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WELL-POSEDNESS OF FRACTIONAL PARABOLIC DIFFERENTIAL AND DIFFERENCE EQUATIONS WITH DIRICHLET-NEUMANN CONDITIONS

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Abstract. We study initial-boundary value problems for fractional parabolic equations with the Dirichlet-Neumann conditions. We obtain a stable difference schemes for this problem, and obtain theorems on coercive stability estimates for the solution of the first order of accuracy difference scheme. A procedure of modified Gauss elimination method is applied for the solution of the first and second order of accuracy difference schemes of one-dimensional fractional parabolic differential equations.

1. Introduction

Theory, applications and methods of solutions of problems for fractional differential equations have been studied extensively by many researchers (see, e.g., [1]–[8], [10]–[16], [18], [19], [21], [23]–[30], [32]–[34], [39]–[46] and the references given therein). In this article, we study the initial-boundary value problem

$$\begin{gathered} D_t^\alpha u(t,x) - a(x)u_{xx}(t,x) + \sigma u(t,x) = f(t,x), \quad 0 < x < l,\ 0 < t < T, \\ u(t,0) = 0, \quad u_x(t,l) = 0, \quad 0 \le t \le T, \\ u(0,x) = 0, \quad 0 \le x \le l \end{gathered} \tag{1.1}$$

for the fractional parabolic equation with the Dirichlet-Neumann conditions. Here $D_t^\alpha = D_{0+}^\alpha$ is the standard Riemann-Louville's derivative of order $\alpha \in [0,1)$. Here $a(x)(x \in (0,l))$ and $f(t,x)(t \in (0,T), x \in (0,l))$ are given smooth functions, $a(x) \ge a > 0$, $\sigma > 0$. Theorem on coercive stability estimates for the solution of the initial-boundary value problem (1.1) is established. Stable difference schemes for the approximate solution of problem (1.1) are considered. Theorem on coercive stability estimates for the solution of the first order of accuracy in t difference scheme is proved. A procedure of modified Gauss elimination method is applied for the solution of the first and second order of accuracy difference schemes for the fractional parabolic equations.

The organization of the present paper as follows. The first section is introduction where we provide the history and formulation of the problem. In Section 2, theorem on coercivity stability of problem (1.1) is established. In Section 3, stable difference

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schemes for the approximate solution of problem (1.1) are considered. Theorem on coercivity stability for the first order of accuracy in t difference scheme is proved. In Section 4, the numerical application is given. Finally, Section 5 is conclusion.

2. Theorems on coercive stability

We will give some statements which will be useful in the sequel.

Let E be a Banach space, and $A : D(A) \subset E \to E$ be a linear unbounded operator densely defined in E. We call A *strongly positive* in the Banach space E, if its spectrum σ_A lies in the interior of the sector of angle ϕ, $0 < 2\phi < \pi$, symmetric with respect to the real axis, and if on the edges of this sector, $S_1(\phi) = \{\rho e^{i\phi} : 0 \le \rho \le \infty\}$ and $S_2(\phi) = \{\rho e^{-i\phi} : 0 \le \rho \le \infty\}$, and outside of the sector the resolvent $(\lambda - A)^{-1}$ is subject to the bound

$$\|(\lambda - A)^{-1}\|_{E\to E} \le \frac{M}{1 + |\lambda|}. \tag{2.1}$$

The infimum of such angles is called spectral angle $\varphi(A, E)$ of A.

Throughout this article, positive constants have different values in time and they will be indicated with M On the other hand $M(\alpha, \beta, \cdots)$ is used to focus on the fact that the constant depends only on $\alpha, \beta, \cdots$.

For a positive operator A in the Banach space E, let us introduce the fractional spaces $E_\beta = E_\beta(E, A)(0 < \beta < 1)$ consisting of those $\nu \in E$ for which the norm

$$\|\nu\|_{E_\beta} = \sup_{\lambda>0} \lambda^\beta \|A(\lambda + A)^{-1}\nu\|_E + \|\nu\|_E$$

is finite.

Theorem 2.1 ([17, 31]). *Let A and B be two commutative positive operators with $\varphi(A, E) + \varphi(B, E) < \pi$. Then it follows that there exists the bounded operator $(A + B)^{-1}$ defined on whole space E. Moreover, for every $\beta \in (0, 1)$ and f, there exists a unique solution $u = u(f)$ of the problem*

$$Au + Bu = f$$

and the following estimates hold

$$\begin{aligned} &\|Au\|_{E_\beta(E,B)} + \|Bu\|_{E_\beta(E,B)} + \|Bu\|_{E_\beta(E,A)} \le M(\beta)\|f\|_{E_\beta(E,B)}, \\ &\|Au\|_{E_\beta(E,A)} + \|Bu\|_{E_\beta(E,A)} + \|Au\|_{E_\beta(E,B)} \le M(\beta)\|f\|_{E_\beta(E,A)}. \end{aligned}$$

Theorem 2.2 ([31]). *Let A be the positive operator with $\varphi(A, E) < \pi$. Then for $\beta \le \frac{1}{2}, A^\beta$ is a positive operator with $\varphi(A^\beta, E) < \frac{\pi}{2}$.*

Theorem 2.3 ([3]). *Let A be the operator acting in $E = C[0, T]$ defined by the formula $Av(t) = v'(t)$, with the domain $D(A) = \{v(t) : v'(t) \in C[0, T], v(0) = 0\}$. Then A is a positive operator in the Banach space $E = C[0, T]$ and*

$$A^\beta f(t) = D_t^\beta f(t)$$

for all $f(t) \in D(A)$.

From the above theorems it follows the following theorem.

Theorem 2.4. *Let A and B be the positive operators with $\varphi(A, E) < \pi$ and $\varphi(B, E) \le \frac{\pi}{2}$. Then for $\beta \le \frac{1}{2}$ it follows that there exists bounded $(D^\beta + B)^{-1}$*

defined on whole space E. Moreover, for every f, there exists a unique solution $u = u(f)$ of the problem

$$D^\beta u + Bu = f$$

and the following estimate holds

$$\|D^\beta u\|_{E_\beta(E,B)} + \|Bu\|_{E_\beta(E,B)} \le M(\beta)\|f\|_{E_\beta(E,B)}.$$

Now, we consider the second order differential operator

$$B^x u(x) = -a(x)u_{xx}(x) + \sigma u(x) \tag{2.2}$$

with the domain $D(B^x) = \{u; u, u', u'' \in C[0,l], u(0) = 0, u'(l) = 0\}$.

Let us introduce the Banach space $C^\gamma[0,l]$, $\gamma \in (0,1]$ of all continuous function $\varphi(x)$ defined on $[0,l]$ and satisfying a Hölder condition for which the following norm is finite

$$\|\varphi\|_{C^\gamma[0,l]} = \|\varphi\|_{C[0,l]} + \sup_{x_1 \neq x_2} \frac{|\varphi(x_1) - \varphi(x_2)|}{|x_1 - x_2|^\gamma},$$

where $C[0,l]$ is the Banach space of all continuous functions $\varphi(x)$ defined on $[0,l]$ with the norm

$$\|\varphi\|_{C[0,l]} = \max_{x\in[0,l]} |\varphi(x)|.$$

The positivity of the operator B^x in the Banach space $C[0,l]$ was established (see, [37, 38]). Moreover, we have that for any $\beta \in (0,1/2)$ the norms in the spaces $E_\beta(E,B)$ and $C^{2\beta}[0,l]$ are equivalent.

Theorem 2.5. *For $\beta \in (0,1/2)$, the norms of the space $E_\beta(C[0,l], B^x)$ and the Hölder space $C^{2\beta}[0,l]$ are equivalent.*

The proof of Theorem 2.5 is based on the following estimates

$$|G^x(x,x_0;\lambda)| \le \frac{M(\sigma,a)}{\sqrt{\sigma+\lambda}} \begin{cases} e^{-\frac{1}{2}\sqrt{\frac{\sigma+\lambda}{a}}(x-s)}, & 0 \le x_0 \le x, \\ e^{-\frac{1}{2}\sqrt{\frac{\sigma+\lambda}{a}}(x_0-x)}, & x \le x_0 \le l, \end{cases}$$

$$|G^x_x(x,x_0;\lambda)| \le M(\sigma,a) \begin{cases} e^{-\frac{1}{2}\sqrt{\frac{\sigma+\lambda}{a}}(x-x_0)}, & 0 \le x_0 \le x, \\ e^{-\frac{1}{2}\sqrt{\frac{\sigma+\lambda}{a}}(x_0-x)}, & x \le x_0 \le l \end{cases}$$

for the Green's function of the differential operator B^x defined by the formula (2.2) and it follows the scheme of the proof of the Theorem of paper [9].

Theorem 2.6. *For the solution of problem* (1.1) *the coercive stability estimate*

$$\max_{0\le t\le T} \|u_{xx}(t,.)\|_{C^\beta[0,l]} \le M(\beta)\|f(t,.)\|_{C^\beta[0,l]}$$

holds, where $M(\beta)$ does not depend on $f(t,x)$ $(0 \le t \le T$, $x \in [0,l])$ and $0 < \beta < 1$.

The proof of Theorem 2.6 is based on the positivity of differential operator B^x defined by formula (2.2), on the Theorem 2.3 on connection of fractional derivatives with fractional powers of positive operators, on the Theorem 2.2 on spectral angle of fractional powers of positive operators, and on the Theorem 2.4 on fractional powers of coercively positive sums two operators.

3. Difference schemes and stability estimates

The discretization of problem (1.1) is carried out in two steps. In the first step, let us define the grid space

$$[0,l]_h = (x_n = nh,\ 0 \le n \le M,\ Mh = l)$$

To the differential space operator B^x generated by formula (2.2), we assign the difference operator B_h^x by the formula

$$B_h^x u^h = -a(x) u^h_{x_n \bar{x}_n} + \sigma u(x)^h \tag{3.1}$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $u^h(x) = 0$ for all $x = 0$ and $D^h u^h(x) = 0$ for $x = l$. Here $D^h u^h(x)$ is the approximation of u_x. With the help of B_h^x we arrive at the initial boundary value problem

$$\begin{gathered} D_t^\alpha v^h(t,x) + B_h^x v^h(t,x) = f^h(t,x), \quad 0 < t < T, \quad x \in [0,l]_h, \\ v^h(0,x) = 0, \quad x \in [0,l]_h \end{gathered} \tag{3.2}$$

for a finite system of ordinary fractional differential equations.

In the second step, applying the first order of approximation formula (see [3])

$$D_\tau^\alpha u_k = \frac{1}{\Gamma(1-\alpha)} \sum_{r=1}^{k} \frac{\Gamma(k-r-\alpha+1)}{(k-r)!} \frac{u_r - u_{r-1}}{\tau^\alpha}, 1 \le k \le N$$

for

$$D_\tau^\alpha u(t_k) = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_k} (t_k - s)^{-\alpha} u'(s) ds$$

and using the first order of accuracy stable difference scheme for parabolic equations, one can present the first order of accuracy difference scheme with respect to t,

$$\begin{gathered} \frac{1}{\Gamma(1-\alpha)} \sum_{r=1}^{k} \frac{\Gamma(k-r-\alpha+1)}{(k-r)!} \frac{u_r^h(x) - u_{r-1}^h(x)}{\tau^\alpha} + B_h^x u_k^h(x) = f_k^h(x), \\ f_k^h(x) = f^h(t_k, x),\ t_k = k\tau, \quad 1 \le k \le N,\ N\tau = T,\ x \in [0,l]_h, \\ u_0^h(x) = 0, \quad x \in [0,l]_h \end{gathered} \tag{3.3}$$

for the approximate solution of problem (1.1). Moreover, applying the second order of approximation formula: for $k = 1$,

$$D_\tau^\alpha u_k = -d \frac{2^{\alpha-1}}{(2-\alpha)(1-\alpha)} u_0 + d \frac{2^{\alpha-1}}{(2-\alpha)(1-\alpha)} u_1,$$

for $k = 2$,

$$\begin{aligned} D_\tau^\alpha u_k &= d\Big[\frac{3^{5-\alpha}}{2^{4-\alpha}} \frac{1}{(1-\alpha)(2-\alpha)(3-\alpha)} - 7 \frac{3^{2-\alpha}}{2^{3-\alpha}} \frac{1}{(1-\alpha)(2-\alpha)}\Big] u_0 \\ &\quad + d\Big[-\frac{3^{4-\alpha}}{2^{2-\alpha}} \frac{1}{(1-\alpha)(2-\alpha)(3-\alpha)} + \frac{3^{2-\alpha}}{2^{-\alpha}} \frac{1}{(1-\alpha)(2-\alpha)}\Big] u_1 \\ &\quad + d\Big[\frac{3^{4-\alpha}}{2^{4-\alpha}} \frac{1}{(1-\alpha)(2-\alpha)(3-\alpha)} - \frac{3^{2-\alpha}}{2^{3-\alpha}} \frac{1}{(1-\alpha)(2-\alpha)}\Big] u_2, \end{aligned}$$

for $3 \le k \le N$,

$$
\begin{aligned}
D_\tau^\alpha u_k = d\sum_{m=2}^{k-1}\Big\{&\Big[\frac{(k-m)}{1-\alpha}\xi(k-m)-\frac{\eta(k-m)}{2-\alpha}\Big]u_{m-2}\\
&+\Big[\frac{(2m-2k-1)}{1-\alpha}\xi(k-m)+\frac{2\eta(k-m)}{2-\alpha}\Big]u_{m-1}\\
&+\Big[\frac{(k-m+1)}{1-\alpha}\xi(k-m)-\frac{\eta(k-m)}{2-\alpha}\Big]u_m\Big\}\\
&+d\Big[-\frac{2^{\alpha-2}}{2-\alpha}u_{k-2}-\Big(\frac{2^{\alpha-1}}{1-\alpha}-\frac{2^{\alpha-1}}{2-\alpha}\Big)u_{k-1}+\Big(\frac{2^{\alpha-1}}{1-\alpha}-\frac{2^{\alpha-2}}{2-\alpha}\Big)u_k\Big]
\end{aligned}
\tag{3.4}
$$

for

$$
D_t^\alpha u(t_k-\tau/2)=\frac{1}{\Gamma(1-\alpha)}\int_0^{t_k-\tau/2}(t_k-\tau/2-s)^{-\alpha}u'(s)ds,
$$

and using a Crank-Nicholson difference scheme for parabolic equations, one can present the second order of accuracy difference scheme with respect to t and x,

$$
\begin{gathered}
D_\tau^\alpha u_k^h(x)+\frac{1}{2}B_h^x\Big(u_k^h(x)+u_{k-1}^h(x)\Big)=f_k^h(x),\quad x\in[0,l]_h,\\
f_k^h(x)=f(t_k-\tau/2,x),\quad t_k=k\tau,\ 1\le k\le N,\ N\tau=T,\\
u_0^h(x)=0,\quad x\in[0,l]_h
\end{gathered}
\tag{3.5}
$$

for the approximate solution of problem (1.1). Here,

$$
\begin{gathered}
d=\frac{\tau^{-\alpha}}{\Gamma(1-\alpha)},\ \xi(r)=\big(r+1/2\big)^{1-\alpha}-\big(r-1/2\big)^{1-\alpha},\\
\eta(r)=\big(r+1/2\big)^{2-\alpha}-\big(r-1/2\big)^{2-\alpha}.
\end{gathered}
$$

Now, we consider the equation

$$
B_h^x u^h+\lambda u^h=f^h \tag{3.6}
$$

in the case $a(x)=1$.

Lemma 3.1. *Let $\lambda>0$. Then (3.6) is uniquely solvable, and the formula*

$$
u^h=(B_h^x+\lambda)^{-1}f^h=\Big\{\sum_{i=1}^{M-1}G(k,i;\lambda+\sigma)f_ih\Big\}_0^M \tag{3.7}
$$

is valid, where

$$
G(k,i;\lambda+\sigma)=\frac{h(R^{M-i}-R^{M+i})(R^{M-k}-R^{M+k})}{(1-R^2)(1+R^{2M-1})}+\frac{h(R^{|k-i|+1}-R^{k+i+1})}{(1-R^2)}
$$

for $1\le i\le M-1$, and $1\le k\le M$,

$$
R=(1+\delta h)^{-1},\delta=\frac{1}{2}\big(h(\lambda+\sigma)+\sqrt{(\lambda+\sigma)(4+h^2(\lambda+\sigma))}\big).
$$

The grid function $G(k,i;\lambda+\sigma)$ is called the Green function of equation (3.6) and by the formulas for R and δ, we get

$$
\sum_{i=1}^{M-1}G(k,i;\lambda+\sigma)h=\frac{1}{\lambda+\sigma}-\frac{1}{\lambda+\sigma}\frac{R^k+R^{2M-k-1}}{1+R^{2M-1}},\quad 1\le k\le M. \tag{3.8}
$$

To prove the positivity on B_h^x in the Banach space C_h, we need the following auxiliary lemmas [13].

Lemma 3.2. *The following estimate holds*

$$|\delta| \geq \max\Big\{\frac{|\lambda+\sigma|h}{2}, \sqrt{|\lambda+\sigma|}\Big\}. \tag{3.9}$$

Lemma 3.3. *The following estimate*

$$|R| \leq \frac{1}{1+\sqrt{|\lambda+\sigma|}h\cos\theta} < 1 \tag{3.10}$$

is valid, where $|\theta| < \pi/2$.

Theorem 3.4. *For all* λ *in the sector* $\Sigma_\theta = \{\lambda : |\arg\lambda| \leq \theta, 0 \leq \theta < \pi/2\}$ *the resolvent* $(\lambda I + B_h^x)^{-1}$ *defined by* (3.7) *satisfies the following estimate*

$$\|(\lambda I + B_h^x)^{-1}\|_{C_h \to C_h} \leq \frac{M(\mu,\theta,\sigma)}{1+|\lambda|}. \tag{3.11}$$

Proof. First, we consider the operator B_h^x defined by formula (3.1) in the case $a(x) = 1$. Let us set $k = M$. Since

$$\begin{aligned} u_M &= \frac{h^2 R(1-R^{M-1})(1+R^{M-1})}{(1-R)(1+R^{2M-1})} f_{M-1} \\ &\quad + \frac{1}{(1-R)(1+R^{2M-1})} \sum_{i=1}^{M-2} \Big(R^{M-i} - R^{M+i}\Big) h^2 f_i, \end{aligned}$$

we have that

$$\begin{aligned} |u_M| &\leq 2\Big|\frac{R}{1-R}\Big| h^2 |f_{M-1}| + \frac{1}{(1-|R|)} \sum_{i=1}^{M-2} \Big(|R|^{M-i} + |R|^{M+i}\Big) h^2 |f_i| \\ &\leq 2h^2 \|f^h\|_{C_h} \Big\{ \Big|\frac{R}{1-R}\Big| + \frac{|R^2|}{(1-|R|)^2} \Big\}. \end{aligned}$$

Now, let us $1 \leq k \leq M-1$. Then by formula (3.7) and the triangle inequality, we obtain

$$\begin{aligned} |u_k| &\leq \frac{\big(|R|^{M-k} + |R|^{M+k}\big)}{|1-R^2|\big|1+R^{2M-1}\big|} \sum_{i=1}^{M-1} \Big(|R|^{M-i} + |R|^{M+i}\Big) h^2 |f_i| \\ &\quad + \frac{1}{|1-R^2|} \sum_{i=1}^{M-1} \Big(|R|^{|k-i|+1} + |R|^{k+i+1}\Big) h^2 |f_i| \\ &\leq \frac{2}{|1-R^2|} \sum_{i=1}^{M-1} \Big(|R|^{M-i+1} + |R|^{M+i+1}\Big) h^2 |f_i| \\ &\quad + \frac{1}{|1-R^2|} \sum_{i=1}^{M-1} \Big(|R|^{|k-i|+1} + |R|^{k+i+1}\Big) h^2 |f_i| \\ &\leq \frac{4h^2}{|1-R^2|} \|f^h\|_{C_h} \sum_{i=1}^{M-1} |R|^{M-i+1} \\ &\quad + \frac{2h^2}{|1-R^2|} \|f^h\|_{C_h} \Big\{ \sum_{i=1}^{k-1} |R|^{k-i+1} + |R| + \sum_{i=k+1}^{M-1} |R|^{i-k+1} \Big\} \end{aligned}$$

$$
\begin{aligned}
&\le \frac{2h^2}{|1-R^2|}\|f^h\|_{C_h}\Big\{\frac{2|R|^2}{1-|R|}+\frac{2|R|^2}{1-|R|}+|R|\Big\}\\
&\le M\Big\{\frac{|R|^2}{\big(1-|R|\big)^2}\frac{h^2}{|1+R|}+\Big|\frac{R}{1-R}\Big|\Big|\frac{1}{1+R}\Big|h^2\Big\}.
\end{aligned}
$$

From estimate (3.10) it follows that

$$
\frac{|R|^2}{\big(1-|R|\big)^2}\le\Big(\frac{\frac{1}{1+\sqrt{|\lambda+\sigma|hcos\theta}}}{1-\frac{1}{1+\sqrt{|\lambda+\sigma|hcos\theta}}}\Big)^2=\Big(\frac{1}{\sqrt{|\lambda+\sigma|hcos\theta}}\Big)^2. \tag{3.12}
$$

Clearly, we have that

$$
\begin{aligned}
|\lambda+\sigma| &= |\rho\cos\theta+i\rho\sin\theta+\sigma| = \sqrt{\rho^2+2\rho\sigma\cos\theta+\sigma^2}\\
&\ge \sqrt{\rho^2\cos^2\theta+2\rho\sigma\cos\theta+\sigma^2}=|\lambda|\cos\theta+\sigma.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{1}{|\lambda+\sigma|} &\le \frac{1}{|\lambda|\cos\theta+\sigma}\le\frac{1}{|\lambda|\cos\theta+\sigma\cos\theta}\\
&=\frac{\frac{1}{\cos\theta}}{|\lambda|+\sigma}=\frac{\frac{1}{\sigma\cos\theta}}{1+\frac{1}{\sigma}|\lambda|}\\
&\le\frac{M(\sigma,\theta)}{1+|\lambda|}.
\end{aligned} \tag{3.13}
$$

Combining estimates (3.12) and (3.13), we obtain that

$$
\frac{h^2|R|^2}{\big(1-|R|\big)^2}\le\frac{\frac{1}{\cos^2\theta}}{|\lambda+\sigma|}\le\frac{M(\sigma,\theta)}{1+|\lambda|}. \tag{3.14}
$$

From the definition of R and estimate (3.9), it follows that

$$
\Big|\frac{R}{1-R}\Big|h^2=\frac{h}{|\delta|}\le\frac{2}{1+|\lambda|}. \tag{3.15}
$$

Combining estimates (3.14) and (3.15), we obtain

$$
\|u^h\|_{C_h}\le\frac{M(\mu,\sigma,\theta)}{1+|\lambda|}\|f^h\|_{C_h}.
$$

This concludes the proof of Theorem 3.4 in the case $a(x)=1$. Second, noted that the proof of this statement is based on estimates for the Green's function. Under one more assumption that $\sigma>0$ is sufficiently large number, applying a fixed point Theorem, same estimates for the Green's function can be obtained. Therefore, this statement of theorem is true also for difference operator B_h^x defined by formula (3.1). Theorem 3.4 is proved. □

Theorem 3.5. *Let $0<\beta<\frac{1}{2}$. Then, the norms of spaces $E_\beta(C_h,B_h^x)$ and $C_h^{2\beta}$ are equivalent uniformly in h, $0<h<h_0$.*

Proof. From (3.7) and (3.8) it follows that

$$
\begin{aligned}
\big(\lambda^\beta B_h^x(B_h^x+\lambda)^{-1}f^h\big)_k &= \frac{\sigma\lambda^\beta}{\lambda+\sigma}f_k+\frac{\lambda^{\beta+1}}{\lambda+\sigma}\frac{R^k+R^{2M-k-1}}{1+R^{2M-1}}f_k\\
&\quad+\lambda^{\beta+1}\sum_{i=1}^{M-1}G(k,i;\lambda+\sigma)h(f_k-f_j).
\end{aligned}
$$

Applying the triangle inequality, we obtain

$$\begin{aligned}
&\big|\big(\lambda^\beta B_h^x(B_h^x+\lambda)^{-1}f^h\big)_k\big| \\
&\leq \frac{\sigma\lambda^\beta}{\lambda+\sigma}\big|f_k\big| + \frac{\lambda^{\beta+1}}{\lambda+\sigma}\big|f_k\big| + \lambda^{\beta+1}\sum_{i=1}^{M-1}|G(k,i;\lambda+\sigma)|h|f_k-f_j| \\
&\leq \Big[\frac{\sigma\lambda^\beta}{\lambda+\sigma} + \frac{\lambda^{\beta+1}}{\lambda+\sigma} + M(\sigma)\frac{\lambda^{\beta+1}}{\sqrt{\lambda+\sigma}}\sum_{i=1}^{M-1}R^{|k-i|}|(k-i)h|^{2\beta}h\Big]\|f^h\|_{C_h^{2\beta}} \\
&\leq M_1(\sigma)\|f^h\|_{C_h^{2\beta}}
\end{aligned}$$

for any $\lambda>0$ and $x\in[0,l]$. Therefore, $f^h\in E_\beta(C_h,B_h^x)$ and

$$\|f^h\|_{E_\beta(C_h,B_h^x)} \leq M_1(\sigma)\|f^h\|_{C_h^{2\beta}}.$$

Now, we prove the reverse inequality. For any positive operator B_h^x, we can write

$$v=\int_0^\infty\sum_{i=1}^{M-1}G(k,i;\lambda+\sigma)B_h^x(B_h^x+\lambda)^{-1}f_ih_1dt.$$

Consequently,

$$f_k-f_{k+r}=\int_0^\infty\sum_{i=1}^{M-1}\lambda^{-\beta}[G(k+r,i;\lambda+\sigma)-G(k,i;\lambda+\sigma)]\lambda^\beta A_h^x(A_h^x+\lambda)^{-1}f_ih_1dt,$$

hence

$$|f_k-f_{k+r}|\leq\int_0^\infty\lambda^{-\beta}\sum_{i=1}^{M-1}|G(k+r,i;\lambda+\sigma)-G(k,i;\lambda+\sigma)|h_1dt\|f^h\|_{E_\beta(C_h,B_h^x)}.$$

Let

$$T_h=|rh_1|^{-2\beta}\int_0^\infty\lambda^{-\beta}\sum_{i=1}^{M-1}|G(k+r,i;\lambda+\sigma)-G(k,i;\lambda+\sigma)|h_1dt.$$

The proof of estimate

$$\frac{|f_k-f_{k+r}|}{|rh_1|^{2\beta}}\leq T_h\|f^h\|_{E_\beta(C_h,B_h^x)}$$

is based on the Lemmas 3.2 and 3.3. Thus, for any $1\leq k<k+r\leq N-1$ we have established the inequality

$$\frac{|f_k-f_{k+r}|}{|rh_1|^{2\beta}}\leq\frac{M}{\beta(1-2\beta)}\|f^h\|_{E_\beta(C_h,B_h^x)}.$$

This means that

$$\|f^h\|_{C_h^{2\beta}}\leq\frac{M}{\beta(1-2\beta)}\|f^h\|_{E_\beta(C_h,B_h^x)}.$$

Theorem 3.5 in the case $a(x)=1$ is proved. Now, let $a(x)$ be continuous functions and let $x,x_0\in[0,1]$ be arbitrary fixed points. Clearly, we have that

$$\|(B_h^x-B_h^{x_0})(B_h^{x_0})\|_{C_h\to C_h}\leq M.$$

From the formula

$$\begin{aligned}
B_h^x(B_h^x+\lambda)^{-1}f^h &= B_h^{x_0}(B_h^{x_0}+\lambda)^{-1}f^h \\
&\quad+\lambda(\lambda+B_h^x)^{-1}[B_h^x-B_h^{x_0}](B_h^{x_0})^{-1}B_h^{x_0}(B_h^{x_0}+\lambda)^{-1}f^h
\end{aligned}$$

it follows that

$$\begin{aligned}
&\big|\lambda^\beta B_h^x(B_h^x+\lambda)^{-1}f^h\big| \\
&\le \|f^h\|_{E_\beta(C_h,B_h^{x_0})}+M\lambda\|(\lambda+B_h^x)^{-1}\|_{C_h\to C_h}\|f^h\|_{E_\beta(C_h,B_h^{x_0})} \\
&\le M_1\|f^h\|_{E_\beta(C_h,B_h^{x_0})}.
\end{aligned}$$

Then

$$\|f^h\|_{E_\beta(C_h,B_h^{x})}\le M_1\|f^h\|_{E_\beta(C_h,B_h^{x_0})}.$$

Theorem 3.5 is proved. □

Theorem 3.6 ([3]). *Let A_τ be the operator acting in $E_\tau=C[0,T]_\tau$ defined by the formula $A_\tau v^\tau=\{\frac{v_k-v_{k-1}}{\tau}\}_1^N$, with $v_0=0$. Then A_τ is a positive operator in the Banach space $E_\tau=C[0,T]_\tau$ and*

$$A_\tau^\beta f^\tau=\Big\{\frac{1}{\Gamma(1-\beta)}\sum_{r=1}^{k}\frac{\Gamma(k-m-\beta+1)}{(k-m)!}\frac{f_m-f_{m-1}}{\tau^\beta}\Big\}_1^N.$$

By the definition of fractional difference derivative

$$D_\tau^\beta f^\tau:=\Big\{\frac{1}{\Gamma(1-\beta)}\sum_{r=1}^{k}\frac{\Gamma(k-m-\beta+1)}{(k-m)!}\frac{f_m-f_{m-1}}{\tau^\beta}\Big\}_1^N.$$

Theorem 3.7. *Let A_τ be the operator acting in $E_\tau=C[0,T]_\tau$ defined by the formula $A_\tau v^\tau(t)=\{\frac{v_k-v_{k-1}}{\tau}\}_1^N$ with the domain*

$$D(A_\tau)=\{v^\tau:\frac{v_k-v_{k-1}}{\tau}\in C[0,T]_\tau, v_0=0\}.$$

Then A is a positive operator in the Banach space $E_\tau=C[0,T]_\tau$, and

$$A_\tau^\beta f^\tau(t)=D_\tau^\beta f^\tau(t)$$

for all $f^\tau(t)\in D(A_\tau)$.

Thus, we have the following result on coercive stability of difference scheme (3.5).

Theorem 3.8. *Let τ and h be sufficiently small positive numbers and $0<\beta<1$. Then the solution of difference scheme (3.5) satisfies the following coercive stability estimate:*

$$\max_{1\le k\le N} Big\|\Big\{\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}\Big\}_{n=1}^{M-1}\|_{C^\beta[0,l]_h}\le M(\beta)\max_{1\le k\le N}\Big\|f_k^h\|_{C^\beta[0,l]_h}.$$

Here, $M(\beta)$ does not depend on τ,h and $f_k^h, 1\le k\le N$.

The proof of Theorem 3.8 is based on the Theorem 3.4 on positivity of difference space operator B_h^x defined by formula ((3.1), on the Theorem 3.5 on the structure of fractional space $E_\beta(C_h,B_h^{x_0})$, on the Theorem 2.3 on connection of fractional derivatives with fractional powers of positive operators, on the Theorem 2.2 on spectral angle of fractional powers of positive operators, and on the Theorem 2.1 on fractional powers of coercively positive sums two operators.

4. A numerical application

For numerical results, we consider the example

$$
\begin{gathered}
D_t^{\alpha} u(t,x) - u_{xx}(t,x) + u(t,x) = f(t,x), \\
f(t,x) = \frac{6\sin^2(\pi x) t^{3-\alpha}}{\Gamma(4-\alpha)} - 2\pi^2 t^3 \cos(2\pi x) + t^3 \sin^2(\pi x), \\
0 < t < 1,\ 0 < x < 1, \\
u(0,x) = 0, 0 \le x \le 1, \\
u(t,0) = u_x(t,1) = 0, \quad 0 \le t \le 1
\end{gathered} \tag{4.1}
$$

for the one-dimensional fractional parabolic partial differential equation with $0 < \alpha < 1$. The exact solution of problem (4.1) is $u(t,x) = t^3 \sin^2 \pi x$. Note that this function is independent of α, but $f(t,x)$ depends on α.

Applying the difference scheme (3.3) for the numerical solution of (4.1), we obtain

$$
\begin{gathered}
\frac{1}{\Gamma(1-\alpha)} \sum_{m=1}^{k} \frac{\Gamma(k-m-\alpha+1)}{(k-m)!} \frac{u_n^m - u_n^{m-1}}{\tau^{\alpha}} - \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} + u_n^k = \phi_n^k, \\
\phi_k^n = f(t_k, x_n), \quad t_k = k\tau,\ 1 \le k \le N, \quad N\tau = T, \\
x_n = nh, \quad 1 \le n \le M-1, \\
u_n^0 = 0, \quad 0 \le n \le M, \\
u_0^k = 0, \quad u_{M-1}^k = u_M^k, \quad 0 \le k \le N.
\end{gathered} \tag{4.2}
$$

We get the system of equations in the matrix form

$$
\begin{gathered}
AU_{n+1} + BU_n + CU_{n-1} = D\phi_n, \quad 1 \le n \le M-1, \\
U_0 = \widetilde{0}, \quad U_{M-1} = U_M,
\end{gathered} \tag{4.3}
$$

where

$$
A = \begin{pmatrix}
0 & 0 & 0 & \dots & 0 & 0 \\
0 & a_n & 0 & \dots & 0 & 0 \\
0 & 0 & a_n & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & a_n & 0 \\
0 & 0 & 0 & \dots & 0 & a_n
\end{pmatrix}_{(N+1)x(N+1)},
$$

$$
B = \begin{pmatrix}
b_{11} & 0 & 0 & \dots & 0 & 0 \\
b_{21} & b_{22} & 0 & \dots & 0 & 0 \\
b_{31} & b_{32} & b_{33} & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
b_{N1} & b_{N2} & b_{N3} & \dots & b_{NN} & 0 \\
b_{N+1,1} & b_{N+1,2} & b_{N+1,3} & \dots & b_{N+1,N} & b_{N+1,N+1}
\end{pmatrix}_{(N+1)x(N+1)},
$$

$$C = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & c_n & 0 & \dots & 0 & 0 \\ 0 & 0 & c_n & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & c_n & 0 \\ 0 & 0 & 0 & \dots & 0 & c_n \end{pmatrix}_{(N+1)x(N+1)},$$

$$D = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}_{(N+1)x(N+1)},$$

$$\phi_n = \begin{pmatrix} \phi_n^0 \\ \phi_n^1 \\ \phi_n^2 \\ \vdots \\ \phi_n^{N-1} \\ \phi_n^N \end{pmatrix}_{(N+1)x(1)}, \quad U_n = \begin{pmatrix} U_q^0 \\ U_q^1 \\ U_q^2 \\ \vdots \\ U_q^{N-1} \\ U_q^N \end{pmatrix}_{(N+1)x(1)}, q = \{n \pm 1, n\},$$

$$a_n = -\frac{1}{h^2},\ c_n = -\frac{1}{h^2}, \quad b_{11} = 1, \quad b_{21} = -\frac{1}{\tau^\alpha}, \quad b_{22} = \frac{1}{\tau^\alpha} + 1 + \frac{2}{h^2},$$

$$b_{31} = -\frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)\tau^\alpha}, \quad b_{32} = \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)\tau^\alpha} - \frac{1}{\tau^\alpha}, \quad b_{33} = \frac{1}{\tau^\alpha} + 1 + \frac{2}{h^2},$$

and

$$b_{ij} = \begin{cases} -\frac{\Gamma(i-1-\alpha)}{\Gamma(1-\alpha)(i-2)!\tau^\alpha}, & j = 1, \\ \frac{1}{\Gamma(1-\alpha)\tau^\alpha}\Big[\frac{\Gamma(i-j+1-\alpha)}{(i-j)!} - \frac{\Gamma(i-j-\alpha)}{(i-j-1)!}\Big], & 2 \le j \le i-2, \\ \frac{\Gamma(2-\alpha)-\Gamma(1-\alpha)}{\Gamma(1-\alpha)\tau^\alpha}, & j = i-1, \\ \frac{1}{\tau^\alpha} + 1 + \frac{2}{h^2}, & j = i, \\ 0, & i < j \le N+1 \end{cases} \tag{4.4}$$

for $i = 4, 5, \dots, N+1$ and

$$\phi_n^k = \frac{6\sin^2(\pi nh)(k\tau)^{3-\alpha}}{\Gamma(4-\alpha)} - 2\pi^2(k\tau)^3\cos(2\pi nh) + (k\tau)^3\sin^2(\pi nh).$$

To solve the difference problem (4.3), a procedure of modified Gauss elimination method is applied. Hence, we seek a solution of the matrix equation in the following form:

$$U_j = \alpha_{j+1}U_{j+1} + \beta_{j+1},\ U_M = (I - \alpha_M)^{-1}\beta_M,\ j = M-1, \dots, 2, 1$$

where α_j $(j = 1, 2, \dots, M)$ are $(N+1)\times(N+1)$ square matrices, and β_j $(j = 1, 2, \dots, M)$ are $(N+1)\times 1$ column matrices defined by

$$\alpha_{j+1} = -(B + C\alpha_j)^{-1}A,$$
$$\beta_{j+1} = (B + C\alpha_j)^{-1}(D\phi - C\beta_j), \quad j = 1, 2, \dots, M-1$$

where $j = 1, 2, \dots, M-1$, α_1 is the $(N+1)\times(N+1)$ zero matrix, and β_1 is the $(N+1)\times 1$ zero matrix.

Second, applying the difference scheme (3.5), we obtain the second order of accuracy difference scheme in t and in x and the Crank-Nicholson difference scheme for parabolic equations, one can represent the second order of accuracy difference scheme with respect in t and in x

$$\begin{gathered}
D_\tau^\alpha u_n^k - \frac{1}{2}\Big[\frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} + \frac{u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}}{h^2}\Big] + \frac{1}{2}\Big[u_n^k + u_n^{k-1}\Big] = \phi_n^k, \\
\phi_n^k = f(t_k - \frac{\tau}{2}, x_n), \quad t_k = k\tau, x_n = nh, \\
1 \le k \le N, \quad 1 \le n \le M-1, \\
u_n^0 = 0, \quad 0 \le n \le M, \\
u_0^k = 0, \quad 3u_M^k - 4u_{M-1}^k + u_{M-2}^k = 0, \quad 0 \le k \le N.
\end{gathered} \tag{4.5}$$

Here $D_\tau^\alpha u_n^k$ is defined by (3.4) for u_n^k. We get the system of equations in the matrix form

$$\begin{gathered}
AU_{n+1} + BU_n + CU_{n-1} = D\phi_n, \quad 1 \le n \le M-1, \\
U_0 = \widetilde{0}, \quad 3U_M - 4U_{M-1} + U_{M-2} = 0,
\end{gathered} \tag{4.6}$$

where

$$A = \begin{pmatrix}
0 & 0 & 0 & \dots & 0 & 0 \\
a_n & a_n & 0 & \dots & 0 & 0 \\
0 & a_n & a_n & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & a_n & 0 \\
0 & 0 & 0 & \dots & a_n & a_n
\end{pmatrix}_{(N+1)\text{x}(N+1)},$$

$$B = \begin{pmatrix}
b_{11} & 0 & 0 & \dots & 0 & 0 \\
b_{21} & b_{22} & 0 & \dots & 0 & 0 \\
b_{31} & b_{32} & b_{33} & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
b_{N1} & b_{N2} & b_{N3} & \dots & b_{NN} & 0 \\
b_{N+1,1} & b_{N+1,2} & b_{N+1,3} & \dots & b_{N+1,N} & b_{N+1,N+1}
\end{pmatrix}_{(N+1)\text{x}(N+1)},$$

$$C = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ c_n & c_n & 0 & \dots & 0 & 0 \\ 0 & c_n & c_n & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & c_n & 0 \\ 0 & 0 & 0 & \dots & c_n & c_n \end{pmatrix}_{(N+1)\text{x}(N+1)},$$

$$D = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}_{(N+1)\text{x}(N+1)},$$

$$\phi_n = \begin{pmatrix} \phi_n^0 \\ \phi_n^1 \\ \phi_n^2 \\ \vdots \\ \phi_n^{N-1} \\ \phi_n^N \end{pmatrix}_{(N+1)x(1)}, \quad U_q = \begin{pmatrix} U_q^0 \\ U_q^1 \\ U_q^2 \\ \vdots \\ U_q^{N-1} \\ U_q^N \end{pmatrix}_{(N+1)x(1)}, \quad q = \{n \pm 1, n\},$$

$$a_n = -\frac{1}{2h^2}, \quad c_n = -\frac{1}{2h^2},$$

$$b_{11} = 1, \quad b_{21} = -d\frac{2^{\alpha-1}}{(2-\alpha)(1-\alpha)} + \frac{1}{h^2} + \frac{1}{2}, \quad b_{22} = d\frac{2^{\alpha-1}}{(2-\alpha)(1-\alpha)} + \frac{1}{h^2} + \frac{1}{2},$$

$$b_{31} = d\Big[(3/2)^{5-\alpha}\Big(\frac{1}{1-\alpha} - \frac{2}{2-\alpha} + \frac{1}{3-\alpha}\Big) - 7\frac{3^{2-\alpha}}{2^{3-\alpha}}\frac{1}{(1-\alpha)(2-\alpha)}\Big],$$

$$b_{32} = d\Big[-\frac{3^{4-\alpha}}{2^{3-\alpha}}\Big(\frac{1}{1-\alpha} - \frac{2}{2-\alpha} + \frac{1}{3-\alpha}\Big) + \frac{3^{2-\alpha}}{2^{-\alpha}}\frac{1}{(1-\alpha)(2-\alpha)}\Big] + \frac{1}{h^2} + \frac{1}{2},$$

$$b_{33} = d\Big[\frac{3^{4-\alpha}}{2^{5-\alpha}}\Big(\frac{1}{1-\alpha} - \frac{2}{2-\alpha} + \frac{1}{3-\alpha}\Big) - \frac{3^{2-\alpha}}{2^{3-\alpha}}\frac{1}{(1-\alpha)(2-\alpha)}\Big] + \frac{1}{h^2} + \frac{1}{2},$$

$$b_{41} = d\Big[\frac{1}{1-\alpha}\xi(1) - \frac{1}{2-\alpha}\eta(1)\Big],$$

$$b_{42} = d\Big[-\frac{5}{1-\alpha}\xi(1) + \frac{2}{2-\alpha}\eta(1) - \frac{2^{\alpha-2}}{2-\alpha}\Big],$$

$$b_{43} = d\Big[\frac{2}{1-\alpha}\xi(1) - \frac{1}{2-\alpha}\eta(1) - \frac{2^{\alpha-1}}{1-\alpha} + \frac{2^{\alpha-1}}{2-\alpha}\Big] + \frac{1}{h^2} + \frac{1}{2},$$

$$b_{44} = d\Big[\frac{2^{\alpha-1}}{1-\alpha} - \frac{2^{\alpha-2}}{2-\alpha}\Big] + \frac{1}{h^2} + \frac{1}{2}, \; b_{51} = d\Big[\frac{2}{1-\alpha}\xi(2) - \frac{1}{2-\alpha}\eta(2)\Big],$$

$$b_{52} = d\Big[-\frac{5}{1-\alpha}\xi(2) + \frac{2}{2-\alpha}\eta(2) + \frac{1}{1-\alpha}\xi(1) - \frac{1}{2-\alpha}\eta(1)\Big],$$
$$b_{53} = d\Big[-\frac{3}{1-\alpha}\xi(1) + \frac{2}{2-\alpha}\eta(1) + \frac{3}{1-\alpha}\xi(2) - \frac{1}{2-\alpha}\eta(2) - \frac{2^{\alpha-2}}{2-\alpha}\Big],$$
$$b_{54} = d\Big[\frac{2}{1-\alpha}\xi(1) - \frac{1}{2-\alpha}\eta(1) - \frac{2^{\alpha-1}}{1-\alpha} + \frac{2^{\alpha-1}}{2-\alpha}\Big] + \frac{1}{h^2} + \frac{1}{2},$$
$$b_{55} = d\Big[\frac{2^{\alpha-1}}{1-\alpha} - \frac{2^{\alpha-2}}{2-\alpha}\Big] + \frac{1}{h^2} + \frac{1}{2},$$

and

$$b_{ij} = \begin{cases} d\Big[\frac{1}{1-\alpha}(i-3)\xi(i-3) - \frac{1}{2-\alpha}\eta(i-3)\Big], & j=1, \\ d\Big[\frac{1}{1-\alpha}(5-2i)\xi(i-3) + \frac{2}{2-\alpha}\eta(i-3) \\ +\frac{1}{1-\alpha}(i-4)\xi(i-4) - \frac{1}{2-\alpha}\eta(i-4)\Big], & j=2, \\ d\Big[\frac{1}{1-\alpha}(i-j+1)\xi(i-j) - \frac{1}{2-\alpha}\eta(i-j) \\ +\frac{1}{1-\alpha}(2j-2i+1)\xi(i-j-1) + \frac{2}{2-\alpha}\eta(i-j-1) \\ +\frac{1}{1-\alpha}(i-j-2)\xi(i-j-2) - \frac{1}{2-\alpha}\eta(i-j-2)\Big], & 3 \le j \le i-3, \\ d\Big[\frac{3}{1-\alpha}\xi(2) - \frac{1}{2-\alpha}\eta(2) - \frac{3}{1-\alpha}\xi(1) \\ +\frac{2}{2-\alpha}\eta(1) - \frac{2^{\alpha-2}}{2-\alpha}\Big], & j=i-2, \\ d\Big[\frac{2\xi(1)}{1-\alpha} - \frac{\eta(1)}{2-\alpha} - \frac{2^{\alpha-1}}{1-\alpha} + \frac{2^{\alpha-1}}{2-\alpha}\Big] + \frac{1}{h^2} + \frac{1}{2}, & j=i-1, \\ d\Big[\frac{2^{\alpha-1}}{1-\alpha} - \frac{2^{\alpha-2}}{2-\alpha}\Big] + \frac{1}{h^2} + \frac{1}{2}, & j=i, \\ 0, & i<j\le N+1 \end{cases}$$

for $i = 6, 7, \dots, N+1$ and

$$\phi_n^k = \frac{6\sin^2(\pi nh)(k\tau)^{3-\alpha}}{\Gamma(4-\alpha)} - 2\pi^2(k\tau)^3\cos(2\pi nh) + (k\tau)^3\sin^2(\pi nh).$$

For solving of the matrix equation (4.6), we use the same algorithm as in the (4.3) with

$$u_M = [3I - 4\alpha_M + \alpha_{M-1}\alpha_M]^{-1}[(4I - \alpha_{M-1})\beta_M - \beta_{M-1}].$$

Applying the difference schemes (3.3) and (3.5) for the numerical solution of (4.1), we constructed first and second order of accuracy difference schemes. The results of computer calculations show that the Crank-Nicholson difference scheme is more accurate than first order of accuracy difference scheme. Tables 11 and 2 are constructed for $N = M = 10, 20, 40, 80$, respectively.

Table 1. Error analysis of first and second order of accuracy difference schemes for $\alpha = 1/2$

Method	N=M=10	N=M=20	N=M=40	N=M=80
1^{st} order of accuracy	1.1110	0.7049	0.3850	0.1998
2^{nd} order of accuracy	0.0953	0.0111	0.0017	3.332×10^{-4}

Table 2. Error analysis of first and second order of accuracy difference schemes for $\alpha = 1/3$

Method	N=M=10	N=M=20	N=M=40	N=M=80
1^{st} order of accuracy	1.1493	0.7333	0.4015	0.2086
2^{nd} order of accuracy	0.1015	0.0121	0.0019	7.5456×10^{-4}

Conclusion. In [12] the multidimensional fractional parabolic equation with the Dirichlet-Neumann conditions was studied. Stability estimates for the solution of the initial-boundary value problem for this fractional parabolic equation were given without proof. The stable difference schemes for this problem were presented. Stability estimates for the solution of the first order of accuracy difference scheme were given without proof. The numerical result was given for the solution of first and second order of accuracy difference schemes of one-dimensional fractional parabolic differential equations without any discuss on the realization.

In the present study, coercive stability estimates for the solution of this initial-value problem for the fractional parabolic equation with the Dirichlet-Neumann conditions are established. Stable the first and second order of approximation in t and first order of approximation in x difference schemes for this problem are considered. Coercive stability estimates for the solution of the first order of accuracy difference scheme are obtained. A procedure of modified Gauss elimination method is applied for the solution of the first and second order of accuracy difference schemes of one-dimensional fractional parabolic differential equations. Moreover, applying this approach we can constructed the first and second of approximation in t and a high order of approximation in x difference schemes. Of course, coercive stability estimates for the solution of the first order of accuracy difference scheme can be obtained.

References

[1] R. P. Agarwal, B. de Andrade, C. Cuevas; *Weighted pseudo-almost periodic solutions of a class of semilinear fractional differential equations.* Nonlinear Analysis-Real World Applications **11** (5) (2010), 3532–3554.

[2] D. Amanov, A. Ashyralyev; *Initial-boundary value problem for fractional partial differential equations of higher order.* Abstract and Applied Analysis (2012), Article ID 973102, doi: 10.1155/2012/973102, 1–14.

[3] A. Ashyralyev; *A note on fractional derivatives and fractional powers of operators.* Journal of Mathematical Analysis and Applications **357** (1) (2009), doi: 10.1016/j.jmaa.2009.04.012, 232–236.

[4] A. Ashyralyev; *Well-posedness of fractional parabolic equations.* Boundary Value Problems (2013), Article ID 31, doi: 10.1186/1687-2770-2013-31, 1–18.

[5] A. Ashyralyev; *Well-Posedness of Parabolic Differential and Difference Equations with the Fractional Differential Operator.* Malaysian Journal of Mathematical Sciences **6(S)** (2012), 73–89.

[6] A. Ashyralyev; *Well-posedness of the Basset problem in spaces of smooth functions.* Applied Mathematics Letters **24** (7) (2011), doi: 10.1016/j.aml.2011.02.002, 1176–1180.

[7] A. Ashyralyev, B. Hicdurmaz; *On the numerical solution of fractional Schrodinger differential equations with the Dirichlet condition.* International Journal of Computer Mathematics **89** (13-14) (2012), doi: 10.1080/00207160.2012.698841, 1927–1936.

[8] A. Ashyralyev, B. Hicdurmaz; *A note on the fractional Schrodinger differential equations.* Kybernetes **40** (5-6) (2011), doi: 10.1108/03684921111142287, 736–750.

[9] A. Ashyralyev, D. Agirseven; *Well-posedness of delay parabolic difference equations.* Advances in Difference Equations **2014** (18) (2014), doi: 10.1186/1687-1847-2014-18, 1–20.

[10] A. Ashyralyev, F. Dal; *Finite difference and iteration methods for fractional hyperbolic partial differential equations with the neumann condition.* Discrete Dynamics in Nature and Society (2012), Article ID 434976, doi: 10.1155/2012/434976.

[11] A. Ashyralyev, F. Dal, Z. Pinar; *A note on the fractional hyperbolic differential and difference equations.* Applied Mathematics and Computation **217** (9) (2011), doi: 10.1016 /j.amc.2010.11.017, 4654–4664.

[12] A. Ashyralyev, N. Emirov, Z. Cakir; *Fractional parabolic differential and difference equations with the dirichlet-neumann conditions.* First International Conference on Analysis and Applied Mathematics **1470** (2012), doi: 10.1063/1.4747641, 69–72.

[13] A. Ashyralyev, N. Nalbant, Y. Sozen; *Structure of fractional spaces generated by second order difference operators.* Journal of the Franklin Institute-Engineering and Applied Mathematics **351** (2) (2014), doi: 10.1016/j.jfranklin.2013.07.009, 713–731.

[14] A. Ashyralyev, Y. A. Sharifov; *Existence and uniqueness of solutions for the system of nonlinear fractional differential equations with nonlocal and integral boundary conditions.* Abstract and Applied Analysis (2012), doi: 10.1155/2012/594802, Article ID 594802.

[15] A. Ashyralyev, Z. Cakir; *On the numerical solution of fractional parabolic partial differential equations with the dirichlet condition.* Discrete Dynamics in Nature and Society (2012), doi: 10.1155/2012/696179, Article ID 696179.

[16] A. S. Berdyshev, A. Cabada, E. T. Karimov; *On a non-local boundary problem for a parabolic-hyperbolic equation involving a Riemann-Liouville fractional differential operator.* Nonlinear Analysis–Theory Methods and Applications **75** (6) (2012), doi: 10.1016/j.na.2011.12.033, 3268–3273.

[17] S. Q. Bu, P. Clement, S. Guerre-Delabriere; *Regularity of pairs of positive operators.* Illinois Journal of Mathematics **42** (3) (1998), 357–370.

[18] Z. Cakir; *Stability of difference schemes for fractional parabolic pde with the dirichlet-neumann conditions.* Abstract and Applied Analysis (2012), doi: 10.1155/2012/463746, Article ID 463746, 1–17.

[19] J. Y. Cao, C. J. Xu; *A high order schema for the numerical solution of the fractional ordinary differential equations.* Journal of Computational Physics **238** (2013), doi: 10.1016/j.jcp.2012.12.013, 154–168.

[20] Ph. Clement; *On (L_p-L_q) Coerciveness for a Class of Integrodifferential Equation on the Line.* Two lectures at the 23rd annual Voronezh Winter School of Mathematics (USSR), Tech. Report 5-4-90, Voronezh, (1990).

[21] E. Cuesta, C. Lubich, C. Palencia; *Convolution quadrature time discretization of fractional diffusion-wave equations.* Mathematics of Computation **75** (254) (2006), doi: 10.1090/S0025-5718-06-01788-1, 673–696.

[22] G. Da Prato, P. Grisvard; *Sums of linear-operators and operational differential equations.* Journal De Mathematiques Pures Et Appliquees **54** (1975), 305–387.

[23] F. Dal; *Application of variational iteration method to fractional hyperbolic partial differential equations.* Mathematical Problems in Engineering (2009), doi: 10.1155/2009/824385, Article ID 824385.

[24] M. De la Sen; *About robust stability of caputo linear fractional dynamic systems with time delays through fixed point theory.* Fixed Point Theory and Applications (2011), doi: 10.1155/2011/867932, Article ID 867932.

[25] M. De la Sen; *Positivity and stability of the solutions of caputo fractional linear time-invariant systems of any order with internal point delays.* Abstract and Applied Analysis (2011), doi: 10.1155/2011/161246, Article ID 161246.

[26] M. De la Sen, R. P. Agarwal, A. Ibeas, et al; *On the existence of equilibrium points, boundedness, oscillating behavior and positivity of a sveirs epidemic model under constant and impulsive vaccination.* Advances in Difference Equations (2011), doi: 10.1155/2011/748608, Article ID 748608.

[27] A. S. Erdogan, H. Uygun; *A note on the inverse problem for a fractional parabolic equation.* Abstract and Applied Analysis (2012), doi: 10.1155/2012/276080, Article ID 276080.

[28] J. Fan, J. H. He; *Fractal derivative model for air permeability in hierarchic porous media.* Abstract and Applied Analysis (2012), doi: 10.1155/2012/354701, Article ID 354701.

[29] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo; *Theory and Applications of Fractional Differential Equations.* Elsevier, Amsterdam (2006).

[30] M. Kirane, Y. Laskri; *Nonexistence of global solutions to a hyperbolic equation with a space-time fractional damping*. Applied Mathematics and Computation **167** (2) (2005), doi: 10.1016/j.amc.2004.08.038, 1304–1310.
[31] S. G. Krein; *Linear Differential Equations in a Banach Space*. Nauka, Moscow, (1966).
[32] V. Lakshmikantham, A. Vatsala, *Basic theory of fractional differential equations*. Nonlinear Analysis-Theory Methods and Applications **69** (8) (2008), doi: 10.1016/j.na.2007.08.042, 2677–2682.
[33] C. G. Li, M. Kostic, M. Li, S. Piskarev; *On a class of time-fractional differential equations*. Fractional Calculus and Applied Analysis **15** (4) (2012), doi: 10.2478/s13540-012-0044-x, 639–668.
[34] I. Podlubny; *Fractional Differential Equations*. Mathematics in Science and Engineering **198**, Academic Press, San Diego, California, (1999).
[35] P. E. Sobolevskii; *Fractional powers of coercively positive sums of operators*. Doklady Akademii Nauk (SSSR) **16** (1975), 1638–1641.
[36] P. E. Sobolevskii; *Fractional powers of coercive-positive sums of operators*. Siberian Mathematical Journal **18** (3) (1977), 637–657.
[37] M. Z. Solomyak; *Estimation of norm of resolvent of elliptic operator in spaces L_p*. Usp. Mat. Nauk **15** (6) (1960), 141–148.
[38] H. B. Stewart; *Generation of analytic semigroups by strongly elliptic operators*. Tran. Amer. Math. Soc. **199** (1974), doi: 10.2307/1996879, 141–162.
[39] L. J. Su, P. Cheng; *A high-accuracy moc/fd method for solving fractional advection-diffusion equations*. Journal of Applied Mathematics (2013), doi: 10.1155/2013/648595, Article ID 648595.
[40] A. Yakar, M. E. Koksal; *Existence results for solutions of nonlinear fractional differential equations*. Abstract and Applied Analysis (2012), doi: 10.1155/2012/267108, Article ID 267108.
[41] H. Yang; *Existence of mild solutions for a class of fractional evolution equations with compact analytic semigroup*. Abstract and Applied Analysis (2012), doi: 10.1155/2012/903518, Article ID 903518.
[42] C. J. Yuan; *Multiple positive solutions for $(n-1,1)$-type semipositone conjugate boundary value problems for coupled systems of nonlinear fractional differential equations*. Electronic Journal of Qualitative Theory of Differential Equations **13** (2011), 1–12.
[43] C. J. Yuan; *Multiple positive solutions for $(n-1,1)$-type semipositone conjugate boundary value problems of nonlinear fractional differential equations*. Electronic Journal of Qualitative Theory of Differential Equations **36** (2010), 1–12.
[44] C. J. Yuan; *Multiple positive solutions for semipositone (n,p)-type boundary value problems of nonlinear fractional differential equations*. Analysis and Applications **9** (1) (2011), doi: 10.1142/S0219530511001753, 97–112.
[45] C. J. Yuan; *Two positive solutions for $(n-1,1)$-type semipositone integral boundary value problems for coupled systems of nonlinear fractional differential equations*. Communications in Nonlinear Science and Numerical Simulation **17** (2) (2012), doi: 10.1016/j.cnsns.2011.06.008, 930–942.
[46] C. J. Yuan, D. Jiang, D. O'Regan, R. P. Agarwal; *Multiple positive solutions to systems of nonlinear semipositone fractional differential equations with coupled boundary conditions*. Electronic Journal of Qualitative Theory of Differential Equations **13** (2012), 1–17.

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