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WELL-POSEDNESS FOR ONE-DIMENSIONAL ANISOTROPIC CAHN-HILLIARD AND ALLEN-CAHN SYSTEMS

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ABSTRACT. Our aim is to prove the existence and uniqueness of solutions for one-dimensional Cahn-Hilliard and Allen-Cahn type equations based on a modification of the Ginzburg-Landau free energy proposed in [8]. In particular, the free energy contains an additional term called Willmore regularization and takes into account strong anisotropy effects.

1. INTRODUCTION

The original Ginzburg-Landau free energy

$$\Psi_{GL} = \int_\Omega \big(\frac{1}{2}|\nabla u|^2 + F(u)\big)\,dx \tag{1.1}$$

plays a fundamental role in phase separation and transition, see, [4, 2]. Here, u is the order parameter, Ω is the domain occupied by the material (we assume that it is a bounded and regular domain of $\mathbb{R}^N$),

$$F(s) = \frac{1}{4}(s^2-1)^2, \tag{1.2}$$

$$f(s) = s^3 - s. \tag{1.3}$$

In [7] (also in [13]), the authors proposed the following modification of the Ginzburg-Landau free energy which takes into account strong anisotropy effects arising during the growth and coarsening of thin films, namely,

$$\Psi_{MGL} = \int_\Omega \Big(\gamma(n)(\frac{1}{2}|\nabla u|^2 + F(u)) + \frac{\beta}{2}\omega^2\Big)\,dx, \tag{1.4}$$

where

$$n = \frac{\nabla u}{|\nabla u|}, \quad \omega = f(u) - \Delta u, \quad F' = f. \tag{1.5}$$

Here, $\gamma(n)$ accounts for anisotropy effects (we also refer the reader to, e.g., [6] for a different approach to account for anisotropy effects in phase-field models) and $G(u) = \omega^2$ is called nonlinear Willmore regularization. Such a regularization is relevant, e.g., in determining the equilibrium shape of a crystal in its own liquid matrix, when anisotropy effects are strong. Indeed, in that case, the equilibrium

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interface may not be a smooth curve, but may present facets and corners with slopes of discontinuities (see, e.g., [12]). In particular, the corresponding Cahn-Hilliard equation

$$\frac{\partial u}{\partial t} = \Delta \frac{D\Psi_{MGL}}{Du}$$

(where $\frac{D}{Du}$ denotes a variational derivative) is an ill-posed problem and requires regularization. The author in [9] proved the well-posedness for a one-dimensional Allen-Cahn system based on (1.4).

In [8], the author introduced another modification of the Ginzburg-Landau free energy, namely,

$$\Psi_{AMGL} = \int_\Omega \big[\frac{1}{2}|\gamma(n)\nabla u|^2 + F(u) + \frac{1}{2}\omega^2\big]\, dx. \tag{1.6}$$

This model describes dendritic pattern formations and plays an important role in crystal growth.

To the best of our knowledge, there is no mathematical result concerning the Cahn-Hilliard (resp. Allen-Cahn) model associated with the free energy (1.6).

In this article, we consider the one dimensional case, i.e., taking $\Omega = (-L, L)$, (1.6) reads

$$\Psi = \int_\Omega \big[\frac{1}{2}|\gamma(n)u_x|^2 + F(u) + \frac{1}{2}\omega^2\big]\, dx, \tag{1.7}$$

where

$$n = \frac{u_x}{|u_x|}, \quad \omega = f(u) - u_{xx}, \quad F' = f. \tag{1.8}$$

In [7, 14], the authors proposed efficient energy stable schemes for the Cahn-Hilliard equation based on (1.4) and (1.6); actually, in [7], the authors considered a slightly different problem and also considered a second regularization, based on the bi-Laplacian, and, in that case, studied the isotropic case $\gamma(n) = 1$ as well. We also mention that, in [10] (resp. [11]), the Cahn-Hilliard (resp. Allen-Cahn) equation based on the Willmore regularization is studied in the isotropic case. There, well-posedness results are obtained.

Our aim in this article is to prove the existence and uniqueness of solutions for the Cahn-Hilliard and Allen-Cahn systems associated with the Ginzburg-Landau free energy (1.7).

Assumptions and notation. As far as the nonlinear term f is concerned, we assume more generally that f is of class C^4 and that

$$f(0) = 0, \quad f'(s) \geq -c_0, \quad c_0 \geq 0, \quad s \in \mathbb{R}, \tag{1.9}$$

$$f(s)s \geq c_1F(s) - c_2 \geq -c_2', \quad c_1 > 0, \quad c_2, c_2' \geq 0, \quad s \in \mathbb{R}, \tag{1.10}$$

where $F(s) = \int_0^s f(\tau)\, d\tau$,

$$sf(s)f'(s) - f(s)^2 \geq c_3f(s)^2 - c_4, \quad c_3 > 0, \quad c_4 \geq 0, \quad s \in \mathbb{R}, \tag{1.11}$$

$$|f'(s)| \leq \epsilon|f(s)| + c_5, \quad \forall \epsilon > 0,\ c_5 \geq 0,\ s \in \mathbb{R}, \tag{1.12}$$

$$sf''(s) \geq 0, \quad s \in \mathbb{R}. \tag{1.13}$$

Note that these assumptions are satisfied by the cubic nonlinear term (1.3).

As far as the bounded function γ is concerned, we introduce the following functions:

$$g(s) = \begin{cases} \gamma^2(-1)s^2 & s < 0, \\ 0 & s = 0, \\ \gamma^2(1)s^2 & s > 0, \end{cases} \tag{1.14}$$

g being a C^1-function, with $g'(0) = 0$, and

$$h(s) = \begin{cases} \gamma^2(-1)s & s < 0, \\ 0 & s = 0, \\ \gamma^2(1)s & s > 0. \end{cases} \tag{1.15}$$

Thus, h is a C^0-function, with $h' \in L^\infty(\mathbb{R})$.

Lemma 1.1. *The function h is Lipschitz continuous on $(-L, L)$.*

Proof. Let s_1 and s_2 belong to $\mathbb{R}$. We have two cases, depending on the sign of s_1 and s_2:

- If s_1 and s_2 have the same sign (or vanish), then it is clear that
$$|h(s_1) - h(s_2)| \le \max\{\gamma^2(1), \gamma^2(-1)\}|s_1 - s_2|.$$
- If s_1 and s_2 have opposite signs, then, assuming that $s_1 > 0$ and $s_2 < 0$ (the case $s_1 < 0$ and $s_2 > 0$ is similar),
$$\begin{aligned} |h(s_1) - h(s_2)| &= \gamma^2(1)s_1 - \gamma^2(-1)s_2 \\ &\le \max\{\gamma^2(1), \gamma^2(-1)\}(s_1 - s_2) \\ &= \max\{\gamma^2(1), \gamma^2(-1)\}|s_1 - s_2|. \end{aligned}$$

The result follows. □

We denote by $((\cdot,\cdot))$ the usual L^2-scalar product, with associated norm $\|\cdot\|$, and we set $\|\cdot\|_{-1} = \|(-\Delta)^{-1/2}\cdot\|$, where $(-\Delta)^{-1}$ is the inverse minus Laplace operator associated with Neumann boundary conditions and acting on functions with null average.

We set, whenever it makes sense, $\langle\cdot\rangle = \frac{1}{\mathrm{Vol}(\Omega)}\int_\Omega \cdot\, dx$, being understood that, for $\varphi \in H^{-1}(\Omega)$, $\langle\varphi\rangle = \frac{1}{\mathrm{Vol}(\Omega)}\langle\varphi, 1\rangle_{H^{-1}(\Omega), H^1(\Omega)}$, and we note that

$$\varphi \mapsto \left(\|\varphi - \langle\varphi\rangle^2\|_{-1}^2 + \langle\varphi\rangle^2\right)^{1/2}$$

is a norm on $H^{-1}(\Omega)$ which is equivalent to the usual one.

Throughout this article, the same letter c (and sometimes c') denotes constants which may vary from line to line. Similarly, the same letter Q denotes monotone increasing (with respect to each argument) functions which may vary from line to line.

Remark 1.2. *We can write, formally, for a small variation,*

$$\begin{aligned} D\Psi &= \int_{-L}^{L} \left[(\gamma(n)u_x)\, D(\gamma(n)u_x) + F'(u)Du + \omega D\omega \right] dx \\ &= \int_{-L}^{L} \left[\gamma(n)u_x D(\gamma(n)u_x) + f(u)Du + \omega f'(u)Du - \omega_{xx}Du\right] dx. \end{aligned}$$

We then note that

$$\Big(\gamma\big(\frac{s}{|s|}\big)s\Big)' = \gamma\big(\frac{s}{|s|}\big) \quad \text{in } \mathcal{D}'.$$

Indeed, we have

$$\Big(\gamma\big(\frac{s}{|s|}\big)s\Big)' = s\gamma'\big(\frac{s}{|s|}\big)\big(\frac{s}{|s|}\big)' + \gamma\big(\frac{s}{|s|}\big) \quad \text{in } \mathcal{D}'.$$

Now, it is sufficient to prove that

$$s\gamma'\big(\frac{s}{|s|}\big)\big(\frac{s}{|s|}\big)' = 0 \quad \text{in } \mathcal{D}'.$$

To do so, we let $\varphi \in \mathcal{D}(-L, L)$ *and have*

$$\begin{aligned}
\langle\big(\frac{s}{|s|}\big)', \varphi\rangle_{\mathcal{D}',\mathcal{D}} &= -\langle\frac{s}{|s|}, \varphi'\rangle_{\mathcal{D}',\mathcal{D}} = -\int_{-L}^{L} \frac{s}{|s|}\varphi'(s)\,ds \\
&= -\int_0^L \varphi'(s)\,ds + \int_{-L}^0 \varphi'(s)\,ds \\
&= [\varphi(s)]_{-L}^0 + [-\varphi(s)]_0^L \\
&= 2\varphi(0) = 2\langle\delta_0, \varphi\rangle_{\mathcal{D}',\mathcal{D}},
\end{aligned}$$

so that

$$s\gamma'\big(\frac{s}{|s|}\big)\big(\frac{s}{|s|}\big)' = 2s\delta_0\gamma'\big(\frac{s}{|s|}\big) \ \text{ in } \mathcal{D}'.$$

Since $s\delta_0 = 0$ *in* $\mathcal{D}'$, *we obtain*

$$\Big(\gamma\big(\frac{s}{|s|}\big)s\Big)' = \gamma\big(\frac{s}{|s|}\big) \quad \text{in } \mathcal{D}'. \tag{1.16}$$

Thus, owing to (1.16), *we obtain, formally,*

$$\begin{aligned}
D\Psi &= \int_{-L}^{L} \big[\gamma^2(n)u_x D(u_x) + f(u)Du + \omega f'(u)Du - \omega_{xx}Du\big]\,dx \\
&= \int_{-L}^{L} \big[-(\gamma^2(n)u_x)_x + f(u) + \omega f'(u) - \omega_{xx}\big]Du\,dx
\end{aligned}$$

and the variational derivative of Ψ *with respect to* u *reads*

$$\frac{D\Psi}{Du} = -\left(h(u_x)\right)_x + f(u) + \omega f'(u) - \omega_{xx}.$$

2. Cahn-Hilliard system

The Cahn-Hilliard equation is an equation of mathematical physics which describes the evolution of different material phases via an order parameter (or multiple order parameters). The equation was initially derived as a model for spinodal decomposition in solid materials [3, 5] and has since been extended to many other physical systems.

Setting of the problem. Writing mass conservation, i.e., $\frac{\partial u}{\partial t} = -h_x$, where h is the mass flux which is related to the chemical potential μ by the constitutive relation $h = -\mu_x$, and that the chemical potential is the variational derivative of Ψ with respect to u, we end up with the following sixth-order Cahn-Hilliard system

$$\frac{\partial u}{\partial t} = \mu_{xx}, \tag{2.1}$$

$$\mu = -(h(u_x))_x + f(u) + \omega f'(u) - \omega_{xx}, \tag{2.2}$$

$$\omega = f(u) - u_{xx}, \tag{2.3}$$

together with the Neumann boundary conditions

$$u_x\big|_{\pm L} = \mu_x\big|_{\pm L} = \omega_x\big|_{\pm L} = 0 \tag{2.4}$$

and the initial condition

$$u\big|_{t=0} = u_0. \tag{2.5}$$

2.1. **A priori estimates.** We first note that, integrating (formally) (2.1) over Ω, we obtain the conservation of mass, namely,

$$\langle u(t)\rangle = \langle u_0\rangle, \quad t \geq 0. \tag{2.6}$$

Multiplying (2.1) by $(-\Delta)^{-1}\frac{\partial u}{\partial t}$, we have, integrating over Ω and by parts,

$$\|\frac{\partial u}{\partial t}\|_{-1}^2 = -((\mu, \frac{\partial u}{\partial t})). \tag{2.7}$$

We then multiply (2.2) by $\frac{\partial u}{\partial t}$ and integrate over Ω to obtain

$$\begin{aligned} &((\mu, \frac{\partial u}{\partial t})) \\ &= \int_\Omega h(u_x)\frac{\partial u_x}{\partial t}\,dx + \frac{d}{dt}\int_\Omega F(u)\,dx + ((\omega f'(u), \frac{\partial u}{\partial t})) - ((\omega_{xx}, \frac{\partial u}{\partial t})). \end{aligned} \tag{2.8}$$

Noting that from (2.3) it follows that

$$((\omega f'(u), \frac{\partial u}{\partial t})) - ((\omega_{xx}, \frac{\partial u}{\partial t})) = \frac{1}{2}\frac{d}{dt}\|\omega\|^2, \tag{2.9}$$

we have, owing to (1.14),

$$\int_\Omega h(u_x)\frac{\partial u_x}{\partial t}\,dx = \frac{1}{2}\frac{d}{dt}\int_\Omega g(u_x)\,dx. \tag{2.10}$$

We finally deduce from (2.7)-(2.10) that

$$\frac{d}{dt}\Big[\int_\Omega g(u_x)\,dx + 2\int_\Omega F(u)\,dx + \|\omega\|^2\Big] + 2\|\frac{\partial u}{\partial t}\|_{-1}^2 = 0. \tag{2.11}$$

In particular, (2.11) yields that the free energy decreases along the trajectories, as expected.

We now multiply (2.1) by $(-\Delta)^{-1}\bar{u}$, where $\bar{u} = u - \langle u\rangle$, and integrate over Ω. We obtain, owing to (2.6),

$$\frac{1}{2}\frac{d}{dt}\|\bar{u}\|_{-1}^2 = -((\mu, u)) + \mathrm{Vol}(\Omega)\langle\mu\rangle\langle u_0\rangle, \tag{2.12}$$

where, owing to (2.2),

$$\langle\mu\rangle = \langle f(u)\rangle + \langle f'(u)\rangle. \tag{2.13}$$

Multiplying then (2.2) by u and integrating over Ω, we have, owing to (2.3),

$$\begin{aligned}((\mu,u)) &= \int_\Omega g(u_x)\,dx + ((f(u),u)) + ((f(u)f'(u),u)) \\ &\quad - ((f'(u)u_{xx},u)) - ((f(u)_{xx},u)) + \|u_{xx}\|^2.\end{aligned} \tag{2.14}$$

Noting that

$$\begin{gathered}((f'(u)u_{xx},u)) = -((f'(u)u_x,u_x)) - ((uf''(u)u_x,u_x)),\\ ((f(u)_{xx},u)) = -((f'(u)u_x,u_x)),\end{gathered}$$

we obtain

$$\begin{aligned}((\mu,u)) &= \int_\Omega g(u_x)\,dx + ((f(u),u)) + \|\omega\|^2 + ((uf''(u)u_x,u_x)) \\ &\quad + \int_\Omega \big(f(u)f'(u)u - f^2(u)\big)\,dx\end{aligned}$$

and finally, owing to (1.10), (1.11), (1.13) and (2.12), we obtain

$$\begin{aligned}&\frac{d}{dt}\|\bar{u}\|_{-1}^2 + c\Big[\int_\Omega g(u_x)\,dx + 2\int_\Omega F(u)\,dx + \|\omega\|^2 + \|f(u)\|^2\Big] \\ &\le 2\operatorname{Vol}(\Omega)\langle\mu\rangle\langle u_0\rangle + c', \quad c>0.\end{aligned} \tag{2.15}$$

We now assume that

$$|\langle u_0\rangle| \le M \quad (\text{hence, } |\langle u(t)\rangle| \le M,\, t\ge 0), \quad M \ge 0. \tag{2.16}$$

Therefore, owing to (1.12) and (2.13),

$$\begin{aligned}|2\operatorname{Vol}(\Omega)\langle u_0\rangle\langle\mu\rangle| &\le c_M\,(|\langle f(u)\rangle| + |\langle \omega f'(u)\rangle|) \\ &\le \frac{c}{2}\Big(\int_\Omega f(u)^2\,dx + \int_\Omega \omega^2\,dx\Big) + c'_M,\end{aligned} \tag{2.17}$$

where c is the constant appearing in (2.15), and we deduce from (2.15) and (2.17) that

$$\frac{d}{dt}\|\bar{u}\|_{-1}^2 + c\Big[\int_\Omega g(u_x)\,dx + 2\int_\Omega F(u)\,dx + \|\omega\|^2\Big] \le c'_M. \tag{2.18}$$

Combining (2.11) and (2.18), we have an inequality of the form

$$\frac{dE}{dt} + c(E + \|\frac{\partial u}{\partial t}\|_{-1}^2) \le c'_M, \tag{2.19}$$

where

$$E = \|\bar{u}\|_{-1}^2 + \langle u\rangle^2 + \int_\Omega g(u_x)\,dx + 2\int_\Omega F(u)\,dx + \|\omega\|^2. \tag{2.20}$$

In particular, we deduce from (2.19) and Gronwall's Lemma that

$$E(t) \le E(0)e^{-ct} + c'_M, \quad c>0,\ t\ge 0. \tag{2.21}$$

Noting that, owing to (1.9),

$$\|\omega\|^2 \ge \|f(u)\|^2 + \|u_{xx}\|^2 - 2c_0\|u_x\|^2, \tag{2.22}$$

we finally deduce from (2.20)-(2.22) and the boundedness of $\gamma(n)$ that

$$\|u\|_{H^2(\Omega)}^2 + \|f(u)\|^2 \le Q(\|u_0\|_{H^2(\Omega)})e^{-ct} + c'_M. \tag{2.23}$$

Rewriting (2.1) in the equivalent form

$$\mu = \langle\mu\rangle - (-\Delta)^{-1}\frac{\partial u}{\partial t}, \tag{2.24}$$

we obtain

$$\|\mu_x\| \le c\|\frac{\partial u}{\partial t}\|_{-1}. \tag{2.25}$$

Noting that, proceeding as in (2.17),

$$|\langle\mu\rangle| \le c\left(\|u\|^2_{H^2(\Omega)} + \|f(u)\|^2 + 1\right),$$

we finally find

$$\|\mu\|_{H^1(\Omega)} \le c\big(\|\frac{\partial u}{\partial t}\|_{-1} + \|u\|^2_{H^2(\Omega)} + \|f(u)\|^2 + 1\big). \tag{2.26}$$

Now, owing to (2.2), we have

$$\omega_{xx} = -(h(u_x))_x - \mu + f(u) + \omega f'(u)$$

and, owing to (1.12), there holds

$$\begin{aligned}\|\omega_{xx}\| &\le c\left(\|(h(u_x))_x\| + \|f(u)\|^2 + \|\omega\|^2 + \|\mu\|\right)\\ &\le c\left(\|h(u_x)\|_{H^1(\Omega)} + \|f(u)\|^2 + \|\omega\|^2 + \|\mu\|\right),\end{aligned} \tag{2.27}$$

where we have used the fact that

$$\left\{\begin{array}{c} h(u_x) = \gamma^2(n)u_x \in L^2(\Omega) \\ (h(u_x))' = h'(u_x)u_{xx} \in L^2(\Omega)\end{array}\right\} \Rightarrow h(u_x) \in H^1(\Omega).$$

Recall that h is Lipschitz continuous, with $h(0) = 0$, and note that

$$\|h(u_x)\|_{H^1(\Omega)} \le c\|u\|_{H^2(\Omega)}.$$

We then have, owing to (1.14) and (2.26)-(2.27),

$$\|\omega\|_{H^2(\Omega)} \le c\Big(\|\frac{\partial u}{\partial t}\|_{-1} + \|u\|^2_{H^2(\Omega)} + \|f(u)\|^2 + 1\Big). \tag{2.28}$$

We now multiply (2.1) by u and integrate over Ω to get

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 = -((\mu_x, u_x)). \tag{2.29}$$

Multiplying then (2.2) by $-u_{xx}$ and integrating over Ω, we obtain, in view of (2.3),

$$\begin{aligned}((\mu_x, u_x)) = &\int_\Omega h(u_x)u_{xxx}\,dx + ((f'(u)u_x, u_x)) - ((\omega f'(u), u_{xx}))\\ &- ((f(u)_{xx}, u_{xx})) + \|u_{xxx}\|^2.\end{aligned} \tag{2.30}$$

We note that

$$\begin{aligned}|((\omega f'(u), u_{xx}))| &\le \|f'(u)\|_{L^\infty(\Omega)}\|\omega\|\|u_{xx}\|\\ &\le \frac{1}{2}\|u_{xx}\|^2 + Q\left(\|u\|_{H^2(\Omega)}\right)\|\omega\|^2,\end{aligned} \tag{2.31}$$

where Q is continuous (here, we have used the fact that $H^2(\Omega)$ is continuously embedded into $C(\bar{\Omega})$), and, proceeding similarly,

$$\begin{aligned}\big|((f(u)_{xx}, u_{xx}))\big| &= \big|((f'(u)u_x, u_{xxx}))\big|\\ &\le \frac{1}{2}\|u_{xxx}\|^2 + Q\left(\|u\|_{H^2(\Omega)}\right)\|u_x\|^2.\end{aligned} \tag{2.32}$$

Finally,

$$\big|\int_\Omega h(u_x)u_{xxx}\,dx\big| \le c[\|u_x\|^2 + \|u_{xxx}\|^2]. \tag{2.33}$$

It thus follows from (1.9) and (2.29)-(2.33) that

$$\frac{d}{dt}\|u\|^2 + \|u\|^2_{H^3(\Omega)} \le Q(\|u\|_{H^2(\Omega)})\big(\|u\|^2_{H^1(\Omega)} + \|\omega\|^2\big), \tag{2.34}$$

where Q is continuous.

2.2. **Existence and uniqueness of solutions.**

Theorem 2.1. *Assume that* (2.16) *holds and that* $u_0 \in H^2(\Omega)$, *with* $\frac{\partial u_0}{\partial x}\big|_{\pm L} = 0$. *Then* (2.1)-(2.5) *admits a unique (variational) solution such that*

$$u \in L^\infty(\mathbb{R}^+; H^2(\Omega)) \cap L^2(0,T; H^3(\Omega)), \quad \frac{\partial u}{\partial t} \in L^2(0,T; H^{-1}(\Omega)),$$
$$\mu \in L^2(0,T; H^1(\Omega)), \quad \omega \in L^\infty(\mathbb{R}^+; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega))$$

for all $T > 0$.

Proof. (a) Existence: The proof of existence is based on a classical Galerkin scheme and on the *a priori* estimates derived in the previous section. We can note that a weak (variational) formulation of (2.1)-(2.5) reads

$$((\frac{\partial u}{\partial t}, v)) = ((\mu_{xx}, v)), \quad \forall v \in H^1(\Omega), \tag{2.35}$$

$$((\mu, v)) = ((h(u_x), v_x)) + ((\omega f'(u), v)) + ((f(u), v)) - ((\omega_{xx}, v)), \quad \forall v \in H^1(\Omega), \tag{2.36}$$

$$((\omega, v)) = ((f(u), v)) - ((u_{xx}, v)), \quad \forall v \in H^1(\Omega), \tag{2.37}$$

$$u\big|_{t=0} = u_0. \tag{2.38}$$

Let $v_0, v_1, \dots$ be an orthonormal (in $L^2(\Omega)$) and orthogonal (in $H^1(\Omega)$) family associated with the eigenvalues $0 = \lambda_0 < \lambda_1 \le \cdots$ of the operator $-\Delta$ associated with Neumann boundary conditions (note that v_0 is a constant). We set

$$V_m = \operatorname{Span}\{v_0, v_1, \dots, v_m\}$$

and consider the approximate problem:

Find $(u_m, \mu_m, \omega_m) : [0,T] \to V_m \times V_m \times V_m$ such that

$$((\frac{\partial u_m}{\partial t}, v)) = -((\mu_{m_x}, v)), \quad \forall v \in V_m, \tag{2.39}$$

$$((\mu_m, v)) = ((h(u_{mx}), v_x)) + ((\omega f'(u_m), v)) + ((f(u_m), v)) - ((\omega_{mxx}, v)), \quad \forall v \in V_m, \tag{2.40}$$

$$((\omega_m, v)) = ((f(u_m), v)) - ((u_{mxx}, v)), \quad \forall v \in V_m, \tag{2.41}$$

$$u_m\big|_{t=0} = u_{0,m}, \tag{2.42}$$

where $u_{0,m} = P_m u_0$, P_m being the orthogonal projector from $L^2(\Omega)$ onto V_m.

The existence of a local (in time) solution to (2.39)-(2.42) is standard. Indeed, we have to solve a Lipschitz continuous finite-dimensional system of ODE's to find u_m, which yields ω_m and then μ_m.

The *a priori* estimates derived in the previous section, which are now justified within the Galerkin approximation, yield that the solution is global and that, up to a subsequence which we do not relabel and owing to classical Aubin-Lions compacteness results,

$u_m \to u$ weak star in $L^\infty(0,T; H^2(\Omega))$, strongly in $C([0,T]; H^{2-\varepsilon}(\Omega))$, and a.e.,

$$\frac{\partial u_m}{\partial t} \to \frac{\partial u}{\partial t} \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)),$$
$$\mu_m \to \mu \quad \text{weakly in } L^2(0,T;H^1(\Omega)),$$
$$\omega_m \to \omega \quad \text{weak star in } L^\infty(0,T;L^2(\Omega)) \text{ and weakly in } L^2(0,T;H^2(\Omega)),$$

as $m \to +\infty, \forall T > 0$.

Note that, owing to (2.19), (2.21) and (2.23), we have $u \in L^\infty(\mathbb{R}^+;H^2(\Omega))$ and, consequently, $\omega \in L^\infty(\mathbb{R}^+;L^2(\Omega))$.

As far as the passage to the limit is concerned, the most delicate part is to prove that

$$\int_0^T \int_\Omega (\omega_m f'(u_m) - \omega f'(u))\varphi \,dx\,dt \to 0 \quad \text{as } m \to +\infty,$$
$$\int_0^T \int_\Omega (h(u_{mx}) - h(u_x))\varphi_x \,dx\,dt \to 0 \quad \text{as } m \to +\infty,$$

for φ regular enough.

We have, say, for $\varphi \in C^2([0,T] \times \bar{\Omega})$ such that $\varphi(T) = \varphi(0) = 0$,

$$\begin{aligned} &\int_0^T \int_\Omega \left(\omega_m f'(u_m) - \omega f'(u)\right) \varphi \,dx\,dt \\ &= \int_0^T \int_\Omega (\omega_m - \omega) f'(u)\varphi \,dx\,dt + \int_0^T \int_\Omega \omega_m \left(f'(u_m) - f'(u)\right) \varphi \,dx\,dt. \end{aligned} \tag{2.43}$$

The passage to the limit in the first integral in the right-hand side of (2.43) is straightforward, while the passage to the limit in the second one follows from the above convergences which yield, in particular, the inequality

$$\Big| \int_0^T \int_\Omega \omega_m \left(f'(u_m) - f'(u)\right) \varphi \,dx\,dt \Big| \le c\|u_m - u\|_{L^2((0,T)\times\Omega)}.$$

Finally, recalling that h is Lipschitz continuous, we have

$$\Big| \int_0^T \int_\Omega \big(h(u_{mx}) - h(u_x)\big)\varphi_x \,dx\,dt \Big| \le c\|u_{mx} - u_x\|_{L^2((0,T)\times\Omega)}.$$

(b) Uniqueness: Let (u_1, μ_1, ω_1) and (u_2, μ_2, ω_2) be two solutions to (2.1)-(2.4) with initial data $u_{1,0}$ and $u_{2,0}$, respectively, such that

$$|\langle u_{i,0} \rangle| \le M, \quad i = 1, 2. \tag{2.44}$$

We set $(u, \mu, \omega) = (u_1, \mu_1, \omega_1) - (u_2, \mu_2, \omega_2)$ and $u_0 = u_{1,0} - u_{2,0}$ and have

$$\frac{\partial u}{\partial t} = \mu_{xx}, \tag{2.45}$$

$$\begin{aligned} \mu = &-\big(h(u_{1x})\big)_x + \big(h(u_{2x})\big)_x + f(u_1) - f(u_2) \\ &+ \omega_1 f'(u_1) - \omega_2 f'(u_2) - \omega_{xx}, \end{aligned} \tag{2.46}$$

$$\omega = f(u_1) - f(u_2) - u_{xx}, \tag{2.47}$$

$$u_x\big|_{\pm L} = \mu_x\big|_{\pm L} = \omega_x\big|_{\pm L} = 0, \tag{2.48}$$

$$u\big|_{t=0} = u_0. \tag{2.49}$$

We multiply (2.45) by $(-\Delta)^{-1}\bar{u}$ and obtain, integrating over Ω and by parts,

$$\frac{1}{2}\frac{d}{dt}\|\bar{u}\|_{-1}^2 = -((\mu,u)) + \mathrm{Vol}(\Omega)\langle\mu\rangle\langle u\rangle, \tag{2.50}$$

where, owing to (2.46),

$$\langle\mu\rangle = \langle f(u_1)-f(u_2)\rangle + \langle\omega_1 f'(u_1)-\omega_2 f'(u_2)\rangle. \tag{2.51}$$

We then multiply (2.46) by u and find, in view of (2.47),

$$\begin{aligned}((\mu,u)) &= \int_\Omega h(u_{1x})u_x\,dx - \int_\Omega h(u_{2x})u_x\,dx\\ &\quad + ((f(u_1)-f(u_2),u)) + ((\omega_1 f'(u_1)-\omega_2 f'(u_2),u))\\ &\quad - ((f(u_1)-f(u_2),u_{xx})) + \|u_{xx}\|^2.\end{aligned} \tag{2.52}$$

We have, owing to (1.9),

$$((f(u_1)-f(u_2),u)) = ((f'(u)u,u)) \ge -c_0\|u\|^2. \tag{2.53}$$

Furthermore,

$$|((f(u_1)-f(u_2),u_{xx}))| \le \frac{1}{8}\|u_{xx}\|^2 + Q(\|u_{1,0}\|_{H^2(\Omega)},\|u_{2,0}\|_{H^2(\Omega)})\|u\|^2 \tag{2.54}$$

and

$$\begin{aligned}&\big|((\omega_1 f'(u_1)-\omega_2 f'(u_2),u))\big|\\ &\le |((\omega_1(f'(u_1)-f'(u_2)),u))| + |((\omega f'(u_2),u))|\\ &\le Q(\|u_{1,0}\|_{H^2(\Omega)},\|u_{2,0}\|_{H^2(\Omega)})\|\omega_1\|_{H^2(\Omega)}\|u\|^2\\ &\quad + |((f'(u_2)u_{xx},u))| + |((f'(u_2)(f(u_1)-f(u_2)),u))|\\ &\le \frac{1}{8}\|u_{xx}\|^2 + Q(\|u_{1,0}\|_{H^2(\Omega)},\|u_{2,0}\|_{H^2(\Omega)})(\|\omega_1\|_{H^2(\Omega)}+1)\|u\|^2.\end{aligned} \tag{2.55}$$

Similarly,

$$\begin{aligned}&|\,\mathrm{Vol}(\Omega)\langle u\rangle\langle\mu\rangle|\\ &\le c(\int_\Omega |f(u_1)-f(u_2)|\,dx + \int_\Omega |\omega_1 f'(u_1)-\omega_2 f'(u_2)|\,dx)|\langle u\rangle|\\ &\le \Big(\int_\Omega |f(u_1)-f(u_2)||f'(u_2)|\,dx\Big)|\langle u\rangle|\\ &\quad + (\int_\Omega |\omega_1||f'(u_1)-f'(u_2)|\,dx + \int_\Omega |u_{xx}||f'(u_2)|\,dx)|\langle u\rangle|\\ &\quad + Q(\|u_{1,0}\|_{H^2(\Omega)},\|u_{2,0}\|_{H^2(\Omega)})\|u\||\langle u\rangle|\\ &\le \frac{1}{8}\|u_{xx}\|^2 + Q(\|u_{1,0}\|_{H^2(\Omega)},\|u_{2,0}\|_{H^2(\Omega)})(\|\omega_1\|+1)(\|u\|^2+|\langle u\rangle|^2).\end{aligned} \tag{2.56}$$

Recalling that h is Lipschitz continuous, we have

$$|((h(u_{1x})-h(u_{2x}),u_x))| \le \int_\Omega |h(u_{1x})-h(u_{2x})||u_x|\,dx \le c\|u_x\|^2. \tag{2.57}$$

We finally deduce from (2.50), (2.52)-(2.57) and the interpolation inequality

$$\|\bar{u}\| \le c\|\bar{u}\|_{-1}^{1/2}\|\nabla\bar{u}\|^{1/2} \le c'\|\bar{u}\|_{-1}^{1/2}\|\Delta\bar{u}\|^{1/2} \tag{2.58}$$

that

$$\begin{aligned} &\frac{d}{dt}(\|\bar{u}\|_{-1}^2 + \langle u\rangle^2) + \|u_{xx}\|^2 \\ &\le Q(\|u_{1,0}\|_{H^2(\Omega)}, \|u_{2,0}\|_{H^2(\Omega)})(1 + \|\omega_1\| + \|\omega_1\|_{H^2(\Omega)})(\|\bar{u}\|_{-1}^2 + |\langle u\rangle|^2). \end{aligned} \tag{2.59}$$

Gronwall's Lemma then yields, owing to (2.19), (2.23) and (2.28) (written for (u_1, μ_1, ω_1)),

$$\|u(t)\|_{H^{-1}(\Omega)} \le ce^{Q(\|u_{1,0}\|_{H^2(\Omega)}, \|u_{2,0}\|_{H^2(\Omega)})t}\|u_0\|_{H^{-1}(\Omega)}, \tag{2.60}$$

hence the uniqueness, as well as the continuous dependence with respect to the initial data in the H^{-1}-norm.

It follows from Theorem 2.1 that we can define the continuous (for the H^{-1}-norm) semigroup

$$S(t) : \Phi_M \to \Phi_M, \quad u_0 \to u(t), \quad t \ge 0$$

(i.e., $S(0) = Id$ and $S(t+s) = S(t) \circ S(s),\ t, s \ge 0$), where

$$\Phi_M = \big\{v \in H^2(\Omega), \frac{\partial v}{\partial x}\big|_{\pm L} = 0, |\langle v\rangle| \le M\big\}, \quad M \ge 0.$$

We then deduce from (2.23) that $S(t)$ is dissipative, i.e., it possesses a bounded absorbing set $\mathcal{B}_0 \subset \Phi_M$ (in the sense that, for all $B \subset \Phi_M$ bounded, there exists $t_0 = t_0(B)$ such that $t \ge t_0 \Rightarrow S(t)B \subset \mathcal{B}_0$). □

3. Allen-Cahn system

The Allen-Cahn equation describes important processes related with phase separation in binary alloys, namely, the ordering of atoms in a lattice (see [1]).

Assuming the relaxation dynamics $\frac{\partial u}{\partial t} = -\frac{D\psi}{Du}$, we obtain the Allen-Cahn system

$$\frac{\partial u}{\partial t} - (h(u_x))_x + f(u) + \omega f'(u) - \omega_{xx} = 0, \tag{3.1}$$

$$\omega = f(u) - u_{xx}, \tag{3.2}$$

together with the Neumann boundary conditions

$$u_x\big|_{\pm L} = \omega_x\big|_{\pm L} = 0 \tag{3.3}$$

and the initial condition

$$u\big|_{t=0} = u_0. \tag{3.4}$$

3.1. A priori estimates.

We Multiply (3.1) by $\frac{\partial u}{\partial t}$ and have, integrating over Ω and by parts,

$$\|\frac{\partial u}{\partial t}\|^2 + \int_\Omega h(u_x)\frac{\partial u_x}{\partial t}\,dx + \frac{d}{dt}\int_\Omega F(u)\,dx + ((\omega f'(u) - \omega_{xx}, \frac{\partial u}{\partial t})) = 0,$$

which yields, noting that it follows from (3.2) that

$$((\omega f'(u), \frac{\partial u}{\partial t})) - ((\omega_{xx}, \frac{\partial u}{\partial t})) = \frac{1}{2}\frac{d}{dt}\|\omega\|^2$$

and from (1.14) that

$$\int_\Omega h(u_x)\frac{\partial u_x}{\partial t}\,dx = \frac{1}{2}\frac{d}{dt}\int_\Omega g(u_x)\,dx,$$

the differential equality

$$\frac{d}{dt}\Big[\int_\Omega g(u_x)\,dx + 2\int_\Omega F(u)\,dx + \|\omega\|^2\Big] + 2\|\frac{\partial u}{\partial t}\|^2 = 0. \tag{3.5}$$

In particular, it follows from (3.5) that the energy decreases along the trajectories, as expected.

We then multiply (3.1) by u and obtain, owing to (3.2),

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u\|^2 + \int_\Omega g(u_x)\,dx + ((f(u),u)) + \int_\Omega uf(u)f'(u)\,dx\\ &+ 2((f'(u)u_x,u_x)) + ((uf''(u)u_x,u_x)) + \|u_{xx}\|^2 = 0,\end{aligned}$$

which yields, owing to (3.2),

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u\|^2 + \int_\Omega g(u_x)\,dx + ((f(u),u)) + \|w\|^2\\ &+ \int_\Omega (uf(u)f'(u) - f^2(u))\,dx + ((uf''(u)u_x,u_x)) = 0,\end{aligned}$$

hence, in view of (1.10), (1.11) and (1.13),

$$\frac{d}{dt}\|u\|^2 + c\Big[\int_\Omega g(u_x)\,dx + 2\int_\Omega F(u)\,dx + \|\omega\|^2\Big] \le c', \quad c>0. \tag{3.6}$$

Summing (3.5) and (3.6), we find an inequality of the form

$$\frac{dE_1}{dt} + c\Big(E_1 + \|\frac{\partial u}{\partial t}\|^2\Big) \le c', \quad c>0, \tag{3.7}$$

where

$$E_1 = \|u\|^2 + \int_\Omega g(u_x)\,dx + 2\int_\Omega F(u)\,dx + \|\omega\|^2. \tag{3.8}$$

In particular, it follows from (3.7) and Gronwall's Lemma that

$$E_1(t) \le E_1(0)e^{-ct} + c', \quad c>0, \tag{3.9}$$

hence, in view of (1.9) (which yields that $\|\omega\|^2 \ge \|u_{xx}\|^2 + \|f(u)\|^2 - 2c_0\|u_x\|^2$), (3.8) and classical elliptic regularity results,

$$\|u(t)\|_{H^2(\Omega)} \le Q(\|u_0\|_{H^2(\Omega)})e^{-ct} + c', \quad c>0,\ t\ge 0. \tag{3.10}$$

Next, we multiply (3.1) by $-u_{xx}$ to have

$$\begin{aligned}&-\int_\Omega \frac{\partial u}{\partial t}u_{xx}\,dx - \int_\Omega h(u_x)u_{xxx}\,dx - \int_\Omega f(u)u_{xx}\,dx\\ &-\int_\Omega \omega f'(u)u_{xx}\,dx + \int_\Omega \omega_{xx}u_{xx}\,dx = 0.\end{aligned} \tag{3.11}$$

It follows from (3.2) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_x\|^2 - \int_\Omega h(u_x)u_{xxx}\,dx + ((f'(u)u_x,u_x))\\ &-((\omega f'(u),u_{xx})) + (((f(u))_{xx},u_{xx})) + \|u_{xxx}\|^2 = 0.\end{aligned} \tag{3.12}$$

Now, owing to the continuous embedding $H^2(\Omega)\subset \mathcal{C}(\bar\Omega)$ and (3.2), there holds

$$\big|((f'(u)u_x,u_x))\big| + \big|((\omega f'(u),u_{xx}))\big| + \big|(((f(u))_{xx},u_{xx}))\big| \le Q(\|u\|_{H^2(\Omega)})$$

(indeed, it follows from (3.2) that $\|\omega\| \leq Q(\|u\|_{H^2(\Omega)})$) and

$$\Big|\int_\Omega h(u_x)u_{xxx}\,dx\Big| \leq c[\|u_x\|^2 + \|u_{xxx}\|^2],$$

hence

$$\frac{d}{dt}\|u_x\|^2 + \|u\|^2_{H^3(\Omega)} \leq Q(\|u\|_{H^2(\Omega)}). \tag{3.13}$$

3.2. **Existence and uniqueness of solutions.**

Theorem 3.1. *Let $u_0 \in H^2(\Omega)\cap H^1_0(\Omega)$. Then,* (3.1)-(3.4) *admits a unique (variational) solution such that $u \in L^\infty(\mathbb{R}^+; H^2(\Omega)\cap H^1_0(\Omega))$ and $\frac{\partial u}{\partial t} \in L^2(0,T;L^2(\Omega))$. Furthermore, $\omega \in L^\infty(\mathbb{R}^+;L^2(\Omega))\cap L^2(0,T;H^2(\Omega)\cap H^1_0(\Omega))$ for all $T>0$. Finally, the associated semigroup is dissipative in $H^2(\Omega)\cap H^1_0(\Omega)$.*

Proof. (a) Uniqueness: Let u_1 and u_2 be two solutions to (3.1)-(3.3) with initial data $u_{1,0}$ and $u_{2,0}$ respectively, where ω_1 and ω_2 are defined from (3.2). We set $u = u_1 - u_2$, $\omega = \omega_1 - \omega_2$, $u_0 = u_{1,0} - u_{2,0}$ and have

$$\begin{gathered}\frac{\partial u}{\partial t} - (h(u_{1x}))_x + (h(u_{2x}))_x + f(u_1) - f(u_2)\\ + \omega_1 f'(u_1) - \omega_2 f'(u_2) - \omega_{xx} = 0,\end{gathered} \tag{3.14}$$

$$\omega = f(u_1) - f(u_2) - u_{xx}, \tag{3.15}$$

$$u_x\big|_{\pm L} = \omega_x\big|_{\pm L} = 0, \tag{3.16}$$

$$u\big|_{t=0} = u_0. \tag{3.17}$$

We multiply (3.14) by u and integrating over Ω, we obtain

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\|u\|^2 + ((h(u_{1x}) - h(u_{2x}), u_x)) + ((f(u_1) - f(u_2), u))\\ + ((\omega_1 f'(u_1) - \omega_2 f'(u_2), u)) - ((f(u_1) - f(u_2), u_{xx})) + \|u_{xx}\|^2 = 0.\end{gathered} \tag{3.18}$$

We note that, by (1.9),

$$((f(u_1) - f(u_2), u)) \geq c_0\|u\|^2$$

and that, owing to (3.15),

$$\begin{aligned}&|((\omega_1 f'(u_1) - \omega_2 f'(u_2), u))|\\ &\leq |((\omega f'(u_1), u))| + |((\omega_2(f'(u_1) - f'(u_2)), u))|\\ &\leq Q(\|u_{1,0}\|_{H^2(\Omega)}, \|u_{2,0}\|_{H^2(\Omega)})(\|\omega\|\|u\| + \|\omega_2\|\|u\|^2_{L^4(\Omega)})\\ &\leq Q(\|u_{1,0}\|_{H^2(\Omega)}, \|u_{2,0}\|_{H^2(\Omega)})(\|u_{xx}\|^2\|u\| + \|u_x\|^2)\\ &\leq \frac{1}{4}\|u_{xx}\|^2 + Q(\|u_{1,0}\|_{H^2(\Omega)}, \|u_{2,0}\|_{H^2(\Omega)})\|u_x\|^2\end{aligned} \tag{3.19}$$

and

$$|((f(u_1) - f(u_2), u_{xx}))| \leq \frac{1}{8}\|u_{xx}\|^2 + Q(\|u_{1,0}\|_{H^2(\Omega)}, \|u_{2,0}\|_{H^2(\Omega)})\|u\|^2. \tag{3.20}$$

Recalling that h is Lipschitz continuous, we have

$$|((h(u_{1x}) - h(u_{2x}), u_x))| \leq \int_\Omega |h(u_{1x}) - h(u_{2x})||u_x|\,dx \leq c\|u_x\|^2. \tag{3.21}$$

We finally deduce from (3.18)-(3.21) and the interpolation inequality

$$\|u_x\| \le c\|u\|^{1/2}\|u_{xx}\|^{1/2}$$

that

$$\frac{d}{dt}\|u\|^2 + \|u_{xx}\|^2 \le Q(\|u_{1,0}\|_{H^2(\Omega)}, \|u_{2,0}\|_{H^2(\Omega)})\|u\|^2. \tag{3.22}$$

Then Gronwall's Lemma yields

$$\|u_1(t) - u_2(t)\| \le ce^{Q(\|u_{1,0}\|_{H^2(\Omega)}, \|u_{2,0}\|_{H^2(\Omega)})t}\|u_0\|, \tag{3.23}$$

hence the uniqueness, as well as the continuous dependence with respect to the initial data in the L^2-norm.

(b) Existence: The proof of existence of solutions is based on the *a priori* estimates derived in the previous section and, e.g., a standard Galerkin scheme.

In particular, it follows from (3.7)-(3.8) and (3.10) that we can construct a sequence of solutions u_m to a proper approximated problem such that

$$u_m \to u \quad \text{weak star in } L^\infty(0,T;H^2(\Omega)), \text{ strongly in } C([0,T];H^{2-\varepsilon}(\Omega)) \text{ and a.e.,}$$

$$\frac{\partial u_m}{\partial t} \to \frac{\partial u}{\partial t} \quad \text{weakly in } L^2(0,T;L^2(\Omega)),$$

$$\omega_m \to \omega \quad \text{weak star in } L^\infty(0,T;L^2(\Omega)) \text{ and weakly in } L^2(0,T;H^2(\Omega)),$$

as $m \to +\infty$ for all $T > 0$.

The passage to the limit is then standard and can be done as in the previous section. Furthermore, it follows from (3.7)-(3.8) and (3.10) that

$$u \in L^\infty(\mathbb{R}^+;H^2(\Omega)), \quad \frac{\partial u}{\partial t} \in L^2(0,T;L^2(\Omega)), \ \forall T > 0,$$

and, consequently, $\omega \in L^\infty(\mathbb{R}^+;L^2(\Omega))$.

It follows from Theorem 3.1 that we can define the continuous (for the L^2-norm) semigroup

$$S(t) : \Phi \to \Phi, \quad u_0 \to u(t)$$

where $\Phi = H^2(\Omega) \cap H_0^1(\Omega)$. Finally, the dissipativity of $S(t)$ follows from (3.10). $\square$

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