half-space, i.e. the following "symmetrized" problem

$$\begin{gathered} a^\sharp(v,\psi-v) \geq \int_{\Omega^\sharp} f^\sharp \varphi(\psi-v)dx, \quad \forall \psi \in H_0^1(\varphi,\Omega^\sharp),\ \psi\geq 0, \\ v\in H_0^1(\varphi,\Omega^\sharp), \quad v\geq 0, \end{gathered} \tag{1.2}$$

where

$$a^\sharp(v,\psi-v) = \int_{\Omega^\sharp} \varphi D_1 v D_1(\psi-v)\,dx - \int_{\Omega^\sharp} B\varphi D_1 v(\psi-v)\,dx + \int_{\Omega^\sharp} c_{0\sharp}\varphi v(\psi-v)\,dx,$$

where $\Omega^\sharp$ is a half space with the same Gauss measure as Ω, $f^\sharp$ is the Gauss symmetrization of f and $c_{0\sharp}$ is the decreasing Gauss symmetrization of c_0. To this end, by following arguments in [1, 19], we first discuss the existence of symmetric solutions to the "symmetrized" problem (1.2), which is a key step for the comparison results. However, in the equation case, the papers [12, 13] always assume that the "symmetrized" problem has a symmetric solution instead of studying the existence conditions for such solutions. Our results (Theorem 3.1) make up for that in large extent. In addition, as an application of the comparison results, we prove an estimates of the Lorentz-Zygmund norm of u in terms of the norm of the symmetric solutions v.

The main tools we use are Gauss symmetrization and the properties of the weighted rearrangement. It is worth noting that the method used in equation case for obtaining the comparison results can not be applied to the variational inequalities (1.1). In this paper, we combine the property of the first eigenvalue (Lemma 4.3) with the maximum principle to overcome the difficulties and get the desired results.

This article is organized as follows: Section 2 is devoted to give some notation and preliminary results; in Section 3, the main results of this paper are stated; in Section 4, we finish the proof of the main results.

2. Notation and preliminary results

In this section, we recall some definitions and results which will be useful in what follows. First, we recall that the wieghted Sobolev space $W_0^{1,p}(\varphi,\Omega)$ is the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{W^{1,p}(\varphi,\Omega)} = \Big(\int_\Omega |\nabla u(x)|^p \varphi\, dx + \int_\Omega |u(x)|^p \varphi\, dx \Big)^{\frac{1}{p}}.$$

When $p = 2$, the space $W_0^{1,2}(\varphi, \Omega)$ is also denoted by $H_0^1(\varphi, \Omega)$.

Let γ_n be the n-dimensional normalized Gauss measure on $\mathbb{R}^n$ defined as

$$d\gamma_n = \varphi(x)dx = (2\pi)^{-\frac{n}{2}} \exp\big(-\frac{|x|^2}{2}\big)\, dx, \quad x \in \mathbb{R}^n.$$

Set

$$\begin{aligned}\Phi(\tau) &= \gamma_n(\{x \in \mathbb{R}^n : x_1 > \tau\}) \\ &= (2\pi)^{-1/2} \int_\tau^{+\infty} \exp(-\frac{t^2}{2})dt, \quad \forall \tau \in \mathbb{R} \cup \{-\infty, +\infty\}.\end{aligned}$$

In [16] we observe that

$$\lim_{t\to 0^+,1^-} (2\pi)^{-1/2} \frac{\exp(-\frac{\Phi^{-1}(t)^2}{2})}{t(2\log\frac{1}{t})^{1/2}} = 1. \tag{2.1}$$

Remark 2.1 ([26]). By $\lim_{t\to 0^+} \frac{t(2\log\frac{1}{t})^{1/2}}{t(1-\log t)^{1/2}} = \sqrt{2}$ and $\lim_{t\to 1^-} \frac{t(2\log\frac{1}{t})^{1/2}}{t(1-\log t)^{1/2}} = 0$ and note that (2.1) and the fact $\gamma_n(\Omega) < 1$, we have

$$\exp\Big(-\frac{\Phi^{-1}(t)^2}{2}\Big) \le \alpha t(1-\log t)^{1/2}, \quad t \in (0, \gamma_n(\Omega)), \tag{2.2}$$

$$\exp\Big(-\frac{\Phi^{-1}(t)^2}{2}\Big) \ge \beta t(1-\log t)^{1/2}, \quad t \in (0, \gamma_n(\Omega)), \tag{2.3}$$

where α and β are two positive constants depending on $\gamma_n(\Omega)$.

Now we give the notion of rearrangement.

Definition 2.2. If u is a measurable function in Ω and $\mu(t) = \gamma_n(\{x \in \Omega : |u| > t\})$ is the distribution function of u, then we define the decreasing rearrangement of u with respect to Gauss measure as

$$u^\star(s) = \inf\{t \ge 0 : \mu(t) \le s\}, \quad s \in [0, \gamma_n(\Omega)].$$

Let $\Omega^\sharp = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1 > \lambda\}$ be the half-space such that $\gamma_n(\Omega) = \gamma_n(\Omega^\sharp)$. Then

$$u^\sharp(x) = u^\star(\Phi(x_1)), \quad x \in \Omega^\sharp$$

denote the increasing Gauss symmetrization of u (or Gauss symmetrization of u).

Similarly, the decreasing Gauss symmetrization of u will be

$$u_\sharp(x) = u_\star(\Phi(x_1)), \quad x \in \Omega^\sharp,$$

with

$$u_\star(s) = u^\star(\gamma_n(\Omega) - s), \quad s \in (0, \gamma_n(\Omega)).$$

Properties of rearrangement with respect to Gauss measure or a positive measure have been widely considered in [9, 22, 23, 25], for instance. Here we just recall the following:

Hardy-Little inequality:

$$\int_0^{\gamma_n(\Omega)} u_\star(s) v^\star(s) ds = \int_{\Omega^\sharp} u_\sharp(x) v^\sharp(x) d\gamma_n \le \int_\Omega |u(x)v(x)| d\gamma_n$$

$$\leq \int_{\Omega^\sharp} u^\sharp(x)v^\sharp(x)d\gamma_n = \int_0^{\gamma_n(\Omega)} u^\star(s)v^\star(s)ds,$$

where u and v are measurable functions.

Polya-Szëgo principle: Let $u \in W_0^{1,p}(\varphi,\Omega)$ with $1 < p < +\infty$. Then

$$\|\nabla u^\sharp\|_{L^p(\varphi,\Omega^\sharp)} \leq \|\nabla u\|_{L^p(\varphi,\Omega)},$$

and equality holds if and only if $\Omega = \Omega^\sharp$ and $|u| = u^\sharp$ modulo a rotation.

Now we recall the definition and main properties of the Lorentz-Zygmund space (see [6]).

Definition 2.3. For any measurable function u, $0 < q, p \leq +\infty$ and $-\infty < \alpha < +\infty$, set

$$\|u\|_{L^{p,q}(\log L)^\alpha(\varphi,\Omega)} = \begin{cases} \big[\int_0^{\gamma_n(\Omega)} \big(t^{\frac{1}{p}}(1-\log t)^\alpha u^\star(t)\big)^q \frac{dt}{t}\big]^{1/q} & \text{if } 0 < q < +\infty, \\ \sup_{t\in(0,\gamma_n(\Omega))} \big[t^{\frac{1}{p}}(1-\log t)^\alpha u^\star(t)\big] & \text{if } q = +\infty. \end{cases} \tag{2.4}$$

We say that u belongs to the Lorentz-Zygmund space $L^{p,q}(\log L)^\alpha(\varphi,\Omega)$ if

$$\|u\|_{L^{p,q}(\log L)^\alpha(\varphi,\Omega)} < +\infty.$$

Remark 2.4. It is clear that the space $L^{p,q}(\log L)^0(\varphi,\Omega)$ is just the Lorentz space $L^{p,q}(\varphi,\Omega)$. As $p = q$, the space $L^{p,p}(\log L)^0(\varphi,\Omega)$ is the Lebesgue space $L^p(\varphi,\Omega)$.

Remark 2.5. For $1 < p \leq +\infty$, $1 \leq q \leq +\infty$ and $-\infty < \alpha < +\infty$, (2.4) is a quasinorm. Replacing $u^\star(t)$ with

$$u^{\star\star}(t) = \frac{1}{t}\int_0^t u^\star(s)ds,$$

we obtain an equivalent norm.

Remark 2.6. If $0 < r < p \leq +\infty$, $0 < q, s \leq +\infty$ and $-\infty < \alpha, \beta < +\infty$, then

$$L^{p,q}(\log L)^\alpha(\varphi,\Omega) \subseteq L^{r,s}(\log L)^\beta(\varphi,\Omega)$$

and when the first exponents are the same,

$$L^{p,q}(\log L)^\alpha(\varphi,\Omega) \subseteq L^{p,s}(\log L)^\beta(\varphi,\Omega),$$

whenever either

$$q \leq s \text{ and } \alpha \geq \beta$$

or

$$q > s \text{ and } \alpha + \frac{1}{q} > \beta + \frac{1}{s}.$$

Remark 2.7. The space $L^{p,q}(\log L)^\alpha(\varphi,\Omega)$ is nontrivial if and only if one of the following conditions holds

$$p < +\infty,$$

$$p = +\infty \text{ and } \alpha + \frac{1}{q} < 0,$$

$$p = +\infty, q = +\infty \text{ and } \alpha = 0.$$

The following imbedding theorem in Lorentz-Zygmund space is a straight consequence of the Sobolev logarithmic inequalities. It has been proved in [13] by using the properties of rearrangement.

Proposition 2.8. *Let Ω be an open subset of $\mathbb{R}^n$ with $\gamma_n(\Omega) < 1$. If $f \in W_0^{1,p}(\varphi,\Omega)$ with $1 \le p < +\infty$, then $f \in L^p(\log L)^{1/2}(\varphi,\Omega)$ and*

$$\|f\|_{L^p(\log L)^{1/2}(\varphi,\Omega)} \le C\|\nabla f\|_{L^p(\varphi,\Omega)}.$$

The following Hardy inequalities are also needed in this article [6].

Proposition 2.9. *Suppose that $r > 0$, $1 \le q \le +\infty$ and $-\infty < \alpha < +\infty$. Let Ψ be a nonnegative measurable function on $(0,1)$. If $1 \le q < +\infty$, then*

$$\Big(\int_0^1 \Big(t^{-r}(1-\log t)^\alpha \int_0^t \Psi(s)ds\Big)^q \frac{dt}{t}\Big)^{1/q} \le C\Big(\int_0^1 \big(t^{1-r}(1-\log t)^\alpha \Psi(t)\big)^q \frac{dt}{t}\Big)^{1/q} \tag{2.5}$$

holds. Moreover if $q = +\infty$, then

$$\sup_{0<t<1} \Big(t^{-r}(1-\log t)^\alpha \int_0^t \Psi(s)ds\Big) \le C \sup_{0<t<1} \big(t^{1-r}(1-\log t)^\alpha \Psi(t)\big), \tag{2.6}$$

where the positive constant $C = C(r,q,\alpha)$ is independent of Ψ.

3. Statement of main results

The main results of this article are the following three theorems. First, the existence result of symmetric solutions to the "symmetrized" variational problem are presented, which is a key step for the comparison results. Let

$$c_0^+(x) = \max\{c_0(x), 0\}, \quad c_0^-(x) = \max\{-c_0(x), 0\},$$
$$c_{0\sharp}^+(x) = (c_0^+(x))_\sharp, \ c_0^{-\sharp}(x) = (c_0^-(x))^\sharp.$$

Theorem 3.1. *Set*

$$A^\sharp v = -D_1(\varphi D_1 v) - B\varphi D_1 v + c_{0\sharp}\varphi v \quad \text{in } \Omega^\sharp.$$

Suppose that: (1) $c_0(x) \ge 0$, or (2) $c_0^-(x) \not\equiv 0$. Let one of the following equivalent conditions be satisfied:

(a) *The first eigenvalue λ_1 of the problem*

$$A^\sharp \Psi = \lambda_1\Big(\frac{B|x_1|}{2} + c_0^{-\sharp}\Big)\varphi\Psi \quad \text{in } \Omega^\sharp, \qquad \Psi \in H_0^1(\varphi,\Omega^\sharp) \tag{3.1}$$

is positive.

(b) *There exists a constant $\xi > 0$ such that*

$$\begin{aligned} &\int_{\Omega^\sharp} \varphi|D_1 Z|^2 dx - \int_{\Omega^\sharp} B\varphi Z D_1 Z dx + \int_{\Omega^\sharp} c_{0\sharp}\varphi Z^2 dx \\ &\ge \xi \int_{\Omega^\sharp} \varphi|D_1 Z|^2 dx, \quad \forall Z \in H_0^1(\varphi,\Omega^\sharp). \end{aligned} \tag{3.2}$$

Then the maximum principle holds for $A^\sharp$; i.e.,

$$A^\sharp v \ge 0 \text{ in } \Omega^\sharp \text{ and } v \ge 0 \text{ on } \partial\Omega^\sharp \text{ imply } v \ge 0 \text{ in } \Omega^\sharp, \tag{3.3}$$

and problem (1.2) has unique symmetric solution.

Now, the comparison result between problems (1.1) and (1.2) can be stated in the following theorem.

Theorem 3.2. *Let* (A1)–(A4) *and one of the conditions in Theorem 3.1 hold. Suppose that u is the solution of problem* (1.1) *and $v = v^\sharp$ is the solution of problem* (1.2). *Then*

(1) *As $c_0(x) \le 0$, we have*
$$u^\star(s) \le v^\star(s), \quad s \in [0, \gamma_n(\Omega)]. \tag{3.4}$$
(2) *As $c_0^+(x) \not\equiv 0$, it follows that*
$$u^\star(s) \le v^\star(s), \;\; s \in [0, s_1], \tag{3.5}$$
$$\int_{s_1}^{s} e^{B\Phi^{-1}(\sigma)} u^\star(\sigma) d\sigma \le \int_{s_1}^{s} e^{B\Phi^{-1}(\sigma)} v^\star(\sigma) d\sigma, \quad s \in [s_1, \gamma_n(\Omega)], \tag{3.6}$$
where $s_1 = \inf\{s \in [0, \gamma_n(\Omega)] : c_{0\star}(s) > 0\}$.

Using the above comparison results, it is possible to obtain estimates of the solutions to problem (1.1) in terms of the solutions to "symmetrized" problem (1.2).

Theorem 3.3. *Suppose that Lorentz-Zygmund spaces $L^{p,q}(\log L)^\alpha(\varphi, \Omega)$ are non-trivial. Under the same assumptions of Theorem 3.2, we have*

(1) If $c_0(x) \le 0$, then
$$\|u\|_{L^{p,q}(\log L)^\alpha(\varphi,\Omega)} \le \|v\|_{L^{p,q}(\log L)^\alpha(\varphi,\Omega^\sharp)} \tag{3.7}$$
with $0 < p, \, q \le +\infty$ and $-\infty < \alpha < +\infty$.

(2) If $c_0(x) > 0$, then for all $0 < \varepsilon < \frac{1}{p}$,
$$\|u\|_{L^{p,q}(\log L)^\alpha(\varphi,\Omega)} \le C e^{[\frac{B^2}{2\varepsilon} - B\Phi^{-1}(\gamma_n(\Omega))]} \|v\|_{L^{\frac{p}{1-p\varepsilon},q}(\log L)^{\alpha-\frac{\varepsilon}{2}}(\varphi,\Omega^\sharp)}, \tag{3.8}$$
where C is a positive constant depending on p, q and $\gamma_n(\Omega)$, $1 < p < +\infty$, $1 \le q \le +\infty$ and $-\infty < \alpha < +\infty$.

(3) Otherwise,
$$\|u\|_{L^{p,q}(\log L)^\alpha(\varphi,\Omega)} \le e^{[B\Phi^{-1}(s_1) - B\Phi^{-1}(\gamma_n(\Omega))]} \|v\|_{L^{p,q}(\log L)^\alpha(\varphi,\Omega^\sharp)}, \tag{3.9}$$
with s_1 defined in Theorem 3.2, $1 < p \le +\infty$, $1 \le q \le +\infty$ and $-\infty < \alpha < +\infty$.

Remark 3.4. As Ω is bounded, by Schwarz symmetrization, it is possible to estimate $L^{p,q}(\log L)^\alpha$ norm of u by the same norm of v, the solutions to the "symmetrized" problems defined on balls with coefficients depending only on the radial (see Corollary 4.1 in [14]). However, in our case that Ω maybe unbounded, it is impossible to get the same result as before. In the above theorem, we estimate the $L^{p,q}(\log L)^\alpha$ norm of u in terms of a little stronger Lorentz-Zygmund norm of v.

4. Proof of main results

Proof of Theorem 3.1. First, we can see that if $|x_1| \in L^\infty(\log L)^{-1/2}(\varphi, \Omega^\sharp)$, every term in the weak form of (3.1) makes sense. In fact, for any $Y \in H_0^1(\varphi, \Omega^\sharp)$, using Hölder inequality and Hardy-Littlewood inequality, we obtain
$$\begin{aligned}
\int_{\Omega^\sharp} |x_1| \Psi Y \varphi \, dx &\le \Big(\int_{\Omega^\sharp} (|x_1|\Psi)^2 \varphi \, dx \Big)^{1/2} \Big(\int_{\Omega^\sharp} Y^2 \varphi \, dx \Big)^{1/2} \\
&\le \Big(\int_0^{\gamma_n(\Omega)} |x_1|^{\star 2}(s) |\Psi|^{\star 2}(s) ds \Big)^{1/2} \|Y\|_{L^2(\varphi,\Omega^\sharp)}
\end{aligned}$$

$$
\begin{aligned}
&\le \Big[\sup_{0<s<\gamma_n(\Omega)} \big(|x_1|^{\star 2}(s)(1-\log s)^{-1}\big) \Big]^{1/2} \\
&\quad \times \Big[\int_0^{\gamma_n(\Omega)} \Big(|\Psi|^\star(s)(1-\log s)^{1/2}\Big)^2 \, ds\Big]^{1/2} \|Y\|_{L^2(\varphi,\Omega^\sharp)} \\
&\le \| |x_1| \|_{L^\infty(\log L)^{-1/2}(\varphi,\Omega^\sharp)} \| |\Psi| \|_{L^2(\log L)^{1/2}(\varphi,\Omega^\sharp)} \|Y\|_{L^2(\varphi,\Omega^\sharp)}.
\end{aligned}
$$

By Proposition 2.8, $|\Psi| \in L^2(\log L)^{1/2}(\varphi,\Omega^\sharp)$. Thus, $\int_{\Omega^\sharp} |x_1|\Psi Y\varphi\, dx$ is finite. The remainder terms are similarly considered. Hence, it remains us to prove that $|x_1| \in L^\infty(\log L)^{-1/2}(\varphi,\Omega^\sharp)$.

If $\lambda \ge 0$, by $\Omega^\sharp = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1 > \lambda\}$, we have $x_1 > \lambda \ge 0$. Therefore,

$$
|x_1|^\star(s) = x_1^\star(s) = \Phi^{-1}(s). \tag{4.1}
$$

Moreover, from Remark 2.1 it follows that

$$
\begin{aligned}
\lim_{s\to 0^+} \frac{\Phi^{-1}(s)}{(1-\log s)^{1/2}} &= \lim_{s\to 0^+} \frac{-\sqrt{2\pi} e^{\frac{\Phi^{-1}(s)^2}{2}}}{\frac{1}{2}(1-\log s)^{-1/2}(-\frac{1}{s})} \\
&= \lim_{s\to 0^+} 2\sqrt{2\pi} \frac{s(1-\log s)^{1/2}}{e^{-\frac{\Phi^{-1}(s)^2}{2}}} \\
&= \lim_{s\to 0^+} 2\sqrt{2\pi}\, \frac{s(1-\log s)^{1/2}}{s(2\log\frac{1}{s})^{1/2}}\, \frac{s(2\log\frac{1}{s})^{1/2}}{e^{-\frac{\Phi^{-1}(s)^2}{2}}} \\
&= 2\sqrt{2\pi}\, \frac{1}{\sqrt{2}}\, \frac{1}{\sqrt{2\pi}} = \sqrt{2}.
\end{aligned}
$$

Then there exist $M_1 > 0$ and $\delta_0 \in (0, \gamma_n(\Omega))$ such that

$$
\Phi^{-1}(s)(1-\log s)^{-1/2} \le M_1, \quad s \in (0, \delta_0).
$$

Since $\Phi^{-1}(s)(1-\log s)^{-1/2}$ is continuous on $[\delta_0, \gamma_n(\Omega)]$, there exists a constant $M_2 > 0$ such that

$$
\Phi^{-1}(s)(1-\log s)^{-1/2} \le M_2, \quad s \in [\delta_0, \gamma_n(\Omega)).
$$

Thus,

$$
\Phi^{-1}(s)(1-\log s)^{-1/2} \le \max\{M_1, M_2\}, \quad s \in (0, \gamma_n(\Omega)).
$$

Recalling (4.1), we have

$$
\| |x_1| \|_{L^\infty(\log L)^{-1/2}(\varphi,\Omega^\sharp)} = \sup_{0<s<\gamma_n(\Omega)} \big[\Phi^{-1}(s)(1-\log s)^{-1/2}\big] \le \max\{M_1, M_2\}.
$$

If $\lambda < 0$, setting

$$
\begin{aligned}
\mu(t) &= \gamma_n(\{x \in \Omega^\sharp : |x_1| > t\}), \\
\nu(t) &= \gamma_n(\{x \in \Omega^\sharp : x_1 > t\}),
\end{aligned}
$$

we have

$$
\mu(t) = \begin{cases} \nu(t) & t \ge -\lambda, \\ 2\nu(t) - (1-\gamma_n(\Omega)) & 0 \le t < -\lambda, \\ \gamma_n(\Omega) & t < 0 \end{cases}
$$

and

$$
|x_1|^\star(s) = \inf\{t \ge 0 : \mu(t) \le s\}
$$

$$
\begin{aligned}
&= \begin{cases} \inf\{t \ge 0 : \nu(t) \le s\} & s \in [0, 1-\gamma_n(\Omega)], \\ \inf\{t \ge 0 : \nu(t) \le \frac{s+1-\gamma_n(\Omega)}{2}\} & s \in (1-\gamma_n(\Omega), \gamma_n(\Omega)] \end{cases} \\
&= \begin{cases} x_1^\star(s) & s \in [0, 1-\gamma_n(\Omega)], \\ x_1^\star\big(\frac{s+1-\gamma_n(\Omega)}{2}\big) & s \in \big(1-\gamma_n(\Omega), \gamma_n(\Omega)\big] \end{cases} \\
&= \begin{cases} \Phi^{-1}(s) & s \in [0, 1-\gamma_n(\Omega)], \\ \Phi^{-1}\big(\frac{s+1-\gamma_n(\Omega)}{2}\big) & s \in \big(1-\gamma_n(\Omega), \gamma_n(\Omega)\big]. \end{cases}
\end{aligned}
$$

Using the same method as in the case $\lambda > 0$, we obtain that there exists $M > 0$ such that

$$
|x_1|^\star(s)(1-\log s)^{-1/2} \le M,
$$

which implies $|x_1| \in L^\infty(\log L)^{-1/2}(\varphi, \Omega^\sharp)$.

Now we give the proof of the equivalence of (a) and (b).

(a)$\Rightarrow$ (b) The first eigenvalue of (3.1) can be characterized by the Rayleigh principle as

$$
\lambda_1 = \min_{Q \in H_0^1(\varphi, \Omega^\sharp), Q \neq 0} \frac{\int_{\Omega^\sharp} \varphi |D_1 Q|^2\,dx - B\int_{\Omega^\sharp} \varphi Q D_1 Q\,dx + \int_{\Omega^\sharp} c_{0\sharp}\varphi Q^2\,dx}{\int_{\Omega^\sharp}\Big(\frac{B|x_1|}{2} + c_0^{-\sharp}(x)\Big)\varphi Q^2\,dx}. \tag{4.2}
$$

In view of (4.2) and the fact that $c_{0\sharp} = c_{0\sharp}^+ - c_0^{-\sharp}$, integrating by parts, we have

$$
\begin{aligned}
&\int_{\Omega^\sharp} \varphi |D_1 Z|^2\,dx - B\int_{\Omega^\sharp} \varphi Z D_1 Z\,dx + \int_{\Omega^\sharp} c_{0\sharp}\varphi Z^2\,dx \\
&\ge (1-\xi)\lambda_1 \int_{\Omega^\sharp}\Big(\frac{B|x_1|}{2} + c_0^{-\sharp}\Big)\varphi Z^2\,dx + \xi\int_{\Omega^\sharp}\varphi|D_1 Z|^2\,dx \\
&\quad - \xi B\int_{\Omega^\sharp}\varphi Z D_1 Z\,dx + \xi\int_{\Omega^\sharp}(c_{0\sharp}^+ - c_0^{-\sharp})\varphi Z^2 \\
&= (1-\xi)\lambda_1 \int_{\Omega^\sharp}\big(\frac{B|x_1|}{2} + c_0^{-\sharp}\big)\varphi Z^2\,dx - \xi\int_{\Omega^\sharp}\frac{B}{2}x_1\varphi Z^2\,dx \\
&\quad + \xi\int_{\Omega^\sharp}(c_{0\sharp}^+ - c_0^{-\sharp})\varphi Z^2\,dx + \xi\int_{\Omega^\sharp}\varphi|D_1 Z|^2\,dx \\
&\ge ((1-\xi)\lambda_1 - \xi)\int_{\Omega^\sharp}\big(\frac{B|x_1|}{2} + c_0^{-\sharp}\big)\varphi Z^2\,dx + \xi\int_{\Omega^\sharp}\varphi|D_1 Z|^2\,dx,
\end{aligned}
$$

for all $Z \in H_0^1(\varphi, \Omega^\sharp)$. Then we can choose $0 < \xi < \frac{\lambda_1}{\lambda_1+1}$ such that (3.2) holds.

(b) $\Rightarrow$ (a) Assume that Ψ is the eigenfunction corresponding to λ_1. Then

$$
\lambda_1 = \frac{\int_{\Omega^\sharp}|D_1\Psi|^2\varphi\,dx - B\int_{\Omega^\sharp}\varphi\Psi D_1\Psi\,dx + \int_{\Omega^\sharp} c_{0\sharp}\Psi^2\varphi\,dx}{\int_{\Omega^\sharp}\big(\frac{B|x_1|}{2} + c_0^{-\sharp}(x)\big)\Psi^2\varphi\,dx}.
$$

Recalling (b) and Proposition 2.8 and noting that $\Psi \not\equiv 0$, we have

$$
\lambda_1 \ge \frac{\xi\int_{\Omega^\sharp}|D_1\Psi|^2\varphi\,dx}{\int_{\Omega^\sharp}\big(\frac{B|x_1|}{2} + c_0^{-\sharp}\big)\Psi^2\varphi\,dx} \ge \frac{C\xi\int_{\Omega^\sharp}|\Psi|^2\varphi\,dx}{\int_{\Omega^\sharp}\big(\frac{B|x_1|}{2} + c_0^{-\sharp}\big)\Psi^2\varphi\,dx} > 0,
$$

where C is a positive constant depending on $\gamma_n(\Omega)$. Thus (a) holds.

Now, we prove that the maximum principle holds for $A^\sharp$. As $c_0(x) \ge 0$, it is obvious. As $c_0^-(x) \not\equiv 0$, taking v^- as a test function of (3.3) and using (b), we obtain

$$0 \le \xi \int_{\Omega^\sharp} \varphi |D_1 v^-|^2 dx \le \int_{\Omega^\sharp} v^- A^\sharp v^- dx \le 0,$$

which implies $v^- = 0$. Thus $v \ge 0$ on $\partial\Omega^\sharp$.

On the other hand, from [17] we see that (1.2) has unique solution v. Thus the solution depends only on the first variable. To prove $v = v^\sharp$, for the sake of simplicity, we suppose that v is sufficiently smooth. Set $E = \{x \in \Omega^\sharp : v > 0\}$. Then v satisfies

$$A^\sharp v = f^\sharp \quad \text{on } E,$$

which gives

$$-\varphi D_{11} v + (x_1 - B)\varphi D_1 v + c_{0\sharp}\varphi v = f^\sharp \varphi \quad \text{on } E. \tag{4.3}$$

Differentiating (4.3) with respect to x_1 and taking $K = D_1 v$, we obtain

$$\begin{aligned}(A^\sharp + \varphi)K &= -\varphi D_{11} K + (x_1 - B)\varphi D_1 K + c_{0\sharp}\varphi K + \varphi K \\ &= \varphi D_1 f^\sharp - v\varphi D_1 c_{0\sharp} \ge 0 \quad \text{on } E.\end{aligned}$$

Observing $E = \{x \in \Omega^\sharp : x_1 > \xi_0\}$, where $\xi_0 = \Phi^{-1}(\gamma_n(\{x \in \Omega^\sharp : v > 0\}))$, it follows $v(\xi_0) = 0$. Thus $K \ge 0$ on ∂E.

Since $A^\sharp + \varphi$ satisfies the property (b), we obtain $K = D_1 v \ge 0$ on E by applying the maximum principle to the operator $A^\sharp + \varphi$. Thus $v = v^\sharp$ on $\Omega^\sharp$. The proof is complete. □

Before proving Theorem 3.2, we need the following lemmas.

Lemma 4.1. *Assume that* (A1)–(A4) *hold. Let u be the solution to problem* (1.1) *and $v = v^\sharp$ be the solution to problem* (1.2). *Then*

$$-u^{\star\prime}(s) \le 2\pi e^{[\Phi^{-1}(s)^2 - B\Phi^{-1}(s)]} \int_0^s e^{B\Phi^{-1}(\sigma)} [f^\star(\sigma) - c_{0\star}(\sigma) u^\star(\sigma)] d\sigma, \tag{4.4}$$

for $s \in [0, \gamma_n(\{u > 0\})]$, and

$$-v^{\star\prime}(s) = 2\pi e^{[\Phi^{-1}(s)^2 - B\Phi^{-1}(s)]} \int_0^s e^{B\Phi^{-1}(\sigma)} [f^\star(\sigma) - c_{0\star}(\sigma) v^\star(\sigma)] d\sigma, \tag{4.5}$$

for $s \in [0, \gamma_n(\{v > 0\})]$.

The proof of the above lemma follows the same lines as in [12, 13]; we omit it.

Lemma 4.2. *Assume that* (A1)–(A4) *hold. Let u and $v = v^\sharp$ be the solutions to problem* (1.1) *and* (1.2) *respectively. If $c_{0\star}$ is continuous at s_1, then $w = u^\star - v^\star$ satisfies*

$$-w'(s) \le -2\pi e^{\left[\Phi^{-1}(s)^2 - B\Phi^{-1}(s)\right]} \int_0^s e^{B\Phi^{-1}(\sigma)} c_{0\star}(\sigma) w(\sigma) d\sigma, \tag{4.6}$$

for $s \in [0, \gamma_n(\{u > 0\})]$.

Proof. As $\gamma_n(\{u > 0\}) \le \gamma_n(\{v > 0\})$, it is obvious that (4.6) holds. As $\gamma_n(\{u > 0\}) > \gamma_n(\{v > 0\})$, (4.6) holds on $[0, \gamma_n(\{v > 0\})]$. Moreover, using the regularity theory (see [15]), v belongs to $H^2(\varphi, \Omega^\sharp)$ and then $v^\star \in \mathcal{C}^1(0, \gamma_n(\Omega)]$. Thus $v^{\star\prime}(\gamma_n(\{v > 0\})) = 0$. It follows from (4.5) that

$$\int_0^{\gamma_n(\{v>0\})} e^{B\Phi^{-1}(\sigma)} [f^\star(\sigma) - c_{0\star}(\sigma) v^\star(\sigma)] d\sigma = 0. \tag{4.7}$$

Noting that $w(s) = u^\star(s)$ on $[\gamma_n(\{v > 0\}), \gamma_n(\{u > 0\})]$, we combine (4.7) and (4.4) to discover that

$$\begin{aligned} -w'(s) &\le 2\pi e^{[\Phi^{-1}(s)^2 - B\Phi^{-1}(s)]}\Big\{ \int_0^s e^{B\Phi^{-1}(\sigma)}[f^\star(\sigma) - c_{0\star}(\sigma) w(\sigma)] d\sigma \\ &\quad - \int_0^{\gamma_n(\{v>0\})} e^{B\Phi^{-1}(\sigma)} c_{0\star}(\sigma) v^\star(\sigma) d\sigma \Big\} \\ &= 2\pi e^{[\Phi^{-1}(s)^2 - B\Phi^{-1}(s)]}\Big\{ - \int_0^s e^{B\Phi^{-1}(\sigma)} c_{0\star}(\sigma) w(\sigma) d\sigma \\ &\quad + \int_{\gamma_n(\{v>0\})}^s e^{B\Phi^{-1}(\sigma)} f^\star(\sigma) d\sigma \Big\}, \quad s \in [\gamma_n(\{v > 0\}), \gamma_n(\{u > 0\})]. \end{aligned}$$

If we show

$$f^\star(s) < 0 \quad \text{on } [\gamma_n(\{v > 0\}), \gamma_n(\Omega)], \tag{4.8}$$

then (4.6) is proved.

Now it remains to prove (4.8). In fact, as $\gamma_n(\{v > 0\}) \le s_1$, (4.7) yields

$$\int_0^{\gamma_n(\{v>0\})} e^{B\Phi^{-1}(\sigma)} f^\star(\sigma) = \int_0^{\gamma_n(\{v>0\})} e^{B\Phi^{-1}(\sigma)} c_{0\star}(\sigma) v^\star(\sigma) d\sigma \le 0.$$

Therefore $f^\star$ can not be nonnegative on $[0, \gamma_n(\{v > 0\})]$.

As $\gamma_n(\{v > 0\}) > s_1$, taking $V(s) = \int_{s_1}^s e^{B\Phi^{-1}(\sigma)} c_{0\star}(\sigma) v^\star(\sigma) d\sigma$, from (4.5) we obtain

$$\begin{aligned} &- \Big(e^{-B\Phi^{-1}(s)}(c_{0\star}(s))^{-1} V'(s)\Big)' + 2\pi e^{[\Phi^{-1}(s)^2 - B\Phi^{-1}(s)]} V \\ &= 2\pi e^{[\Phi^{-1}(s)^2 - B\Phi^{-1}(s)]}\Big[\int_0^s e^{B\Phi^{-1}(\sigma)} f^\star(\sigma) d\sigma - \int_0^{s_1} e^{B\Phi^{-1}(\sigma)} c_{0\star}(\sigma) v^\star(\sigma) d\sigma \Big] \\ &\ge 2\pi e^{[\Phi^{-1}(s)^2 - B\Phi^{-1}(s)]} \int_0^s e^{B\Phi^{-1}(\sigma)} f^\star(\sigma) d\sigma, \quad s \in (s_1, \gamma_n(\{v > 0\})). \end{aligned}$$

If $f^\star \ge 0$ on $[0, \gamma_n(\{v > 0\})]$, observing $c_{0\star}(s_1) = 0$, V satisfies

$$\begin{gathered} -\Big(e^{-B\Phi^{-1}(s)}(c_{0\star}(s))^{-1} V'(s)\Big)' + 2\pi e^{[\Phi^{-1}(s)^2 - B\Phi^{-1}(s)]} V \ge 0 \quad \text{on } (s_1, \gamma_n(\{v > 0\})), \\ V(s_1) = 0, \quad V'(s_1) = 0. \end{gathered} \tag{4.9}$$

By the maximum principle (see [21]), we obtain $V \le 0$ in $(s_1, \gamma_n(\{v > 0\}))$ which contradict with the fact that $V > 0$ in $(s_1, \gamma_n(\{v > 0\}))$. Thus $f^\star$ can not be nonnegative in $[0, \gamma_n(\{v > 0\})]$ and we obtain the desired result. □

Lemma 4.3 (P.163 in [4]). *Let*

$$L = - \sum_{i,j=1}^n D_i(a_{ij}(x) D_j) + c(x).$$

Consider the eigenvalue problem

$$\begin{gathered} L\Psi = \lambda_1 P\Psi \quad \text{in } D, \\ \Psi = 0 \quad \text{on } \Gamma_0, \\ \frac{\partial \Psi}{\partial \nu} + \eta\Psi = 0 \quad \text{on } \Gamma_1, \end{gathered} \tag{4.10}$$

where $D \subseteq \mathbb{R}^n$, Γ_0 is a subset of ∂D, $\Gamma_1 = \partial D - \Gamma_0$ and P is a positive function in D. If λ_1 is the first eigenvalue of (4.10), then

$$\lambda_1 \le \sup_{x\in D} \frac{Lh}{Ph},$$

where h is any positive function in D satisfying the same boundary conditions as Ψ.

$$\lambda_1 \ge \inf_{x\in D} \frac{Lh}{Ph},$$

where h is positive in D, $h \ge 0$ on Γ_0 and $\frac{\partial h}{\partial \nu} + \eta h \ge 0$ on Γ_1.

Lemma 4.4 ([2])**.** *Let f, g be measurable positive functions such that*

$$\int_0^r f(\sigma)d\sigma \le \int_0^r g(\sigma)d\sigma,\ r \in [0,\rho].$$

If $h \ge 0$ is a decreasing function in $[0,\rho]$, then

$$\int_0^r f(\sigma)h(\sigma)d\sigma \le \int_0^r g(\sigma)h(\sigma)d\sigma, \quad r \in [0,\rho].$$

Proof of Theorem 3.2. Suppose that $c_0(x)$ is smooth in Ω.

(1) As $c_0(x) \le 0$, we assume $c_0(x) < 0$. In this case, $c_0^{-\star} = -c_{0\star}$. Set $W(s) = \int_0^s e^{B\Phi^{-1}(\sigma)} c_0^{-\star}(\sigma)w(\sigma)d\sigma$. By (4.6), we have

$$\begin{gathered} -\big[e^{-B\Phi^{-1}(s)}(c_0^{-\star}(s))^{-1}W'\big]' \le 2\pi e^{\left[\Phi^{-1}(s)^2 - B\Phi^{-1}(s)\right]} W \quad \text{in } (0,\gamma_n(\{u>0\})), \\ W(0)=0, \quad W'(\gamma_n(\{u>0\})) \le 0. \end{gathered} \tag{4.11}$$

Thus, $w \le 0$ on $[0,\gamma_n(\{u>0\})]$. In fact, let E be a half space whose Gauss measure is $\gamma_n(\{u>0\})$. Consider the eigenvalue problem

$$\begin{gathered} A^\sharp \Psi = \widetilde{\lambda} c_0^{-\sharp} \varphi \Psi \quad \text{in } E, \\ \Psi \in H_0^1(\varphi, E). \end{gathered} \tag{4.12}$$

The first eigenvalue of (4.12) can be characterized as

$$\widetilde{\lambda} = \min_{Q\in H_0^1(\varphi,E), Q\ne 0} \frac{\int_E \varphi |D_1 Q|^2\,dx - B\int_E \varphi Q D_1 Q\,dx + \int_E c_{0\sharp}\varphi Q^2\,dx}{\int_E c_0^{-\sharp}\varphi Q^2\,dx}. \tag{4.13}$$

By (b) of Theorem 3.1, we have

$$\begin{aligned} &\int_E \varphi |D_1 Q|^2\,dx - B\int_E \varphi Q D_1 Q\,dx + \int_E c_{0\sharp}\varphi Q^2\,dx \\ &\ge \beta \int_E \varphi |D_1 Q|^2\,dx > 0, \quad Q \in H_0^1(\varphi,E) \text{ and } Q \not\equiv 0 \text{ in } E \end{aligned}$$

Hence, $\widetilde{\lambda} > 0$.

Now if Ψ is an eigenfunction corresponding to $\widetilde{\lambda}$, then both Ψ and $|\Psi|$ minimize (4.13). $|\Psi|$ is also an eigenfunction. Then we can take $\Psi \ge 0$. Moreover, Ψ satisfies

$$\widetilde{\lambda} = \frac{\int_E \varphi |D_1 \Psi|^2\,dx - B\int_E \varphi \Psi D_1 \Psi\,dx + \int_E c_{0\sharp}\varphi \Psi^2\,dx}{\int_E c_0^{-\sharp}\varphi \Psi^2\,dx}. \tag{4.14}$$

Integrating by parts and using Hardy-Littlewood inequality and Polya-Szëgo principle, we obtain

$$\begin{aligned} \widetilde{\lambda} &= \frac{\int_E |D_1\Psi|^2\varphi\,dx - B\int_E \frac{x_1}{2}\Psi^2\varphi\,dx + \int_E c_{0\sharp}\Psi^2\varphi\,dx}{\int_E c_0^{-\sharp}\Psi^2\varphi\,dx} \\ &\geq \frac{\int_E |D_1\Psi^\sharp|^2\varphi\,dx - B\int_E (\frac{x_1}{2})^\sharp\Psi^{\sharp 2}\varphi\,dx + \int_E c_{0\sharp}\Psi^{\sharp 2}\varphi\,dx}{\int_E c_0^{-\sharp}\Psi^{\sharp 2}\varphi\,dx}. \end{aligned} \tag{4.15}$$

Noting that $x_1^\sharp = x_1$, we conclude from (4.15) that

$$\widetilde{\lambda} \geq \frac{\int_E |D_1\Psi^\sharp|^2\varphi\,dx - B\int_E \frac{x_1}{2}\Psi^{\sharp 2}\varphi\,dx + \int_E c_{0\sharp}\Psi^{\sharp 2}\varphi\,dx}{\int_E c_0^{-\sharp}\Psi^{\sharp 2}\varphi\,dx}.$$

Thus, the above inequality, (4.13) and (4.14) imply $\Psi = \Psi^\sharp$. In addition,

$$-\Psi^{\star'}(s) = 2\pi(\widetilde{\lambda}+1)e^{[\Phi^{-1}(s)^2 - B\Phi^{-1}(s)]}\int_0^s e^{B\Phi^{-1}(\sigma)}c_0^{-\star}(\sigma)\Psi^\star(\sigma)d\sigma, \tag{4.16}$$

for $s \in [0, \gamma_n(\{u>0\})]$. Let $\Theta(s) = \int_0^s e^{B\Phi^{-1}(\sigma)}c_0^{-\star}(\sigma)\Psi^\star(\sigma)d\sigma$. Then (4.16) can be written as

$$\begin{gathered} -\big[e^{-B\Phi^{-1}(s)}(c_0^{-\star}(s))^{-1}\Theta'\big]' = 2\pi(\widetilde{\lambda}+1)e^{\left[\Phi^{-1}(s)^2 - B\Phi^{-1}(s)\right]}\Theta \quad \text{in } (0, \gamma_n(\{u>0\})), \\ \Theta(0) = 0, \quad \Theta'(\gamma_n(\{u>0\})) = 0. \end{gathered} \tag{4.17}$$

If λ is the first eigenvalue of the problem

$$\begin{gathered} -\big[e^{-B\Phi^{-1}(s)}(c_0^{-\star}(s))^{-1}U'\big]' = \lambda 2\pi e^{\left[\Phi^{-1}(s)^2 - B\Phi^{-1}(s)\right]}U \quad \text{in } (0, \gamma_n(\{u>0\})), \\ U(0) = 0, \quad U'(\gamma_n(\{u>0\})) = 0, \end{gathered} \tag{4.18}$$

it follows from Lemma 4.3 that

$$\lambda \geq \inf_{s\in(0,\gamma_n(\{u>0\}))} \frac{-[e^{-B\Phi^{-1}(s)}(c_0^{-\star}(s))^{-1}\Theta']'}{2\pi e^{[\Phi^{-1}(s)^2 - B\Phi^{-1}(s)]}\Theta} = \widetilde{\lambda}+1.$$

Therefore, $\lambda \geq \widetilde{\lambda}+1 > 1$. By [4, Lemma 4.7], we have $W' \leq 0$ on $[0, \gamma_n(\{u>0\})]$. That is $w \leq 0$ on $[0, \gamma_n(\{u>0\})]$. The result in the case $c(x) \leq 0$ can be proved by approximation techniques (see [2]).

(2) As $c_0^+(x) \not\equiv 0$, we let

$$s_2 = \inf\{s \in [0, \gamma_n(\Omega)] : c_{0\star}(s) \geq 0\}.$$

Then $s_2 \leq s_1$ and $c_{0\star}(s_2) = 0$. Moreover, noting that $c_{0\star}(s) < 0$ on $(0, s_2)$, we obtain $c_0^{-\star}(s) = -c_{0\star}(s)$ on $(0, s_2)$. Take

$$W_1(s) = \int_0^s e^{B\Phi^{-1}(\sigma)}c_0^{-\star}(\sigma)w(\sigma)d\sigma.$$

By (4.6), we obtain

$$\begin{gathered} -\big[e^{-B\Phi^{-1}(s)}(c_0^{-\star}(s))^{-1}W_1'(s)\big]' \leq 2\pi e^{[\Phi^{-1}(s)^2 - B\Phi^{-1}(s)]}W_1(s) \quad \text{in } (0, s_2), \\ W_1(0) = 0, \quad W_1'(s_2) = 0. \end{gathered}$$

Proceeding as in case (1), it follows that

$$u^\star(s) \leq v^\star(s), \quad s \in [0, s_2]. \tag{4.19}$$

On the other hand, letting

$$W_2(s)=\int_{s_1}^{s} e^{B\Phi^{-1}(\sigma)}c_{0\star}(\sigma)w(\sigma)d\sigma,$$

Inequalities (4.6) and (4.19) yield

$$\begin{gathered}-\big[e^{-B\Phi^{-1}(s)}(c_{0\star}(s))^{-1}W_2'(s)\big]'\le -2\pi e^{[\Phi^{-1}(s)^2-B\Phi^{-1}(s)]}W_2(s)\\ \text{in } (s_1,\gamma_n(\{u>0\})],\\ W_2(s_1)=0,\quad W_2'(\gamma_n(\{u>0\}))\le 0.\end{gathered}$$

By the maximum principle, we have $W_2(s)\le 0$ on $[s_1,\gamma_n(\{u>0\})]$. That is,

$$\int_{s_1}^{s} e^{B\Phi^{-1}(\sigma)}c_{0\star}(\sigma)u^\star(\sigma)d\sigma\le \int_{s_1}^{s} e^{B\Phi^{-1}(\sigma)}c_{0\star}(\sigma)v^\star(\sigma)d\sigma,\quad s\in[s_1,\gamma_n(\{u>0\})].$$

According to Lemma 4.4, we obtain

$$\int_{s_1}^{s} e^{B\Phi^{-1}(\sigma)}u^\star(\sigma)d\sigma\le \int_{s_1}^{s} e^{B\Phi^{-1}(\sigma)}v^\star(\sigma)d\sigma,\quad s\in[s_1,\gamma_n(\{u>0\})],$$

which implies $u^\star(s_1)\le v^\star(s_1)$. Finally applying (4.19) to (4.6), it follows

$$-u^{\star'}(s)\le -v^{\star'}(s),\quad s\in[s_2,s_1] \tag{4.20}$$

Integrating (4.20) from s to s_1, we obtain

$$u^\star(s)\le v^\star(s)\quad \text{in } [s_2,s_1],$$

which completes the proof. □

At last, we can remove the smooth assumption on $c_0(x)$ by approximations.

Remark 4.5. For the variational problem, since $f^\star$ maybe negative in $[0,\gamma_n(\{u>0\})]$, the method in the equation case are failed to obtain (3.4)–(3.6). Here, we use the properties of the first eigenvalue (Lemma 4.3) and maximal principle to obtain the desired results.

Proof of Theorem 3.3. (1) If $c_0(x)\le 0$, inequality (3.7) follows from (3.4).

(2) If $c_0(x)>0$, we have

$$\int_0^{s} e^{B\Phi^{-1}(\sigma)}u^\star(\sigma)d\sigma\le \int_0^{s} e^{B\Phi^{-1}(\sigma)}v^\star(\sigma)d\sigma,\quad s\in[0,\gamma_n(\Omega)], \tag{4.21}$$

Set

$$\|u\|^\star_{L^{p,q}(\log L)^\alpha(\varphi,\Omega)}=\begin{cases}\big[\int_0^{\gamma_n(\Omega)}\big(t^{\frac1p}(1-\log t)^\alpha u^{\star\star}(t)\big)^q\frac{dt}{t}\big]^{1/q} & \text{if } 0<q<+\infty,\\ \sup_{t\in(0,\gamma_n(\Omega))}\big[t^{\frac1p}(1-\log t)^\alpha u^{\star\star}(t)\big] & \text{if } q=+\infty.\end{cases}$$

Remark 2.5 implies that the quasinorm $\|\cdot\|_{L^{p,q}(\log L)^\alpha(\varphi,\Omega)}$ is equivalent to the norm $\|\cdot\|^\star_{L^{p,q}(\log L)^\alpha(\varphi,\Omega)}$ when $p>1$ and $q\ge 1$.

As $q<+\infty$, using (4.21), (2.3) and (2.5), we obtain

$$\begin{aligned}&\|u\|^{\star q}_{L^{p,q}(\log L)^\alpha(\varphi,\Omega)}\\ &=\int_0^{\gamma_n(\Omega)}\Big(t^{\frac1p}(1-\log t)^\alpha\frac1t\int_0^t u^\star(\sigma)d\sigma\Big)^q\frac{dt}{t}\\ &\le \int_0^{\gamma_n(\Omega)}\Big(t^{\frac1p}(1-\log t)^\alpha e^{-B\Phi^{-1}(t)}\frac1t\int_0^t e^{B\Phi^{-1}(\sigma)}u^\star(\sigma)d\sigma\Big)^q\frac{dt}{t}\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^{\gamma_n(\Omega)} \Big(t^{\frac{1}{p}}(1-\log t)^{\alpha} e^{-B\Phi^{-1}(t)}\frac{1}{t}\int_0^t e^{B\Phi^{-1}(\sigma)} v^\star(\sigma)d\sigma\Big)^q \frac{dt}{t}\\
&= \int_0^{\gamma_n(\Omega)} \Big(t^{\frac{1}{p}}(1-\log t)^{\alpha} e^{-B\Phi^{-1}(t)}\\
&\quad\times \frac{1}{t}\int_0^t e^{[-(\sqrt{\frac{\varepsilon}{2}}\Phi^{-1}(\sigma)-\frac{B}{\sqrt{2\varepsilon}})^2+\frac{B^2}{2\varepsilon}]} e^{\frac{\varepsilon\Phi^{-1}(\sigma)^2}{2}} v^\star(\sigma)d\sigma\Big)^q \frac{dt}{t}\\
&\le e^{[q\frac{B^2}{2\varepsilon}-Bq\Phi^{-1}(\gamma_n(\Omega))]}\int_0^{\gamma_n(\Omega)} \Big(t^{\frac{1}{p}}(1-\log t)^{\alpha}\frac{1}{t}\int_0^t e^{\frac{\varepsilon\Phi^{-1}(\sigma)^2}{2}} v^\star(\sigma)d\sigma\Big)^q \frac{dt}{t}\\
&\le Ce^{[q\frac{B^2}{2\varepsilon}-Bq\Phi^{-1}(\gamma_n(\Omega))]}\int_0^{\gamma_n(\Omega)} \Big(t^{\frac{1}{p}}(1-\log t)^{\alpha}\frac{1}{t}\int_0^t \frac{1}{\sigma^{\varepsilon}(1-\log\sigma)^{\frac{\varepsilon}{2}}} v^\star(\sigma)d\sigma\Big)^q \frac{dt}{t}\\
&\le Ce^{[q\frac{B^2}{2\varepsilon}-Bq\Phi^{-1}(\gamma_n(\Omega))]}\int_0^{\gamma_n(\Omega)} \Big(t^{\frac{1}{p}-\varepsilon}(1-\log t)^{\alpha-\frac{\varepsilon}{2}} v^\star(t)\Big)^q \frac{dt}{t}\\
&= Ce^{[q\frac{B^2}{2\varepsilon}-Bq\Phi^{-1}(\gamma_n(\Omega))]}\|v\|^q_{L^{\frac{p}{1-p\varepsilon},q}(\log L)^{\alpha-\frac{\varepsilon}{2}}(\varphi,\Omega^\sharp)},
\end{aligned}
$$

where C is a positive constant depending on p, q and $\gamma_n(\Omega)$.

As $q=\infty$, (3.8) can be obtained by the same method as before with (2.5) replaced by (2.6).

(3) It follows from Theorem 3.2 that

$$
\begin{gathered}
u^\star(s)\le v^\star(s),\quad s\in[0,s_1],\\
\int_{s_1}^s e^{B\Phi^{-1}(\sigma)}u^\star(\sigma)d\sigma \le \int_{s_1}^s e^{B\Phi^{-1}(\sigma)}v^\star(\sigma)d\sigma,\quad s\in[s_1,\gamma_n(\Omega)],
\end{gathered}
$$

where $0<s_1<\gamma_n(\Omega)$.

If $q<+\infty$,

$$
\begin{aligned}
\|u\|^{\star q}_{L^{p,q}(\log L)^{\alpha}(\varphi,\Omega)} &= \int_0^{\gamma_n(\Omega)} \Big(t^{\frac{1}{p}}(1-\log t)^{\alpha}\frac{1}{t}\int_0^t u^\star(\sigma)d\sigma\Big)^q \frac{dt}{t}\\
&= \int_0^{s_1} \Big(t^{\frac{1}{p}}(1-\log t)^{\alpha}\frac{1}{t}\int_0^t u^\star(\sigma)d\sigma\Big)^q \frac{dt}{t}\\
&\quad + \int_{s_1}^{\gamma_n(\Omega)} \Big(t^{\frac{1}{p}}(1-\log t)^{\alpha}\frac{1}{t}\int_0^t u^\star(\sigma)d\sigma\Big)^q \frac{dt}{t}\\
&\le \int_0^{s_1} \Big(t^{\frac{1}{p}}(1-\log t)^{\alpha}\frac{1}{t}\int_0^t v^\star(\sigma)d\sigma\Big)^q \frac{dt}{t}\\
&\quad + \int_{s_1}^{\gamma_n(\Omega)} \Big(t^{\frac{1}{p}}(1-\log t)^{\alpha}\frac{1}{t}\int_0^t u^\star(\sigma)d\sigma\Big)^q \frac{dt}{t}
\end{aligned}
$$

Denote by I_2 the second term on the right-hand side of the above inequality. Then

$$
\begin{aligned}
I_2 &= \int_{s_1}^{\gamma_n(\Omega)} \Big[t^{\frac{1}{p}}(1-\log t)^{\alpha}\frac{1}{t}\Big(\int_0^{s_1} u^\star(\sigma)d\sigma+\int_{s_1}^t u^\star(\sigma)d\sigma\Big)\Big]^q \frac{dt}{t}\\
&\le \int_{s_1}^{\gamma_n(\Omega)} \Big[t^{\frac{1}{p}}(1-\log t)^{\alpha}\frac{1}{t}\Big(\int_0^{s_1} v^\star(\sigma)d\sigma+e^{-B\Phi^{-1}(t)}\int_{s_1}^t e^{B\Phi^{-1}(\sigma)}v^\star(\sigma)d\sigma\Big)\Big]^q \frac{dt}{t}\\
&\le \int_{s_1}^{\gamma_n(\Omega)} \Big[t^{\frac{1}{p}}(1-\log t)^{\alpha}\frac{1}{t}\Big(\int_0^{s_1} v^\star(\sigma)d\sigma+e^{[B\Phi^{-1}(s_1)-B\Phi^{-1}(t)]}\int_{s_1}^t v^\star(\sigma)d\sigma\Big)\Big]^q \frac{dt}{t}
\end{aligned}
$$

$$\le e^{[qB\Phi^{-1}(s_1)-qB\Phi^{-1}(\gamma_n(\Omega))]}\int_{s_1}^{\gamma_n(\Omega)}\Big(t^{\frac1p}(1-\log t)^\alpha\frac1t\int_0^t v^\star(\sigma)d\sigma\Big)^q\frac{dt}{t}.$$

Hence,

$$\|u\|^{\star q}_{L^{p,q}(\log L)^\alpha(\varphi,\Omega)}\le e^{[qB\Phi^{-1}(s_1)-qB\Phi^{-1}(\gamma_n(\Omega))]}\|v\|^{\star q}_{L^{p,q}(\log L)^\alpha(\varphi,\Omega)}.$$

If $q=+\infty$,

$$\begin{aligned}
&\|u\|^{\star}_{L^{p,\infty}(\log L)^\alpha(\varphi,\Omega)}\\
&=\sup_{t\in(0,\gamma_n(\Omega))}\big[t^{\frac1p}(1-\log t)^\alpha u^{\star\star}(t)\big]\\
&=\max\Big\{\sup_{t\in(0,s_1)}[t^{\frac1p}(1-\log t)^\alpha u^{\star\star}(t)],\sup_{t\in(s_1,\gamma_n(\Omega))}[t^{\frac1p}(1-\log t)^\alpha u^{\star\star}(t)]\Big\}\\
&\le\max\Big\{\sup_{t\in(0,s_1)}[t^{\frac1p}(1-\log t)^\alpha v^{\star\star}(t)],\sup_{t\in(s_1,\gamma_n(\Omega))}[t^{\frac1p}(1-\log t)^\alpha\frac1t\int_0^t u^\star(\sigma)d\sigma]\Big\}.
\end{aligned}$$

By the same method as the case $q<+\infty$, we have

$$\begin{aligned}
&\sup_{t\in(s_1,\gamma_n(\Omega))}\Big[t^{\frac1p}(1-\log t)^\alpha\frac1t\int_0^t u^\star(\sigma)d\sigma\Big]\\
&\le e^{[B\Phi^{-1}(s_1)-B\Phi^{-1}(\gamma_n(\Omega))]}\sup_{t\in(s_1,\gamma_n(\Omega))}\Big[t^{\frac1p}(1-\log t)^\alpha\frac1t\int_0^t v^\star(\sigma)d\sigma\Big].
\end{aligned}$$

Thus,

$$\|u\|^{\star}_{L^{p,\infty}(\log L)^\alpha(\varphi,\Omega)}\le e^{\big[B\Phi^{-1}(s_1)-B\Phi^{-1}(\gamma_n(\Omega))\big]}\|v\|^{\star}_{L^{p,\infty}(\log L)^\alpha(\varphi,\Omega)}.$$

□

Conclusions. This article studies a class of linear elliptic variational inequalities which are defined on a possibly unbounded domain and whose ellipticity condition is given in terms of the density of Gauss measure. Using the notion of rearrangement with respect to the Gauss measure, we prove a comparison result with the symmetric solutions of a "symmetrized" problem in which the data depend only on the first variable and the domain is a half-space. To this end, we first discuss the existence of symmetric solutions to the "symmetrized" problem, which make up for the previous results. In addition, as an application of the comparison results, we prove an estimates of the Lorentz-Zygmund norm of u in terms of the norm of the symmetric solutions v.

Acknowledgements. This research was supported by the Tianyuan Special Funds of the National Natural Science Foundation of China (Grant no. 11426146), by the Promotive Research Fund for Excellent Young and Middle-Aged Scientists of Shandong Province (Grant no. BS2012SF026), and by the Doctoral Fund of University of Jinan (Grant no. XBS1337).

References

[1] A. Alvino, S. Matarasso, G. Trombetti; *Variational inequalities and rearrangements*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl., vol. 3, no. 4, pp. 271-285, 1992.

[2] A. Alvino, P. L. Lions, G. Trombetti; *Comparison results for elliptic and parabolic equations via Schwarcz symmetrization*, Ann. Inst. H. Pincaré Anal. Non Linéaire, vol. 7, pp. 37-65, 1990.

[3] A. Alvino, G. Trombetti, P. L. Lions, S. Matarasso; *Comparison results for solutions of elliptic problems via symmetrization*, Annales de l'I. H. P., section C, vol. 16, no. 2, pp. 167-188, 1999.
[4] C. Bandle; *Isopermetric Inequalities and Application*, Monographs and Studies in Math., No. 7, Pitman, London, 1980.
[5] C. Bandle,J. Mossino; *Rearrangement in variational inequalities*, Ann. Mat. Pura Appl., vol. 138, no. 4, pp. 1-14, 1984.
[6] C. Bennett, K. Rudnick; *On Lorentz-Zygmund spaces*, Dissertatioes Math. (Rozprawy Mat.), vol. 175, pp. 67, 1980.
[7] M. F. Betta, F. Brock, A. Mercaldo, M. R. Posteraro; *A comparison result related to Gauss measure*, C. R. Acad. Sci. Pairs., vol. 334, pp. 451-456, 2002.
[8] F. Chiacchio; *Comparison results for linear parabolic equation in unbounded domains via Gaussian symmetrization*, Differential and Integral Equations, vol. 17, pp. 241-258, 2004.
[9] K. M. Chong, N. M. Rice; *Equimeasurable rearrangements of Sobolev functions*, Queen's Papers in Pure and Applied Mathematics, No. 28. Queen's University, Kingston, Ontario, 1971.
[10] J. I. Díaz; *Symmetrization of nonlinear elliptic and parabolic problems and applications: a particular overview*, "Progress in Partial Differential Equations: Elliptic and Parabolic Problems," Pitman Research Notes in Math., n.266. Longman, Harlow, U.K., pp. 1-16, 1992.
[11] J. I. Díaz, J. Mossino; *Isoperimetric inequalities in the parabolic obstacle problems*, J. Math. Pures Appl., vol. 71, pp. 233-266, 1992.
[12] G. Di Blasio; *Linear elliptic equations and Gauss measure*, J. Inequal. Pure Appl. Math., vol. 4, no. 5, pp. 1-11, 2003.
[13] G. Di Blasio, F. Feo, M. R. Posterato; *Regularity results for degenerate elliptic equations related to Gauss measure*, Mathematical Inequalities and Applications, vol. 10, pp. 771-797, 2007.
[14] A. Ferone, R. Volpicelli; *Symmetrization in parabolic obstacle problems*, Bull. Sci. Math., vol. 120, pp. 555-572, 1996.
[15] D. Gilbarg, N. S. Trudinger; *Elliptic Partial Differential Equations of Second Order*, Springer Verlag, New York, 1983.
[16] M. Ledoux; *Isoperimetry and Gaussian analysis*, Lectures on probability theory and statistics (Saint-Flour, 1994), Lecture Notes in Math., 1648, Springer, Berlin, pp. 165-294, 1996.
[17] J. L. Lions; *Quelques méthodes de résolution des problémes aux limites non linéaires*, Dunod, Paris, 1969.
[18] C. Maderna, S. Salsa; *Some special properties of solutions to obstacle problems*, Rend. Sem. Mat. Univ. Padova, vol. 71, pp. 121-129, 1984.
[19] M. R. Posteraro, R. Volpicelli; *Comparison results for a class of variational inequalities*, Revista Mat. Complutense., vol. 6, no. 2, pp. 295-310, 1993.
[20] M. R. Posteraro; *Estimates for solutions of nonlinear variational inequalities*, Ann. Inst. H. Poincar Anal. Non Linaire, vol. 12, pp. 577-597, 1995.
[21] M. H. Protter, Hans F. Weinberger; *Maximum Principles in Differential Equations*, Prentice-Hall, Inc., Englewood Cliffs, N.J. 1967.
[22] J. M. Rakotoson, B. Simon; *Relative rearrangement on a finite measure space. Application to the regularity of weighted monotone rearrangement. I.*, Rev. R. Acad. Cienc. Exactas Fs. Nat. (Esp.), vol. 91, no. 1, pp. 17-31, 1997.
[23] J. M. Rakotoson, B. Simon; *Relative rearrangement on a finite measure space. Application to weighted spaces and to P.D.E.*, Rev. R. Acad. Cienc. Exactas Fs. Nat. (Esp.), vol. 91, no. 1, pp. 33-45, 1997.
[24] G. Talenti; *Elliptic equations and rearrangement*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., vol. 3, pp. 697-718, 1976.
[25] G. Talenti; *A weighted version of a rearrangement inequality*, Ann. Univ. Ferrara, vol. 43, pp. 121-133, 1997.
[26] Y. J. Tian, F. Q. Li; *Comparison and regularity results for degenerate elliptic equations related to Gauss measure*, Applicable Analysis, vol. 89, no. 6, pp. 915-933, 2010.
[27] G. Trombetti; *Metodi di simmetrizzazione nelle equazioni alle derivate parziali*, Bollettino dell'Unione Matematica Italiana, vol. 8, pp. 601-634, 2000.
[28] H. Weinberger; *Symmetrization in uniformly elliptic problems*, Studies in Math Anal., Stanford University Press, 1962.

Yujuan Tian (corresponding author)
School of Mathematical Sciences, Shandong Normal University, Jinan, Shandong 250014, China
E-mail address: tianyujuan0302@126.com

Chao Ma
School of Mathematical Sciences, University of Jinan, Jinan, Shandong 250022, China
E-mail address: chaos_ma@163.com