Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 94, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF PERIODIC SOLUTION TO SINGULAR PROBLEMS WITH IMPULSES

JUNTAO SUN, JIFENG CHU

Abstract. In this article we give a necessary sufficient conditions for the existence of periodic solutions to impulsive periodic solution for a singular differential equation. The proof is based on the variational method.

1. Introduction

In this article we discuss the T-periodic solution for the second-order singular problem with impulsive effects

$$\begin{gathered} u''(t) - \frac{1}{u^{\alpha}(t)} = e(t), \quad \text{a.e. } t \in (0,T), \\ \Delta u'(t_j) = b_j, \quad j = 1,2,\dots,p-1, \end{gathered} \tag{1.1}$$

where $\alpha \geq 1$, $e \in L^1([0,T],\mathbb{R})$ is T-periodic, $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-)$ with $u'(t_j^{\pm}) = \lim_{t \to t_j^{\pm}} u'(t)$; t_j, $j = 1,2,\dots,p-1$, are the instants where the impulses occur and $0 = t_0 < t_1 < t_2 < \dots < t_{p-1} < t_p = T$, $t_{j+p} = t_j + T$; and b_j $(j = 1,2,\dots,p-1)$ are constants.

Impulsive effects occur widely in many evolution processes in which their states are changed abruptly at certain moments of time. In the past few decades, impulsive differential equations have been extensively studied by many researchers [6, 11, 13, 14, 15, 16, 17]. In particular, In 2008, Tian and Ge [17] studied the existence of solutions for impulsive differential equations by using a variational method. Later, Nieto and O'Regan [11] further developed the variational framework for impulsive problems and established existence results for a class of impulsive differential equations with Dirichlet boundary conditions. From then on, the variational method has been a powerful tool in the study of impulsive differential equations. On the other hand, singular differential equations with different kinds of boundary conditions have also been investigated widely in the literature by using either topological methods or variational methods; see [1, 2, 3, 4, 5, 7, 8, 9] and the references therein.

2000 *Mathematics Subject Classification.* 34B15.
Key words and phrases. Positive periodic solution; singular differential equations; impulses; variational methods.

Submitted October 30, 2013. Published April 10, 2014.

In 1987, Lazer and Solimini [10] considered a the second order singular problem

$$u''(t) - \frac{1}{u^\alpha(t)} = e(t), \quad t \in (0, T). \tag{1.2}$$

By using the method of upper and lower solutions, they obtained a famous sufficient and necessary condition on positive T-periodic solution for Problem (1.2) as follows

Theorem 1.1 ([10]). *Assume that $e \in L^1([0,T], \mathbb{R})$ is T-periodic. Then Problem* (1.2) *has a positive T-periodic weak solution if and only if $\int_0^T e(t)dt < 0$.*

Motivated by the above fact, in the present paper we shall consider Problem (1.2) with impulsive effects, i.e., Problem (1.1), and also obtain a sufficient and necessary condition on T-periodic solution. It is worth emphasizing that the method used by us is a variational method, which is different from that in Theorem 1.1. Furthermore, we also point out the dynamical differences between singular problems and singular problems with impulses.

Our results are presented as follows.

Theorem 1.2. *Assume that $e \in L^1([0,T], \mathbb{R})$ is T-periodic. Then Problem* (1.1) *has a positive T-periodic weak solution $u \in H_T^1$ if and only if $\sum_{j=1}^{p-1} b_j + \int_0^T e(t)dt < 0$.*

Remark 1.3. From Theorem 1.2 we can see that if $\int_0^T e(t)dt \geq 0$, but $\sum_{j=1}^{p-1} b_j + \int_0^T e(t)dt < 0$, then Problem (1.1) still admits a positive T-periodic solution. This shows that the existence of positive T-periodic solution for Problem (1.1) depends on the forced term e and impulsive functions b_j together, not single one.

2. Preliminaries

Set

$$\begin{aligned} H_T^1 = \big\{ &u : \mathbb{R} \to \mathbb{R} | u \text{ is absolutely continuous, } u' \in L^2((0,T), \mathbb{R}) \\ &\text{and } u(t) = u(t+T) \text{ for } t \in \mathbb{R} \big\} \end{aligned}$$

with the inner product

$$(u, v) = \int_0^T u(t)v(t)dt + \int_0^T u'(t)v'(t)dt, \quad \forall u, v \in H_T^1.$$

The corresponding norm is defined by

$$\|u\|_{H_T^1} = \Big(\int_0^T |u(t)|^2 dt + \int_0^T |u'(t)|^2 dt \Big)^{1/2}, \quad \forall u \in H_T^1.$$

Then H_T^1 is a Banach space (in fact it is a Hilbert space).

To study Problem (1.1), for any $\lambda \in (0,1)$ we consider the following modified problem

$$\begin{gathered} u''(t) + f_\lambda(u(t)) = e(t), \quad \text{a.e. } t \in (0, T), \\ \Delta u'(t_j) = b_j, \quad j = 1, 2, \dots, p-1, \end{gathered} \tag{2.1}$$

where $f_\lambda : \mathbb{R} \to \mathbb{R}$ is defined by

$$f_\lambda(s) = \begin{cases} -\frac{1}{s^\alpha}, & s \geq \lambda, \\ -\frac{1}{\lambda^\alpha}, & s < \lambda. \end{cases}$$

Now we introduce the following concept of a weak solution for Problem (2.1).

Definition 2.1. We say that a function $u \in H_T^1$ is a weak solution of Problem (2.1) if

$$\int_0^T u'(t)v'(t)dt + \sum_{j=1}^{p-1} b_j v(t_j) - \int_0^T f_\lambda(u(t))v(t)dt + \int_0^T e(t)v(t)dt = 0$$

holds for any $v \in H_T^1$.

Let $F_\lambda \in C^1(\mathbb{R}, \mathbb{R})$ be defined by

$$F_\lambda(s) = \int_1^s f_\lambda(t)dt$$

and consider the functional $\Phi_\lambda : H_T^1 \to \mathbb{R}$ defined by

$$\Phi_\lambda(u) := \frac{1}{2}\int_0^T |u'(t)|^2 dt + \sum_{j=1}^{p-1} b_j u(t_j) - \int_0^T F_\lambda(u(t))dt + \int_0^T e(t)u(t)dt. \tag{2.2}$$

Clearly, Φ_λ is well defined on H_T^1, continuously Gâteaux differentiable functional whose derivative is

$$\Phi'_\lambda(u)v = \int_0^T u'(t)v'(t)dt + \sum_{j=1}^{p-1} b_j v(t_j) - \int_0^T f_\lambda(u(t))v(t)dt + \int_0^T e(t)v(t)dt,$$

for any $v \in H_T^1$. Moreover, it is easy to verify that Φ_λ is weakly lower semicontinuous. Furthermore, by the standard discussion, the critical points of Φ_λ are the weak solutions of Problem (2.1).

3. Proof of Theorem 1.2

Proof. First we show that if $u \in H_T^1$ is a positive T-periodic weak solution of Problem (1.1), then $\sum_{j=1}^{p-1} b_j + \int_0^T e(t)dt < 0$.

Integrating the first equation of Problem (1.1) from 0 to T, one has

$$\int_0^T u''(t)dt - \int_0^T \frac{1}{u^\alpha(t)}dt = \int_0^T e(t)dt. \tag{3.1}$$

The first term one the left-hand side satisfies

$$\int_0^T u''(t)dt = \sum_{j=0}^{p-1} \int_{t_j}^{t_{j+1}} u''(t)dt,$$

and

$$\int_{t_j}^{t_{j+1}} u''(t)dt = u'(t_{j+1}^-) - u'(t_j^+).$$

Thus,

$$\begin{aligned}
\int_0^T u''(t)dt &= \sum_{j=0}^{p-1}(u'(t_{j+1}^-) - u'(t_j^+)) \\
&= -\sum_{j=1}^{p-1} \Delta u'(t_j) + u'(T) - u'(0) \qquad (3.2)\\
&= -\sum_{j=1}^{p-1} b_j.
\end{aligned}$$

By (3.1) and (3.2) we have

$$0 > -\int_0^T \frac{1}{u^\alpha(t)}dt = \sum_{j=1}^{p-1} b_j + \int_0^T e(t)dt.$$

Now we prove that if $\sum_{j=1}^{p-1} b_j + \int_0^T e(t)dt < 0$, then Problem (1.1) has a positive T-periodic weak solution $u \in H_T^1$. The proof is based on the mountain pass theorem, see [12]. We divide it into four steps.

Step 1. Let a sequence $\{u_n\}$ in H_T^1 satisfy $\Phi_\lambda(u_n)$ be bounded and $\Phi'_\lambda(u_n) \to 0$, i.e., there exist a constant $c_1 > 0$ and a sequence $\{\epsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ with $\epsilon_n \to 0$ as $n \to +\infty$ such that for all n,

$$\Big| \int_0^T \big[\frac{1}{2}|u_n'(t)|^2 - F_\lambda(u_n(t)) + e(t)u_n(t)\big]dt + \sum_{j=1}^{p-1} b_j u_n(t_j)\Big| \le c_1, \tag{3.3}$$

and for every $v \in H_T^1$,

$$\Big| \int_0^T [u_n'(t)v'(t)) - f_\lambda(u_n(t))v(t) + e(t)v(t)]dt + \sum_{j=1}^{p-1} b_j v(t_j)\Big| \le \epsilon_n \|v\|_{H_T^1}. \tag{3.4}$$

Now we show that $\{u_n\}$ is bounded in H_T^1. Taking $v(t) \equiv -1$ in (3.4) one has

$$\Big| \int_0^T [f_\lambda(u_n(t)) - e(t)]dt - \sum_{j=1}^{p-1} b_j\Big| \le \epsilon_n\sqrt{T} \quad \text{for all } n,$$

which implies

$$\big| \int_0^T f_\lambda(u_n(t))dt\big| \le \epsilon_n\sqrt{T} + \int_0^T e(t)dt + \sum_{j=1}^{p-1} |b_j| := c_2.$$

Note that for any $t \in [0,T]$, $f_\lambda(u_n(t)) < 0$. Thus

$$\int_0^T |f_\lambda(u_n(t))|dt = \big| \int_0^T f_\lambda(u_n(t))dt\big| \le c_2.$$

On the other hand, take, in (3.4), $v(t) \equiv w_n(t) := u_n(t) - \bar{u}_n$, where $\bar{u}_n = \frac{1}{T}\int_0^T u_n(t)dt$, by [12, Proposition 1.1] we have

$$\begin{aligned} c_3\|w_n\|_{H_T^1} &\ge \int_0^T [w_n'(t)^2 - f_\lambda(u_n(t))w_n(t) + e(t)w_n(t)]dt + \sum_{j=1}^{p-1} b_j w_n(t_j) \\ &\ge \|w_n'\|_{L^2}^2 - (c_2 + \|e\|_{L^1})\|w_n\|_{L^\infty} - \sum_{j=1}^{p-1} |b_j|\|w_n\|_{L^\infty} \\ &= \|w_n'\|_{L^2}^2 - (c_2 + \|e\|_{L^1} + \sum_{j=1}^{p-1} |b_j|)\|w_n\|_{L^\infty} \\ &\ge \|w_n'\|_{L^2}^2 - c_4\|w_n\|_{H_T^1}, \end{aligned}$$

where c_3 and c_4 are two positive constants. Thus,

$$\|w_n'\|_{L^2}^2 \le (c_3 + c_4)\|w_n\|_{H_T^1}.$$

Consequently, using the Wirtinger inequality, we obtain the existence of a positive constant c_5 such that

$$\|u_n'\|_{L^2}^2 \le c_5. \tag{3.5}$$

Now, suppose that $\|u_n\|_{H_T^1} \to +\infty$ as $n \to +\infty$. Since (3.5) holds, we have, passing to subsequence if necessary, that either

$$\begin{gathered} M_n := \max u_n \to +\infty \quad \text{as } n \to +\infty, \quad \text{or} \\ m_n := \min u_n \to -\infty \quad \text{as } n \to +\infty. \end{gathered}$$

(i) Assume that the first possibility occurs. In view to the fact that $f_\lambda < 0$, one has

$$\begin{aligned} &\int_0^T [F_\lambda(u_n(t)) - e(t)u_n(t)]dt - \sum_{j=1}^{p-1} b_j u_n(t_j) \\ &= \int_0^T \Big[\Big(\int_1^{M_n} f_\lambda(s)ds - \int_{u_n(t)}^{M_n} f_\lambda(s)ds\Big) - e(t)u_n(t)\Big]dt - M_n \sum_{j=1}^{p-1} b_j \\ &\quad - \sum_{j=1}^{p-1} b_j\,(u_n(t_j) - M_n) \\ &\ge \int_0^T F_\lambda(M_n)dt - \int_0^T M_n e(t)dt - \max_{t\in[0,T]} |M_n - u_n(t)| \int_0^T |e(t)|dt - M_n \sum_{j=1}^{p-1} b_j \\ &\quad - \max_{t\in[0,T]} |M_n - u_n(t)| \sum_{j=1}^{p-1} |b_j| \\ &\ge TF_\lambda(M_n) - M_n\Big(\int_0^T e(t)dt + \sum_{j=1}^{p-1} b_j\Big) - \Big(\|e\|_{L^1} + \sum_{j=1}^{p-1} |b_j|\Big)|M_n - m_n| \\ &= TF_\lambda(M_n) - M_n\Big(\int_0^T e(t)dt + \sum_{j=1}^{p-1} b_j\Big) - \Big(\|e\|_{L^1} + \sum_{j=1}^{p-1} |b_j|\Big)\Big|\int_{\bar t_n}^{\hat t_n} u_n'(t)dt\Big| \\ &\ge TF_\lambda(M_n) - M_n\Big(\int_0^T e(t)dt + \sum_{j=1}^{p-1} b_j\Big) - \Big(\|e\|_{L^1} + \sum_{j=1}^{p-1} |b_j|\Big)\int_{\bar t_n}^{\hat t_n} |u_n'(t)|dt, \end{aligned}$$

where $u_n(\hat t_n) = M_n$ and $u_n(\bar t_n) = m_n$. Thus, using the Hölder inequality, one has

$$\begin{aligned} &- M_n\Big(\int_0^T e(t)dt + \sum_{j=1}^{p-1} b_j\Big) + TF_\lambda(M_n) \\ &\le \int_0^T [F_\lambda(u_n(t)) - e(t)u_n(t)]dt - \sum_{j=1}^{p-1} b_j u_n(t_j) + \sqrt{T}\Big(\|e\|_{L^1} + \sum_{j=1}^{p-1} |b_j|\Big)\|u_n'\|_{L^2}. \end{aligned} \tag{3.6}$$

If $\alpha = 1$, then $F_\lambda(M_n) = -\ln M_n$. By (3.6) one has

$$-M_n\Big(\int_0^T e(t)dt + \sum_{j=1}^{p-1} b_j\Big) - T\ln M_n \to +\infty \quad \text{as } n \to +\infty.$$

If $\alpha > 1$, then $F_\lambda(M_n) = -\frac{1}{\alpha-1}(\frac{1}{M_n^{\alpha-1}} - 1)$. By (3.6) we obtain

$$-M_n\Big(\int_0^T e(t)dt + \sum_{j=1}^{p-1} b_j\Big) - \frac{1}{\alpha-1}(\frac{1}{M_n^{\alpha-1}} - 1) \to +\infty \quad \text{as } n \to +\infty.$$

From (3.3) and (3.5), we see that the right hand side of (3.6) is bounded, which is a contradiction.

(ii) Assume the second possibility occurs; i.e., $m_n \to -\infty$ as $n \to +\infty$. We replace M_n by $-m_n$ in the preceding arguments, and we also get a contradiction. So $\{u_n\}$ is bounded in H_T^1.

Since H_T^1 is a reflexive Banach space, there exists a subsequence of $\{u_n\}$, denoted again by $\{u_n\}$ for simplicity, and $u \in H_T^1$ such that $u_n \rightharpoonup u$ in H_T^1; then, by the Sobolev embedding theorem, we get $u_n \to u$ in $C([0,T])$ and $u_n \to u$ in $L^2([0,T])$. So

$$\begin{gathered}
\int_0^T (f_\lambda(u_n(t)) - f_\lambda(u(t)))(u_n(t) - u(t))dt \to 0,\\
\sum_{j=1}^{p-1} b_j(u_n(t_j) - u(t_j)) \to 0,\\
\int_0^T e(t)(u_n(t) - u(t))dt \to 0,\\
(\Phi'_\lambda(u_n) - \Phi'_\lambda(u))(u_n - u) \to 0, \quad \text{as } n \to \infty.
\end{gathered} \tag{3.7}$$

By (3.6), (3.7) and the fact that $u_n \to u$ in $L^2([0,T])$, we have $\|u_n - u\|_{H_T^1} \to 0$ as $n \to \infty$. That is, $\{u_n\}$ strongly converges to u in H_T^1, which means that the Palais-Smale condition holds for Φ_λ.

Step 2. Let

$$\Omega = \big\{u \in H_T^1 | \min_{t\in[0,T]} u(t) > 1\big\},$$

and

$$\partial\Omega = \{u \in H_T^1 | u(t) \ge 1 \text{ for all } t \in (0,T),\ \exists t_u \in (0,T) : u(t_u) = 1\}.$$

We show that there exists $d > 0$ such that $\inf_{u\in\partial\Omega} \Phi_\lambda(u) \ge -d$ whenever $\lambda \in (0,1)$.

For any $u \in \partial\Omega$, there exists some $t_u \in (0,T)$ such that $\min_{t\in[0,T]} u(t) = u(t_u) = 1$. By (2.2), and extending the functions by T-periodicity, we have

$$\begin{aligned}
\Phi_\lambda(u) &= \int_{t_u}^{t_u+T} \big[\frac{1}{2}|u'(t)|^2 - F_\lambda(u(t)) + e(t)u(t)\big]dt + \sum_{j=1}^{p-1} b_j u(t_j)\\
&\ge \frac{1}{2}\int_{t_u}^{t_u+T} |u'(t)|^2 dt + \int_{t_u}^{t_u+T} e(t)(u(t)-1)dt + \int_{t_u}^{t_u+T} e(t)dt\\
&\quad + \sum_{j=1}^{p-1} b_j(u(t_j) - 1) + \sum_{j=1}^{p-1} b_j\\
&\ge \frac{1}{2}\|u'\|_{L^2}^2 - \Big(\|e\|_{L^1} + \sum_{j=1}^{p-1}|b_j|\Big)\max_{t\in[0,T]}(u(t)-1) - \|e\|_{L^1} + \sum_{j=1}^{p-1} b_j\\
&= \frac{1}{2}\|u'\|_{L^2}^2 - \Big(\|e\|_{L^1} + \sum_{j=1}^{p-1}|b_j|\Big)\int_{t_u}^{\check{t}_u} u'(t)dt - \|e\|_{L^1} + \sum_{j=1}^{p-1} b_j
\end{aligned}$$

$$\geq \frac{1}{2}\|u'\|_{L^2}^2 - \Big(\|e\|_{L^1} + \sum_{j=1}^{p-1}|b_j|\Big)\int_{t_u}^{t_u+T}|u'(t)|dt - \|e\|_{L^1} + \sum_{j=1}^{p-1} b_j,$$

where $\check{t}_u \in [0,T]$ and $\max_{t\in[0,T]} u(t) = u(\check{t}_u)$. Applying the Hölder inequality, we get

$$\Phi_\lambda(u) \geq \frac{1}{2}\|u'\|_{L^2}^2 - \sqrt{T}\Big(\|e\|_{L^1} + \sum_{j=1}^{p-1}|b_j|\Big)\|u'\|_{L^2} - \|e\|_{L^1} + \sum_{j=1}^{p-1} b_j.$$

The above inequality shows that

$$\Phi_\lambda(u) \to +\infty \quad as\|u'\|_{L^2} \to +\infty.$$

For any $u \in \partial\Omega$, it is easy to verify the fact that $\|u\|_{H_T^1} \to +\infty$ is equivalent to $\|u'\|_{L^2} \to +\infty$. Indeed, when $\|u'\|_{L^2} \to +\infty$, it is clear that $\|u\|_{H_T^1} \to +\infty$. When $\|u\|_{H_T^1} \to +\infty$. Assume that $\|u'\|_{L^2}$ is bounded, then $\|u\|_{L^2} \to +\infty$. Since $\min_{t\in[0,T]} u(t) = 1$, we have

$$u(t) - 1 = \int_{t_u}^{t} u'(s)ds \leq \int_0^T |u'(s)|ds \leq \sqrt{T}\Big(\int_0^T |u'(t)|^2 dt\Big)^{1/2}.$$

Therefore, u is bounded in $L^2(0,T)$, which is a contradiction. Hence

$$\Phi_\lambda(u) \to +\infty \quad \text{as } \|u\|_{H_T^1} \to +\infty,\ \forall u \in \partial\Omega,$$

which shows that Φ_λ is coercive. Thus it has a minimizing sequence. The weak lower semi-continuity of Φ_λ yields

$$\inf_{u\in\partial\Omega} \Phi_\lambda(u) > -\infty.$$

It follows that there exists $d > 0$ such that $\inf_{u\in\partial\Omega} \Phi_\lambda(u) > -d$ for all $\lambda \in (0,1)$.

Step 3. We prove that there exists $\lambda_0 \in (0,1)$ with the property that for every $\lambda \in (0,\lambda_0)$, any solution u of Problem (2.1) satisfying $\Phi_\lambda(u) > -d$ is such that $\min_{u\in[0,T]} u(t) \geq \lambda_0$, and hence u is a solution of Problem (1.1).

Assume on the contrary that there are sequences $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ such that

(i) $\lambda_n \leq \frac{1}{n}$;
(ii) u_n is a solution of Problem (2.2) with $\lambda = \lambda_n$;
(iii) $\Phi_{\lambda_n}(u_n) \geq -d$;
(iv) $\min_{t\in[0,T]} u_n(t) < \frac{1}{n}$.

Since $f_{\lambda_n} < 0$ and $\int_0^T [f_{\lambda_n}(u_n(t)) - e(t)]dt = 0$, one has

$$\|f_{\lambda_n}(u_n(\cdot))\|_{L^1} \leq c_7, \quad \text{for some constant } c_7 > 0.$$

Hence

$$\|u_n'\|_{L^\infty} \leq c_8, \quad \text{for some constant } c_8 > 0. \tag{3.8}$$

From $\Phi_{\lambda_n}(u_n) \geq -d$ it follows that there must exist two constants l_1 and l_2, with $0 < l_1 < l_2$ such that

$$\max\{u_n(t); t \in [0,T]\} \subset [l_1, l_2].$$

If not, u_n would tend uniformly to 0 or $+\infty$. In both cases, by (3.8), we have

$$\Phi_{\lambda_n}(u_n) \to -\infty \quad \text{as } n \to +\infty,$$

which contradicts $\Phi_{\lambda_n}(u_n) \geq -d$.

Let τ_n^1, τ_n^2 be such that, for n large enough

$$u_n(\tau_n^1) = \frac{1}{n} < l_1 = u_n(\tau_n^2).$$

Multiplying the differential equation in (2.1) by u_n' and integrating it on $[\tau_n^1, \tau_n^2]$, or on $[\tau_n^2, \tau_n^1]$, we get

$$\Psi := \int_{\tau_n^1}^{\tau_n^2} u_n''(t)u_n'(t)dt + \int_{\tau_n^1}^{\tau_n^2} f_{\lambda_n}(u_n(t))u_n'(t)dt = \int_{\tau_n^1}^{\tau_n^2} e(t)u_n'(t)dt. \tag{3.9}$$

It is easy to verify that

$$\Psi = \Psi_1 + \frac{1}{2}[u_n'^2(\tau_n^2) - u_n'^2(\tau_n^1)],$$

where

$$\Psi_1 = \int_{\tau_n^1}^{\tau_n^2} f_{\lambda_n}(u_n(t))u_n'(t)dt.$$

From (3.5) and (3.9) it follows that Ψ is bounded, and consequently Ψ_1 is bounded.

On the other hand, it is easy to see that

$$f_{\lambda_n}(u_n(t))u_n'(t) = \frac{d}{dt}[F_{\lambda_n}(u_n(t))].$$

Thus, we have

$$\Psi_1 = F_{\lambda_n}(l_1) - F_{\lambda_n}\big(\frac{1}{n}\big).$$

From the fact that $F_{\lambda_n}\left(\frac{1}{n}\right) \to +\infty$ as $n \to +\infty$, we obtain $\Psi_1 \to -\infty$, i.e., Ψ_1 is unbounded. This is a contradiction.

Step 4. Φ has a mountain-pass geometry for $\lambda \le \lambda_0$. Fix $\lambda \in (0, \lambda_0]$, one has

$$\begin{aligned} F_\lambda(0) &= \int_1^0 f_\lambda(s)ds = -\int_0^1 f_\lambda(s)ds \\ &= -\int_0^\lambda f_\lambda(s)ds - \int_\lambda^1 f_\lambda(s)ds \\ &= \frac{1}{\lambda^{\alpha-1}} - \int_\lambda^1 f_\lambda(s)ds, \end{aligned} \tag{3.10}$$

which implies that

$$F_\lambda(0) > -\int_\lambda^1 f_\lambda(s)ds = \int_1^\lambda f_\lambda(s)ds = F_\lambda(\lambda).$$

Thus we have

$$\Phi_\lambda(0) = -TF_\lambda(0) < -TF_\lambda(\lambda) = \begin{cases} T\ln\lambda, & \text{if } \alpha = 1, \\ -\frac{T}{\alpha-1}\big(\frac{1}{\lambda^{\alpha-1}} - 1\big), & \text{if } \alpha > 1. \end{cases} \tag{3.11}$$

We choose $\lambda \in (0, \lambda_0] \cap (0, e^{-d}) \cap (0, [\frac{T}{T+d(\alpha-1)}]^{1/(\alpha-1)})$, then it follows from (3.11) that $\Phi_\lambda(0) < -d$.

Also, we can choose a constant $R > 1$ enough large such that

$$-\Big(\sum_{j=1}^{p-1} b_j + \int_0^T e(t)dt\Big)R - \frac{T}{\alpha-1}\big(1 - \frac{1}{R^{\alpha-1}}\big) > d,$$

and
$$-\Big(\sum_{j=1}^{p-1} b_j + \int_0^T e(t)dt\Big)R - T\ln R > d.$$
Thus, $R \in H_T^1$ and
$$\begin{aligned}
\Phi_\lambda(R) &= R\sum_{j=1}^{p-1} b_j - TF_\lambda(R) + R\int_0^T e(t)dt \\
&\le \begin{cases} \sum_{j=1}^{p-1} b_jR + T\ln R + R\int_0^T e(t)dt, & \text{if } \alpha = 1, \\ \sum_{j=1}^{p-1} b_jR + \frac{T}{\alpha-1}\big(1-\frac{1}{R^{\alpha-1}}\big) + R\int_0^T e(t)dt, & \text{if } \alpha > 1. \end{cases} \\
&\le \begin{cases} \big(\sum_{j=1}^{p-1} b_j + \int_0^T e(t)dt\big)R + T\ln R, & \text{if } \alpha = 1, \\ \big(\sum_{j=1}^{p-1} b_j + \int_0^T e(t)dt\big)R + \frac{T}{\alpha-1}\big(1-\frac{1}{R^{\alpha-1}}\big), & \text{if } \alpha > 1. \end{cases} \\
&< -d.
\end{aligned}$$
Since Ω is a neighborhood of R, $0 \notin \Omega$ and
$$\max\{\Phi_\lambda(0), \Phi_\lambda(R)\} < \inf_{x\in\partial\Omega} \Phi_\lambda(u),$$
Step 1 and Step 2 imply that Φ_λ has a critical point u_λ such that
$$\Phi_\lambda(u_\lambda) = \inf_{h\in\Gamma}\max_{s\in[0,1]} \Phi_\lambda(h(s)) \ge \inf_{x\in\partial\Omega} \Phi_\lambda(u),$$
where
$$\Gamma = \{h \in C([0,1], H_T^1) : h(0) = 0, h(1) = R\}.$$
Since $\inf_{u\in\partial\Omega} \Phi_\lambda(u_\lambda) \ge -d$, it follows from Step 3 that u_λ is a positive solution of Problem (1.1). The proof of the main result is complete. $\square$

Acknowledgments. Juntao Sun was supported by the National Natural Science Foundation of China (Grant No. 11201270), Shandong Natural Science Foundation (Grant No. ZR2012AQ010), and Young Teacher Support Program of Shandong University of Technology. Jifeng Chu was supported by the National Natural Science Foundation of China (Grant Nos. 11171090, 11271078, and 11271333), the Program for New Century Excellent Talents in University (Grant No. NCET-10-0325), and China Postdoctoral Science Foundation funded project (Grant Nos. 20110491345 and 2012T50431).

References

[1] R. P. Agarwal, D. O'Regan; Existence criteria for singular boundary value problems with sign changing nonlinearities. *J. Differential Equations.* **183**, 409-433(2002).
[2] R. P. Agarwal, K. Perera, D. O'Regan; Multiple positive solutions of singular problems by variational methods. *Proc. Amer. Math. Soc.* **134**, 817-824(2005).
[3] A. Boucherif, N. Daoudi-Merzagui; Periodic solutions of singular nonautonomous second order differential equations. *NoDEA Nonlinear Differential Equations Appl.* **15**, 147-158(2008).
[4] L. Chen, C. C. Tisdell, R. Yuan; On the solvability of periodic boundary value problems with impulse. *J. Math. Anal. Appl.* **331**, 233-244(2007).
[5] J. Chu, N. Fan, P. J. Torres; Periodic solutions for second order singular damped differential equations. *J. Math. Anal. Appl.* **388**, 665-675(2012).
[6] J. Chu, J. J. Nieto; Impulsive periodic solution of first-order singular differential equations. *Bull. London. Math. Soc.* **40**, 143-150(2008).
[7] J. Chu, D. O'Regan; Multiplicity results for second order non-autonomous singular Dirichlet systems. *Acta Appl. Math.* **105**, 323-338(2009).

[8] J. Chu, P.J. Torres, M. Zhang; Periodic solutions of second order non-autonomous singular dynamical systems. *J. Differential Equations* **239**, 196-212(2007).
[9] R. Hakl, P. J. Torres; On periodic solutions of second-order differential equations with attractive-repulsive singularities. *J. Differential Equations* **248**, 111-126(2010).
[10] A.C. Lazer, S. Solimini; On periodic solutions of nonlinear differential equations with singularities. *Proc. Amer. Math. Soc.* **99**, 109-114(1987).
[11] J.J. Nieto, D. O'Regan; Variational approach to impulsive differential equations. *Nonlinear Anal. Real World Appl.* **10**, 680-690(2009).
[12] J. Mawhin, M. Willem; *Critical Point Theory and Hamiltonian Systems*. Springer, 1989.
[13] J. Sun, J. Chu, H. Chen; Periodic solution generated by impulses for singular differential equations. *J. Math. Anal. Appl.* **404**, 562-569(2013).
[14] J. Sun, D. O'Regan; Impulsive periodic solutions for singular problems via variational methods. *Bull. Aust. Math. Soc.* **86**, 193-204(2012).
[15] J. Sun, H. Chen, J. J. Nieto, M. Otero-Novoa; Multiplicity of solutions for perturbed second-order Hamiltonian systems with impulsive effects. *Nonlinear Anal.* **72**, 4575-4586(2010).
[16] J. Sun, H. Chen, J. J. Nieto; Infinitely many solutions for second-order Hamiltonian system with impulsive effects. *Math. Comput. Modelling* **54**, 544-555(2011).
[17] Y. Tian, W. Ge; Applications of variational methods to boundary value problem for impulsive differential equation. *Proc. Edin. Math. Soc.* **51**, 509-527(2008).

Juntao Sun
School of Science, Shandong University of Technology, Zibo, 255049 Shandong, China
E-mail address: `sunjuntao2008@163.com`

Jifeng Chu
College of Science, Hohai University, Nanjing, 210098 Jiangsu, China
E-mail address: `jifengchu@126.com`