Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 83, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE AND STABILITY OF SOLUTIONS TO NONLINEAR IMPULSIVE DIFFERENTIAL EQUATIONS IN β-NORMED SPACES

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Abstract. In this article, we consider nonlinear impulsive differential equations in β-normed spaces. We give new concepts of β-Ulam's type stability. Also we present sufficient conditions for the existence of solutions for impulsive Cauchy problems. Then we obtain generalized β-Ulam-Hyers-Rassias stability results for the impulsive problems on a compact interval. An example illustrates our main results.

1. Introduction

In the past decades, many researchers studied differential equations with instantaneous impulses of the type

$$\begin{gathered} x'(t) = f(t, x(t)), \quad t \in J' := J \setminus \{t_1, \dots, t_m\},\ J := [0, T], \\ x(t_k^+) = x(t_k^-) + I_k(x(t_k^-)), \quad k = 1, 2, \dots, m. \end{gathered} \tag{1.1}$$

where $f : J \times \mathbb{R} \to \mathbb{R}$ and $I_k : \mathbb{R} \to \mathbb{R}$ and t_k satisfy $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $x(t_k^+) = \lim_{\epsilon \to 0^+} x(t_k + \epsilon)$ and $x(t_k^-) = \lim_{\epsilon \to 0^-} x(t_k + \epsilon)$ represent the right and left limits of $x(t)$ at $t = t_k$ respectively. Here, I_k is a sequence of instantaneously impulse operators and have been used to describe abrupt changes such as shocks, harvesting, and natural disasters. For more existence, stability and periodic solutions on (1.1) and other impulsive models, one can read the monographs of [3, 6, 29].

In pharmacotherapy, the above instantaneous impulses can not describe the certain dynamics of evolution processes. For example, one considers the hemodynamic equilibrium of a person, the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous process. So we do not expect to use (1.1) to describe such process. In fact, the above situation should be shown by a new case of impulsive action, which starts at an arbitrary fixed point and stays active on a finite time interval. From the viewpoint of general theories, Hernández and O'Regan [12] initially offered to study a new class of abstract semilinear impulsive differential equations with not instantaneous impulses in a PC-normed Banach space. Meanwhile, Pierri et al. [24] continue the work in a PC_α-normed Banach space and develop the results in [12].

2000 *Mathematics Subject Classification.* 34A37, 34D10.
Key words and phrases. Nonlinear impulsive differential equations; existence; stability.

Submitted February 3, 2014. Published March 26, 2014.

Motivated by [12, 24, 27, 28, 31], we continue to study existence and uniqueness of solutions to differential equations with not instantaneous impulses in a $P\beta$-normed Banach space (see Section 2) of the form

$$
\begin{gathered}
x'(t) = f(t, x(t)), \quad t \in (s_i, t_{i+1}],\ i = 0, 1, 2, \dots, m, \\
x(t) = g_i(t, x(t)), \quad t \in (t_i, s_i],\ i = 1, 2, \dots, m,
\end{gathered} \tag{1.2}
$$

where t_i, s_i are pre-fixed numbers satisfying $0 = s_0 < t_1 \le s_1 \le t_2 < \dots < s_{m-1} \le t_m \le s_m \le t_{m+1} = T$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous and $g_i : [t_i, s_i] \times \mathbb{R} \to \mathbb{R}$ is continuous for all $i = 1, 2, \dots, m$. An improved existence and uniqueness result is obtained.

It is remarkable that Ulam type stability problems [30] have attracted many famous researchers. The readers can refer to monographs of Cădariu [7], Hyers [13, 14], Jung [17, 18], Rassias [26] and other recent works [1, 2, 9, 10, 11, 15, 16, 19, 20, 21, 22, 23, 25] in standard normed spaces and [8, 32] in β-normed spaces.

We introduce some auxiliary facts and offer four new concepts of β-Ulam's type stability for (1.2) (see Definitions 2.3–2.6). This is our main original contribution of this paper. It is quite useful in many applications such as numerical analysis, optimization, biology and economics, where finding the exact solution is quite difficult. As a result, existence and uniqueness and a generalized β-Ulam's type stability result on a compact interval are established. An example is given to illustrate our main results.

2. Preliminaries

Definition 2.1. (see Jung et al. [16] or Balachandran [4]) Suppose E is a vector space over $\mathbb{K}$. A function $\|\cdot\|_\beta$ $(0 < \beta \le 1) : E \to [0, \infty)$ is called a β-norm if and only if it satisfies (i) $\|x\|_\beta = 0$ if and only if $x = 0$; (ii) $\|\lambda x\|_\beta = |\lambda|^\beta \|x\|_\beta$ for all $\lambda \in \mathbb{K}$ and all $x \in E$; (iii) $\|x + y\|_\beta \le \|x\|_\beta + \|y\|_\beta$. The pair $(E, \|\cdot\|_\beta)$ is called a β-normed space. A β-Banach space is a complete β-normed space.

Throughout this paper, let $J = [0, T]$, $\beta \in (0, 1)$ be a fixed constant and $C(J, \mathbb{R})$ be the Banach space of all continuous functions from J into $\mathbb{R}$ with the new norm $\|x\|_\beta := \max\{|x(t)|^\beta : t \in J\}$ for $x \in C(J, \mathbb{R})$. For example, $\|z\|_{\frac{1}{2}} = \sqrt{e}$ for $z = t$, $t \in [0, e]$. We need the $P\beta$-Banach space $PC(J, \mathbb{R}) := \{x : J \to \mathbb{R} : x \in C((t_k, t_{k+1}], \mathbb{R}),\ k = 0, 1, \dots, m$ and there exist $x(t_k^-)$ and $x(t_k^+)$, $k = 1, \dots, m$, with $x(t_k^-) = x(t_k)\}$ with the norm $\|x\|_{P\beta} := \sup\{|x(t)|^\beta : t \in J\}$. For example, $\|z\|_{P\frac{1}{2}} = e$ for $z = t$, $t \in [0, 1]$ and $z = e^t$, $t \in (1, 2]$. Meanwhile, we set $PC^1(J, \mathbb{R}) := \{x \in PC(J, \mathbb{R}) : x' \in PC(J, \mathbb{R})\}$ with $\|x\|_{P\beta^1} := \max\{\|x\|_\beta, \|x'\|_\beta\}$. Clearly, $PC^1(J, \mathbb{R})$ endowed with the norm $\|\cdot\|_{P\beta^1}$ is a $P\beta$-Banach space.

Definition 2.2 ([12])**.** A function $x \in PC^1(J, \mathbb{R})$ is called a solution of the problem

$$
\begin{gathered}
x'(t) = f(t, x(t)), \quad t \in (s_i, t_{i+1}],\ i = 0, 1, 2, \dots, m, \\
x(t) = g_i(t, x(t)), \quad t \in (t_i, s_i],\ i = 1, 2, \dots, m, \\
x(0) = x_0, \quad x_0 \in \mathbb{R},
\end{gathered} \tag{2.1}
$$

if x satisfies

$$
\begin{gathered}
x(0) = x_0; \\
x(t) = g_i(t, x(t)), \quad t \in (t_i, s_i],\ i = 1, 2, \dots, m;
\end{gathered}
$$

$$x(t) = x_0 + \int_0^t f(s, x(s))ds, \quad t \in [0, t_1];$$
$$x(t) = g_i(s_i, x(s_i)) + \int_{s_i}^t f(s, x(s))ds, \quad t \in (s_i, t_{i+1}],\ i = 1, 2, \dots, m.$$

In general, we do not expect to get a precise solution of (2.1). However, we can try to get a function which satisfies some suitable approximation inequalities.

Let $0 < \beta < 1$, $\epsilon > 0$, $\psi \geq 0$ and $\varphi \in PC(J, \mathbb{R}_+)$. We consider the following inequalities:

$$\begin{gathered} |y'(t) - f(t, y(t))| \leq \epsilon, \quad t \in (s_i, t_{i+1}],\ i = 0, 1, 2, \dots, m, \\ |y(t) - g_i(t, y(t))| \leq \epsilon, \quad t \in (t_i, s_i],\ i = 1, 2, \dots, m, \end{gathered} \tag{2.2}$$

and

$$\begin{gathered} |y'(t) - f(t, y(t))| \leq \varphi(t), \quad t \in (s_i, t_{i+1}],\ i = 0, 1, 2, \dots, m, \\ |y(t) - g_i(t, y(t))| \leq \psi, \quad t \in (t_i, s_i],\ i = 1, 2, \dots, m, \end{gathered} \tag{2.3}$$

and

$$\begin{gathered} |y'(t) - f(t, y(t))| \leq \epsilon\varphi(t), \quad t \in (s_i, t_{i+1}],\ i = 0, 1, 2, \dots, m, \\ |y(t) - g_i(t, y(t))| \leq \epsilon\psi, \quad t \in (t_i, s_i],\ i = 1, 2, \dots, m. \end{gathered} \tag{2.4}$$

Next, our aim is to find a solution $y(\cdot)$ close to the measured output $x(\cdot)$ and whose closeness is defined in the sense of β-Ulam's type stabilities.

Definition 2.3. Equation (1.2) is β-Ulam-Hyers stable if there exists a real number $c_{f,\beta,g_i,\varphi} > 0$ such that for each $\epsilon > 0$ and for each solution $y \in PC^1(J, \mathbb{R})$ of (2.2) there exists a solution $x \in PC^1(J, \mathbb{R})$ of (1.2) with

$$|y(t) - x(t)|^\beta \leq c_{f,\beta,g_i,\varphi}\epsilon^\beta, \quad t \in J.$$

Definition 2.4. *Equation* (1.2) *is generalized β-Ulam-Hyers stable if there exists $\theta_{f,\beta,g_i,\varphi} \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\theta_{f,\beta,g_i,\varphi}(0) = 0$ such that for each solution $y \in PC^1(J, \mathbb{R})$ of* (2.2) *there exists a solution $x \in PC^1(J, \mathbb{R})$ of* (1.2) *with*

$$|y(t) - x(t)|^\beta \leq \theta_{f,\beta,g_i,\varphi}(\epsilon^\beta),\ t \in J.$$

Definition 2.5. Equation (1.2) is β-Ulam-Hyers-Rassias stable with respect to (φ, ψ) if there exists $c_{f,\beta,g_i,\varphi} > 0$ such that for each $\epsilon > 0$ and for each solution $y \in PC^1(J, \mathbb{R})$ of (2.4) there exists a solution $x \in PC^1(J, \mathbb{R})$ of (1.2) with

$$|y(t) - x(t)|^\beta \leq c_{f,\beta,g_i,\varphi}\epsilon^\beta(\psi^\beta + \varphi^\beta(t)),\ t \in J.$$

Definition 2.6. Equation (1.2) is generalized β-Ulam-Hyers-Rassias stable with respect to (φ, ψ) if there exists $c_{f,\beta,g_i,\varphi} > 0$ such that for each solution $y \in PC^1(J, \mathbb{R})$ of (2.3) there exists a solution $x \in PC^1(J, \mathbb{R})$ of (1.2) with

$$|y(t) - x(t)|^\beta \leq c_{f,\beta,g_i,\varphi}(\psi^\beta + \varphi^\beta(t)), \quad t \in J.$$

Obviously, (i) Definition 2.3 implies Definition 2.4; (ii) Definition 2.5 implies Definition 2.6; (iii) Definition 2.5 for $\varphi(\cdot) = \psi = 1$ implies Definition 2.3; (iv) Definitions 2.3-2.6 become to Ulam's stability concepts in Wang et al. [31] when $\beta = 1$ and $s_i = t_i$.

Remark 2.7. A function $y \in PC^1(J, \mathbb{R})$ is a solution of (2.3) if and only if there is $G \in PC(J, \mathbb{R})$ and a sequence G_i, $i = 1, 2, \dots, m$ (which depend on y) such that

(i) $|G(t)| \leq \varphi(t)$, $t \in J$ and $|G_i| \leq \psi$, $i = 1, 2, \dots, m$;
(ii) $y'(t) = f(t, y(t)) + G(t)$, $t \in (s_i, t_{i+1}]$, $i = 0, 1, 2, \dots, m$;

(iii) $y(t) = g_i(t, y(t)) + G_i$, $t \in (t_i, s_i]$, $i = 1, 2, \dots, m$.

By Remark 2.7 we get the following results.

Remark 2.8. If $y \in PC^1(J, \mathbb{R})$ is a solution of (2.3) then y is a solution of the integral inequality

$$\begin{gathered}
|y(t) - g_i(t, y(t))| \le \psi, \quad t \in (t_i, s_i],\ i = 1, 2, \dots, m; \\
\big|y(t) - y(0) - \int_0^t f(s, y(s))ds\big| \le \int_0^t \varphi(s)ds, \quad t \in [0, t_1]; \\
\big|y(t) - g_i(s_i, y(s_i)) - \int_{s_i}^t f(s, y(s))ds\big| \le \psi + \int_{s_i}^t \varphi(s)ds, \\
t \in [s_i, t_{i+1}],\ i = 1, 2, \dots, m.
\end{gathered} \tag{2.5}$$

We can give similar remarks for the solutions of the inequalities (2.2) and (2.4). To study Ulam's type stability, we need the following integral inequality results (see [5, Theorem 16.4]).

Lemma 2.9. *(i) Let the following inequality holds*

$$u(t) \le a(t) + \int_0^t b(s)u(s)ds, \quad t \ge 0,$$

where u, $a, \in PC(\mathbb{R}_+, \mathbb{R}_+)$, a is nondecreasing and $b(t) > 0$. Then, for $t \in \mathbb{R}_+$,

$$u(t) \le a(t) \exp\Big(\int_0^t b(s)ds\Big).$$

(ii) Assume

$$u(t) \le a(t) + \int_0^t b(s)u(s)ds + \sum_{0<t_k<t} \beta_k u(t_k^-), \quad t \ge 0,$$

where u, $a, b \in PC(\mathbb{R}_+, \mathbb{R}_+)$, a is nondecreasing and $b(t) > 0$, $\beta_k > 0$, $k \in \{1, \dots, m\}$. Then, for $t \in \mathbb{R}_+$,

$$u(t) \le a(t)(1 + \beta)^k \exp\Big(\int_0^t b(s)ds\Big), \quad t \in (t_k, t_{k+1}],\ k \in \{1, \dots, m\},$$

where $\beta = \sup_{k \in \{1, \dots, m\}}\{\beta_k\}$.

3. Main results

We use the following assumptions:

(H1) $f \in C(J \times \mathbb{R}, \mathbb{R})$.

(H2) There exists a positive constant L_f such that

$$|f(t, u_1) - f(t, u_2)| \le L_f|u_1 - u_2|,$$

for each $t \in J$ and all $u_1, u_2 \in \mathbb{R}$.

(H3) $g_i \in C([t_i, s_i] \times \mathbb{R}, \mathbb{R})$ and there are positive constants L_{g_i}, $i = 1, 2, \dots, m$ such that

$$|g_i(t, u_1) - g_i(t, u_2)| \le L_{g_i}|u_1 - u_2|,$$

for each $t \in [t_i, s_i]$ and all $u_1, u_2 \in \mathbb{R}$.

(H4) : Let $\varphi \in C(J, \mathbb{R}_+)$ be a nondecreasing function. There exists $c_\varphi > 0$ such that
$$\int_0^t \varphi(s)ds \le c_\varphi \varphi(t),$$
for each $t \in J$.

Concerning the existence and uniqueness result for the solutions to (2.1), we give the following theorem.

Theorem 3.1. *Assume that* (H1)–(H3) *are satisfied. Then* (2.1) *has a unique solution x provided that*
$$\varrho := \max\{L_{g_i}^\beta + L_f^\beta (t_{i+1} - s_i)^\beta, L_f^\beta t_1^\beta : i = 1, 2, \dots, m\} < 1. \tag{3.1}$$

Proof. Consider a mapping $F : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ defined by
$$\begin{gathered}
(Fx)(0) = x_0;\\
(Fx)(t) = g_i(t, x(t)), \quad t \in (t_i, s_i],\ i = 1, 2, \dots, m;\\
(Fx)(t) = x_0 + \int_0^t f(s, x(s))ds, \quad t \in [0, t_1];\\
(Fx)(t) = g_i(s_i, x(s_i)) + \int_{s_i}^t f(s, x(s))ds, \quad t \in (s_i, t_{i+1}],\ i = 1, 2, \dots, m.
\end{gathered}$$
Obviously, F is well defined.

For any $x, y \in PC(J, \mathbb{R})$ and $t \in (s_i, t_{i+1}]$, $i = 1, 2, \dots, m$, we have
$$\begin{aligned}
|(Fx)(t) - (Fy)(t)| &\le L_{g_i}|x(s_i) - y(s_i)| + L_f \int_{s_i}^t |x(s) - y(s)|ds\\
&\le L_{g_i}\|x - y\|_C + L_f \int_{s_i}^t \max_{t \in [s_i, t_{i+1}]} |x(s) - y(s)|ds\\
&\le L_{g_i}\|x - y\|_C + L_f (t_{i+1} - s_i)\|x - y\|_{PC},
\end{aligned}$$
which implies
$$|(Fx)(t) - (Fy)(t)|^\beta \le L_{g_i}^\beta \|x - y\|_{P\beta} + L_f^\beta (t_{i+1} - s_i)^\beta \|x - y\|_{P\beta}.$$
This reduces to
$$\|Fx - Fy\|_{P\beta} \le \big(L_{g_i}^\beta + L_f^\beta (t_{i+1} - s_i)^\beta\big)\|x - y\|_{P\beta}, \quad t \in (s_i, t_{i+1}].$$
Proceeding as above, we obtain that
$$\begin{gathered}
\|Fx - Fy\|_{P\beta} \le L_f^\beta t_1^\beta \|x - y\|_{P\beta}, \quad t \in [0, t_1],\\
\|Fx - Fy\|_{P\beta} \le L_{g_i}^\beta \|x - y\|_{P\beta}, \quad t \in (t_i, s_i],\ i = 1, 2, \dots, m.
\end{gathered}$$
From the above facts, we have
$$\|Fx - Fy\|_{P\beta} \le \varrho \|x - y\|_{P\beta},$$
where ϱ is defined in (3.1). Finally, we can deduce that F is a contraction mapping. Then, one can derive the result immediately. $\square$

Next, we discuss hte stability of (1.2) by using the concept of generalized β-Ulam-Hyers-Rassias in the above section.

Theorem 3.2. *Assume that* (H1)-(H4) *and* (3.1) *are satisfied. Then* (1.2) *is generalized β-Ulam-Hyers-Rassias stable with respect to (φ, ψ).*

Proof. Let $y \in PC^1(J,\mathbb{R})$ be a solution of (2.3). Denote by x the unique solution of the impulsive Cauchy problem

$$\begin{gathered} x'(t)=f(t,x(t)), \quad t\in(s_i,t_{i+1}],\ i=0,1,2,\dots,m,\\ x(t)=g_i(t,x(t)), \quad t\in(t_i,s_i],\ i=1,2,\dots,m,\\ x(0)=y(0). \end{gathered} \tag{3.2}$$

Then we obtain

$$x(t)=\begin{cases} g_i(t,x(t)), & t\in(t_i,s_i],\ i=1,2,\dots,m;\\ y(0)+\int_0^t f(s,x(s))ds, & t\in[0,t_1];\\ g_i(s_i,x(s_i))+\int_{s_i}^t f(s,x(s))ds, & t\in(s_i,t_{i+1}],\ i=1,2,\dots,m. \end{cases}$$

Keeping in mind (2.5), for each $t\in(s_i,t_{i+1}]$, $i=1,2,\dots,m$, we have

$$\Big|y(t)-g_i(s_i,y(s_i))-\int_{s_i}^t f(s,y(s))ds\Big| \le \psi+\int_{s_i}^t \varphi(s)ds \le \psi + c_\varphi\varphi(t),$$

and for $t\in(t_i,s_i]$, $i=1,2,\dots,m$, we have

$$|y(t)-g_i(t,y(t))|\le\psi,$$

and for $t\in[0,t_1]$, we have

$$\Big|y(t)-y(0)-\int_0^t f(s,y(s))ds\Big|\le c_\varphi\varphi(t).$$

Hence, for each $t\in(s_i,t_{i+1}]$, $i=1,2,\dots,m$, we have

$$\begin{aligned} &|y(t)-x(t)|\\ &= \Big|y(t)-g_i(s_i,x(s_i))-\int_{s_i}^t f(s,x(s))ds\Big|\\ &\le \Big|y(t)-g_i(s_i,y(s_i))-\int_{s_i}^t f(s,y(s))ds\Big|\\ &\quad + \Big|g_i(s_i,y(s_i))-g_i(s_i,x(s_i))\Big| + \Big(\int_{s_i}^t |f(s,y(s))-f(s,x(s))|ds\Big)\\ &\le (1+c_\varphi)[\psi+\varphi(t)] + L_{g_i}|y(s_i)-x(s_i)| + \int_{s_i}^t L_f|y(s)-x(s)|ds\\ &\le (1+c_\varphi)[\psi+\varphi(t)] + \sum_{0<s_i<t} L_{g_i}|y(s_i)-x(s_i)| + \int_0^t L_f|y(s)-x(s)|ds. \end{aligned}$$

Clearly, $a(t):=(1+c_\varphi)[\psi+\varphi(t)]$, $t\in(s_i,t_{i+1}]$, is nondecreasing and $a\in PC(\mathbb{R}_+,\mathbb{R}_+)$. By Lemma 2.9 (ii), we obtain

$$\begin{aligned} |y(t)-x(t)| &\le (1+c_\varphi)[\psi+\varphi(t)](1+L_g)^i\exp\Big(\int_0^t L_f ds\Big)\\ &\le (1+c_\varphi)[\psi+\varphi(t)](1+L_g)^i\exp\big(L_f t_{i+1}\big) \end{aligned}$$

where $L_g = \max\{L_{g_1}, L_{g_2}, \dots, L_{g_m}\}$. Thus,

$$\begin{aligned} |y(t) - x(t)|^\beta &\le \big[(1+c_\varphi)[\psi + \varphi(t)](1+L_g)^i \exp\big(L_f t_{i+1}\big)\big]^\beta \\ &\le \big[(1+c_\varphi)(1+L_g)^i \exp\big(L_f t_{i+1}\big)\big]^\beta [\psi + \varphi(t)]^\beta \\ &\le \big[(1+c_\varphi)(1+L_g)^i \exp\big(L_f t_{i+1}\big)\big]^\beta (\psi^\beta + \varphi(t)^\beta), \end{aligned} \tag{3.3}$$

for $t \in (s_i, t_{i+1}]$, $i = 1, 2, \dots, m$.

Further, for $t \in (t_i, s_i]$, $i = 1, 2, \dots, m$, we have

$$\begin{aligned} |y(t) - x(t)|^\beta &\le |y(t) - g_i(t, x(t))|^\beta \\ &\le |y(t) - g_i(t, y(t))|^\beta + |g_i(t, y(t)) - g_i(t, x(t))|^\beta \\ &\le \psi^\beta + L_{g_i}^\beta |y(t) - x(t)|^\beta, \end{aligned}$$

which yields

$$|y(t) - x(t)|^\beta \le \frac{1}{1 - L_{g_i}^\beta} \psi^\beta. \quad ((3.1) \text{ implies } L_{g_i}^\beta < 1) \tag{3.4}$$

Moreover, for $t \in [0, t_1]$, we have

$$\begin{aligned} |y(t) - x(t)| &= \big|y(t) - y(0) - \int_0^t f(s, x(s)) ds\big| \\ &\le \big|y(t) - y(0) - \int_0^t f(s, y(s)) ds\big| + \Big(\int_0^t |f(s, y(s)) - f(s, x(s))| ds\Big) \\ &\le c_\varphi \varphi(t) + \int_0^t L_f |y(s) - x(s)| ds. \end{aligned}$$

By Lemma 2.9 (i), we obtain

$$\begin{aligned} |y(t) - x(t)| &\le c_\varphi \varphi(t) \exp\Big(\int_0^t L_f ds\Big) \\ &\le c_\varphi \varphi(t) \exp\big(L_f t_1\big). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} |y(t) - x(t)|^\beta &\le \big[c_\varphi \varphi(t) \exp\big(L_f t_1\big)\big]^\beta \\ &\le \big[c_\varphi \exp\big(L_f t_1\big)\big]^\beta \varphi(t)^\beta, \quad t \in [0, t_1]. \end{aligned} \tag{3.5}$$

Summarizing, we combine (3.3), (3.4) and (3.5) and derive that

$$\begin{aligned} |y(t) - x(t)|^\beta &\le \Big(\big[(1+c_\varphi)(1+L_g)^i \exp\big(L_f t_{i+1}\big)\big]^\beta \\ &\quad + \frac{1}{1 - L_{g_i}^\beta} + \big[c_\varphi \exp\big(L_f t_1\big)\big]^\beta\Big)(\psi^\beta + \varphi^\beta(t)) \\ &:= c_{f,\beta,g_i,\varphi}(\psi^\beta + \varphi^\beta(t)), \quad t \in J, \end{aligned}$$

which implies that (1.2) is generalized β-Ulam-Hyers-Rassias stable with respect to (φ, ψ). The proof is complete. $\square$

4. An example

Consider the nonlinear differential equation, without instantaneous impulses,

$$\begin{aligned} x'(t) &= \frac{1}{1+15e^t}\arctan(t^2+x(t)), \quad t\in(0,1],\\ x(t) &= \frac{1}{15+t^2}\ln(x(t)+1), \quad t\in(1,2], \end{aligned} \tag{4.1}$$

and inequalities

$$\begin{aligned} \big|y'(t) - \frac{1}{1+15e^t}\arctan(t^2+y(t))\big| &\le e^t, \quad t\in(0,1],\\ \big|y(t) - \frac{1}{15+t^2}\ln(y(t)+1)\big| &\le 1, \quad t\in(1,2]. \end{aligned} \tag{4.2}$$

Set $J=[0,2]$, $0=s_0<t_1=1<s_1=2$ and $\beta=\frac{1}{2}$. Denote $f(t,x(t)) = \frac{1}{(1+15e^t)}\arctan(t^2+x(t))$ with $L_f=1/16$ for $t\in(0,1]$ and $g_1(t,x(t)) = \frac{1}{15+t^2}\ln(x(t)+1)$ with $L_{g_1}=1/16$ for $t\in(1,2]$. We put $\varphi(t)=e^t$ and $\psi=1$.

Let $y\in PC^1([0,2],\mathbb{R})$ be a solution of the inequality (4.2). Then there exist $G(\cdot)\in PC^1([0,2],\mathbb{R})$ and $G_1\in\mathbb{R}$ such that $|G(t)|\le e^t$, $t\in(0,1]$, $|G_1|\le 1$, $t\in(1,2]$, and

$$\begin{aligned} y'(t) &= \frac{1}{1+15e^t}\arctan(t^2+y(t)) + G(t), \quad t\in(0,1],\\ y(t) &= \frac{1}{15+t^2}\ln(y(t)+1) + G_1, \quad t\in(1,2]. \end{aligned} \tag{4.3}$$

For $t\in[0,1]$, integrating (4.3) from 0 to t, we have

$$y(t) = y(0) + \int_0^t \big(\frac{1}{1+15e^s}\arctan(s^2+y(s)) + G(s)\big)ds.$$

For $t\in(1,2]$, we have

$$y(t) = \frac{1}{15+t^2}\ln(y(t)+1) + G_1.$$

For

$$\begin{gathered} x'(t) = \frac{1}{1+15e^t}\arctan(t^2+x(t)),\ t\in(0,1],\\ x(t) = \frac{1}{15+t^2}\ln(x(t)+1),\ t\in(1,2],\\ x(0)=y(0), \end{gathered} \tag{4.4}$$

all the conditions in Theorem 3.1 are satisified. Thus, (4.4) has a unique solution.

Let us take the solution x of (4.4) given by

$$\begin{aligned} x(t) &= y(0) + \int_0^t \frac{1}{1+15e^s}\arctan(s^2+x(s))ds, \quad t\in(0,1],\\ x(t) &= \frac{1}{15+t^2}\ln(x(t)+1), \quad t\in(1,2]. \end{aligned}$$

For $t\in(0,1]$, we have

$$\begin{aligned} |y(t)-x(t)|^{1/2} &\le [c_\varphi \exp\big(L_f t_1\big)]^\beta \varphi(t)^\beta\\ &\le [c_\varphi \exp\big(L_f t_1\big)]^{1/2} e^{t/2}\\ &= e^{1/32}e^{t/2}. \end{aligned}$$

For $t \in (1,2]$, we have

$$|y(t)-x(t)|^{1/2} \le \frac{1}{4}|y(t)-x(t)|^{1/2}+1,$$

which yields

$$|y(t)-x(t)|^{1/2} \le \frac{4}{3}.$$

Summarizing, we have

$$|y(t)-x(t)|^{1/2} \le \frac{4}{3}(1+e^{t/2}), \quad t \in J.$$

So the equation (4.1) is generalized $\frac{1}{2}$-Ulam-Hyers-Rassias stable with respect to $(e^{t/2},1)$.

Acknowledgments. This work is supported by National Natural Science Foundation of China (11201091) and Doctoral Project of Guizhou Normal College (13BS010). The authors want to thank the anonymous referees for their careful reading of the manuscript and the insightful comments, which help us improve this article. Also we would like to acknowledge the valuable comments and suggestions from Professor Julio G. Dix.

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