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MULTIPLE POSITIVE SOLUTIONS FOR A CRITICAL ELLIPTIC PROBLEM WITH CONCAVE AND CONVEX NONLINEARITIES

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Abstract. In this article, we study the multiplicity of positive solutions for a semi-linear elliptic problem involving critical Sobolev exponent and concave-convex nonlinearities. With the help of Nehari manifold and Ljusternik-Schnirelmann category, we prove that problem admits at least $\operatorname{cat}(\Omega)+1$ positive solutions.

1. Introduction and main result

Let us consider the semi-linear problem

$$\begin{gathered}-\Delta u=\lambda|u|^{q-2}u+|u|^{2^*-2}u,\quad x\in\Omega,\\ u>0,\quad x\in\Omega,\\ u=0,\quad x\in\partial\Omega,\end{gathered}\tag{1.1}$$

where Ω is an open bounded domain in $\mathbb{R}^N$ with smooth boundary, $1<q<2$, $2^*=\frac{2N}{N-2}$ $(N\geq 3)$ and λ is a positive real parameter.

Under the assumption $\lambda\not\equiv 0$, (1.1) can be regarded as a perturbation problem of the equation

$$\begin{gathered}-\Delta u=|u|^{2^*-2}u,\quad x\in\Omega,\\ u>0,\quad x\in\Omega,\\ u=0,\quad x\in\partial\Omega.\end{gathered}\tag{1.2}$$

It is well known that the existence of solutions of (1.2) is affected by the shape of the domain Ω. This has been the focus of a great deal of research by several authors. In particular, the first striking result is due to Pohozaev [13] who proved that if Ω is star-shaped with respect to some point, (1.2) has no solution. However, if Ω is an annulus, Kazdan and Warner [11] pointed out that (1.2) has at least one solution. For a non-contractible domain Ω, Coron [8] proved that (1.2) has a solution. Further existence results for "rich topology" domain, we refer to [2, 10, 11, 12, 13, 14, 15, 16].

The fact that the number of solutions of (1.1) is affected by the concave-convex nonlinearities and the domain Ω has been the focus of a great deal of research in recent years. In particular, Ambrosetti, Brezis and Cerami [3] showed that there

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exists $\lambda_0 > 0$ such that (1.1) admits at least two solutions for $\lambda \in (0, \lambda_0)$, one solution for $\lambda = \lambda_0$ and no solution for $\lambda > \lambda_0$. Actually, Adimurthi et al. [5], Ouyang and Shi [12] and Tang [16] proved that there exists $\lambda_0 > 0$ such that (1.1) in unit ball $B^N(0;1)$ has exactly two solutions for all $\lambda \in (0, \lambda_0)$, exactly one solution for $\lambda = \lambda_0$ and no solution for all $\lambda > \lambda_0$. Recently, when Ω is a non-contractible domain, Wu [18] showed that (1.1) admits at least three solutions if λ is small enough.

In this work we aim to get a better information on the number of solutions of (1.1), for small value of parameter λ, via the Nehari manifold and Ljusternik-Schnirelmann category. Our main result is as follows.

Theorem 1.1. *There exists $\lambda_0 > 0$ such that, for each $\lambda \in (0, \lambda_0)$, problem* (1.1) *has at least* $\mathrm{cat}(\Omega) + 1$ *solutions.*

Here cat means the Ljusternik-Schnirelmann category and for properties of it we refer to Struwe [14].

Remark 1.2. If Ω is a general domain, $\mathrm{cat}(\Omega) \geq 1$ and Theorem 1.1 is the result of [3]. If Ω is non-contractible, $\mathrm{cat}(\Omega) \geq 2$ and Theorem 1.1 is the result of Wu [18].

Associated with (1.1), we consider the energy functional J_λ for each $H_0^1(\Omega)$,

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{q}\int_\Omega (u_+)^q dx - \frac{1}{p^*}\int_\Omega (u_+)^{2^*} dx,$$

where $u_+ = \max\{u, 0\}$. From the assumption, it is easy to prove that J_λ is well defined in $H_0^1(\Omega)$ and $J_\lambda \in C^2(H_0^1(\Omega), \mathbb{R})$. Furthermore, the critical points of J_λ are weak solutions of (1.1). We consider the behaviors of J_λ on the Nehari manifold

$$S_\lambda = \{u \in H_0^1(\Omega) \setminus \{0\}; u_+ \not\equiv 0 \text{ and } \langle J_\lambda'(u), u\rangle = 0\},$$

where $\langle , \rangle$ denotes the usual duality between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. This enables us to construct homotopies between Ω and certain levels of J_λ. Clearly, $u \in S_\lambda$ if and only if

$$\int_\Omega |\nabla u|^2 dx - \lambda\int_\Omega (u_+)^q dx - \int_\Omega (u_+)^{2^*} dx = 0.$$

On the Nehari manifold S_λ, from the Sobolev embedding theorem and the Young inequality, we have

$$\begin{aligned} J_\lambda(u) &= \big(\frac{1}{2} - \frac{1}{2^*}\big)\int_\Omega |\nabla u|^2 dx - \lambda\big(\frac{1}{q} - \frac{1}{2^*}\big)\int_\Omega (u_+)^q dx \\ &\geq \big(\frac{1}{2} - \frac{1}{2^*}\big)\int_\Omega |\nabla u|^2 dx - \lambda\big(\frac{1}{q} - \frac{1}{2^*}\big)S_q^{-q}\Big(\int_\Omega |\nabla u|^2 dx\Big)^{q/2} \\ &\geq \big(\frac{1}{2} - \frac{1}{2^*}\big)\int_\Omega |\nabla u|^2 dx - \big(\frac{1}{2} - \frac{1}{2^*}\big)\int_\Omega |\nabla u|^2 dx - D\lambda^{\frac{2}{2-q}}, \end{aligned} \tag{1.3}$$

where S_q is the best Sobolev constant for the embedding of $H_0^1(\Omega)$ into $L^q(\Omega)$ and D is a positive constant depending on q and S_q.

Thus J_λ is coercive and bounded below on S_λ. It is useful to understand S_λ in terms of the fibrering maps $\phi_u(t) = J_\lambda(tu)(t > 0)$. It is clear that, if $u \in S_\lambda$, then ϕ_u has a critical point at $t = 1$. Furthermore, we will discuss the essential nature of ϕ_u in Section 2.

This article is organized as follows: In Section 2, we give some notations and preliminary results. In Section 3, we discuss some concentration behavior. In Section 4, we give the proof of the main theorem.

2. Preliminaries

Throughout the paper by $|\cdot|_r$ we denote the L^r-norm. On the space $H_0^1(\Omega)$ we consider the norm

$$\|u\| = \Big(\int_\Omega |\nabla u|^2 dx\Big)^{1/2}.$$

Set also

$$\mathcal{D}^{1,2}(\mathbb{R}^N) := \big\{u \in L^{2^*}(\mathbb{R}^N); \frac{\partial u}{\partial x_i} \in L^2(\mathbb{R}^N) \text{ for } i = 1, \dots, N\big\}$$

equipped with the norm

$$\|u\|_* = \Big(\int_{\mathbb{R}^N} |\nabla u|^2 dx\Big)^{1/2}.$$

We will denote by S the best Sobolev constant of the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ given by

$$S := \inf\big\{\int_\Omega |\nabla u|^2 dx; u \in H_0^1(\Omega), |u|_{2^*} = 1\big\}.$$

It is known that S is independent of Ω and is never achieved except when $\Omega = \mathbb{R}^N$ (see [15]).

We then define the Palais-Smale(simply by (PS)) sequences, (PS)-values, and (PS)-conditions in $H_0^1(\Omega)$ for J_λ as follows.

Definition 2.1. (i) For $\beta \in \mathbb{R}$, a sequence $\{u_k\}$ is a $(PS)_\beta$-sequence in $H_0^1(\Omega)$ for J_λ if $J_\lambda(u_k) = \beta + o(1)$ and $J'_\lambda(u_k) = o(1)$ strongly in $H^{-1}(\Omega)$ as $k \to \infty$.
(ii) J_λ satisfies the $(PS)_\beta$-condition in $H_0^1(\Omega)$ if every $(PS)_\beta$-sequence in $H_0^1(\Omega)$ for J_λ contains a convergent subsequence.

We now define

$$\psi_\lambda(u) := \langle J'_\lambda(u), u\rangle = \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega (u_+)^q dx - \int_\Omega (u_+)^{2^*} dx. \tag{2.1}$$

Then for $u \in S_\lambda$,

$$\langle \psi'_\lambda(u), u\rangle = (2-q)\int_\Omega |\nabla u|^2 dx - (2^*-q)\int_\Omega (u_+)^{2^*} dx \tag{2.2}$$

$$= (2-2^*)\int_\Omega |\nabla u|^2 dx + \lambda(2^*-q)\int_\Omega (u_+)^q dx. \tag{2.3}$$

Similarly to the method used in [6], we split S_λ into three parts:

$$S_\lambda^+ = \{u \in S_\lambda; \langle \psi'_\lambda(u), u\rangle > 0\},$$
$$S_\lambda^0 = \{u \in S_\lambda; \langle \psi'_\lambda(u), u\rangle = 0\},$$
$$S_\lambda^- = \{u \in S_\lambda; \langle \psi'_\lambda(u), u\rangle < 0\}.$$

Then we have the following results.

Lemma 2.2. *Suppose that u_0 is a local minimum for J_λ on S_λ. Then, if $u_0 \notin S_\lambda^0$, u_0 is a critical point of J_λ.*

Proof. Since u_0 is a local minimum for J_λ on S_λ, then u_0 is a solution of the optimization problem

$$\text{minimize } J_\lambda(u) \text{ subject to } \psi_\lambda(u) = 0.$$

Hence, by the theory of Lagrange multipliers, there exists $\mu \in \mathbb{R}$ such that $J'_\lambda(u_0) = \mu\psi'_\lambda(u_0)$ in $H^{-1}(\Omega)$. Thus,

$$\langle J'_\lambda(u_0), u_0\rangle = \mu\langle \psi'_\lambda(u_0), u_0\rangle. \tag{2.4}$$

Since $u_0 \in S_\lambda$, we obtain $\langle J'_\lambda(u_0), u_0\rangle = 0$. However, $u_0 \notin S^0_\lambda$ and so by (2.4) $\mu = 0$ and $J'_\lambda(u_0) = 0$. This completes the proof. □

Lemma 2.3. *There exists $\lambda_1 > 0$ such that for each $\lambda \in (0, \lambda_1)$, we have $S^0_\lambda = \emptyset$.*

Proof. Suppose otherwise, that is $S^0_\lambda \neq \emptyset$ for all $\lambda > 0$. Then for $u \in S^0_\lambda$, we from (2.2), (2.3) and the Sobolev embedding theorem obtain that there are two positive numbers c_1, c_2 independent of u and λ such that

$$\int_\Omega |\nabla u|^2 dx \le c_1\Big(\int_\Omega |\nabla u|^2 dx\Big)^{2^*/2}, \quad \int_\Omega |\nabla u|^2 dx \le \lambda c_2\Big(\int_\Omega |\nabla u|^2 dx\Big)^{q/2}$$

or

$$\int_\Omega |\nabla u|^2 dx \ge c_1^{-\frac{2}{2^*-2}}, \quad \int_\Omega |\nabla u|^2 dx \le (\lambda c_2)^{\frac{2}{2-q}}.$$

If λ is sufficiently small, this is impossible. Thus we can conclude that there exists $\lambda_1 > 0$ such that for each $\lambda \in (0, \lambda_1)$, we have $S^0_\lambda = \emptyset$. □

By Lemma 2.3, for $\lambda \in (0, \lambda_1)$, we write $S_\lambda = S^+_\lambda \cup S^-_\lambda$ and define

$$\alpha^+_\lambda = \inf_{u \in S^+_\lambda} J_\lambda(u), \quad \alpha^-_\lambda = \inf_{u \in S^-_\lambda} J_\lambda(u).$$

We now discuss the nature of the fibrering maps $\phi_u(t)$. It is useful to consider the function

$$M_u(t) = t^{2-q}\int_\Omega |\nabla u|^2 dx - t^{2^*-q}\int_\Omega (u_+)^{2^*} dx. \tag{2.5}$$

Clearly, for $t > 0$, $tu \in S_\lambda$ if and only if t is a solution of

$$M_u(t) = \lambda \int_\Omega (u_+)^q dx. \tag{2.6}$$

Moreover, we have from $M'_u(t) = 0$ know that there is a unique critical point $t_{\max}$:

$$t_{\max} = \Big(\frac{(2-q)\int_\Omega |\nabla u|^2 dx}{(2^*-q)\int_\Omega (u_+)^{2^*} dx}\Big)^{1/(2^*-2)}.$$

Furthermore, the direct computation gives that

$$M''_u(t_{\max}) = (2^*-q)(2-p^*)t_{\max}^{2^*-q-2}\int_\Omega (u_+)^{2^*} dx < 0.$$

This shows that $M_u(t)$ is increasing in $(0, t_{\max})$ and decreasing for $t \ge t_{\max}$.

Suppose $tu \in S_\lambda$. It follows from (2.2) and (2.5) that if $M'_u(t) > 0$, then $tu \in S^+_\lambda$, and if $M'_u(t) < 0$, then $tu \in S^-_\lambda$. If $\lambda > 0$ is sufficiently large, (2.6) has no solution and so $\phi_u(t)$ has no critical point, in this case $\phi_u(t)$ is a decreasing function. Hence no multiple of u lies in S_λ. If, on the other hand, $\lambda > 0$ is sufficiently small, there are exactly two solutions $t_1(u) < t_2(u)$ of (2.6) with $M'_u(t_1(u)) > 0$ and $M'_u(t_2(u)) < 0$. Thus there are exactly two multiples of $u \in S_\lambda$, that is, $t_1(u)u \in S^+_\lambda$ and $t_2(u)u \in S^-_\lambda$. It follows that $\phi_u(t)$ has exactly two critical points, a local minimum at $t_1(u)$ and a local maximum at $t_2(u)$. Moreover, $\phi_u(t)$ is decreasing in $(0, t_1(u))$, increasing in $(t_1(u), t_2(u))$ and decreasing in $(t_2(u), \infty)$. Then we have the following result.

Lemma 2.4. (i) $\alpha_\lambda^+ < 0$.
(ii) *There exist* $\lambda_2, \delta > 0$ *such that* $\alpha_\lambda^- \geq \delta$ *for all* $\lambda \in (0, \lambda_2)$.

Proof. (i) Given $u \in S_\lambda^+$, from (2.3) and the definition of S_λ^+, we obtain

$$\begin{aligned} J_\lambda(u) &= \big(\frac{1}{2} - \frac{1}{2^*}\big) \int_\Omega |\nabla u|^2 dx - \lambda\big(\frac{1}{q} - \frac{1}{2^*}\big) \int_\Omega (u_+)^q dx \\ &\leq \big[\big(\frac{1}{2} - \frac{1}{2^*}\big) - \big(\frac{1}{q} - \frac{1}{2^*}\big)\frac{2^* - 2}{2^* - q}\big] \int_\Omega |\nabla u|^2 dx \\ &= \frac{2^* - 2}{2^*}\big(\frac{1}{2} - \frac{1}{q}\big) \int_\Omega |\nabla u|^2 dx < 0. \end{aligned}$$

This yields $\alpha_\lambda^+ < 0$.

(ii) For $u \in S_\lambda^-$, by (2.2) and the Sobolev embedding theorem, we obtain

$$\begin{aligned} (2 - q) \int_\Omega |\nabla u|^2 dx &< (2^* - q) \int_\Omega (u_+)^{2^*} dx \\ &\leq (2^* - q) S^{-\frac{2^*}{2}} \Big(\int_\Omega |\nabla u|^2 dx\Big)^{2^*/2}. \end{aligned}$$

Thus there exists $c > 0$ such that

$$\int_\Omega |\nabla u|^2 dx \geq c.$$

Moreover,

$$\begin{aligned} J_\lambda(u) &= \big(\frac{1}{2} - \frac{1}{2^*}\big) \int_\Omega |\nabla u|^2 dx - \lambda\big(\frac{1}{q} - \frac{1}{2^*}\big) \int_\Omega (u_+)^q dx \\ &\geq \big(\frac{1}{2} - \frac{1}{2^*}\big) \int_\Omega |\nabla u|^2 dx - \lambda\big(\frac{1}{q} - \frac{1}{2^*}\big) S_q^{-q} \Big(\int_\Omega |\nabla u|^2 dx\Big)^{q/2} \\ &= \Big(\int_\Omega |\nabla u|^2 dx\Big)^{q/2} \Big[\big(\frac{1}{2} - \frac{1}{2^*}\big)\Big(\int_\Omega |\nabla u|^2 dx\Big)^{1 - \frac{q}{2}} - \lambda\big(\frac{1}{q} - \frac{1}{2^*}\big) S_q^{-q}\Big]. \end{aligned}$$

Hence, there exist $\lambda_2, \delta > 0$ such that $\alpha_\lambda^- \geq \delta$ for all $\lambda \in (0, \lambda_2)$. □

We establish that J_λ satisfies the $(PS)_\beta$-condition under some condition on the level of $(PS)_\beta$-sequences in the following.

Lemma 2.5. *For each* $\lambda \in (0, \lambda_2)$, J_λ *satisfies the* $(PS)_\beta$*-condition with* β *in* $(-\infty, \alpha_\lambda^+ + \frac{1}{N} S^{N/2})$.

Proof. Let $\{u_k\} \subset H_0^1(\Omega)$ be a $(PS)_\beta$-sequence for J_λ and $\beta \in (-\infty, \alpha_\lambda^+ + \frac{1}{N} S^{N/2})$. After a standard argument(see [19]), we know that $\{u_k\}$ is bounded in $H_0^1(\Omega)$. Thus, there exists a subsequence still denoted by $\{u_k\}$ and $u \in H_0^1(\Omega)$ such that $u_k \rightharpoonup u$ weakly in $H_0^1(\Omega)$. By the compactness of Sobolev embedding and the Brezis-Lieb Lemma [19], we obtain

$$\begin{gathered} \lambda \int_\Omega (u_k)_+^q dx = \lambda \int_\Omega (u_+)^q dx + o(1), \\ \int_\Omega |\nabla u_k - \nabla u|^2 dx = \int_\Omega |\nabla u_k|^2 dx - \int_\Omega |\nabla u|^2 dx + o(1), \end{gathered}$$

$$\int_\Omega (u_k - u)_+^{2^*} dx = \int_\Omega (u_k)_+^{2^*} dx - \int_\Omega (u_+)^{2^*} dx + o(1).$$

Moreover, we can obtain $J'_\lambda(u) = 0$ in $H^{-1}(\Omega)$. Since $J_\lambda(u_k) = \beta + o(1)$ and $J'_\lambda(u_k) = o(1)$ in $H^{-1}(\Omega)$, we deduce that

$$\frac{1}{2}\int_\Omega |\nabla u_k - \nabla u|^2 dx - \frac{1}{2^*}\int_\Omega (u_k - u)_+^{2^*} dx = \beta - J_\lambda(u) + o(1) \tag{2.7}$$

and

$$\int_\Omega |\nabla u_k - \nabla u|^2 dx - \int_\Omega (u_k - u)_+^{2^*} dx = o(1).$$

Now we may assume that

$$\int_\Omega |\nabla u_k - \nabla u|^2 dx \to l, \quad \int_\Omega (u_k - u)_+^{2^*} dx \to l \quad \text{as } k \to \infty,$$

for some $l \in [0, +\infty)$.

Suppose $l \neq 0$. Using the Sobolev embedding theorem and passing to the limit as $k \to \infty$, we have $l \geq S l^{2/2^*}$; that is,

$$l \geq S^{N/2}. \tag{2.8}$$

Then by (2.7), (2.8) and $u \in S_\lambda$, we have

$$\beta = J_\lambda(u) + \frac{1}{N} l \geq \frac{1}{N} S^{N/2} + \alpha_\lambda^+,$$

which contradicts the definition of β. Hence $l = 0$, that is, $u_k \to u$ strongly in $H_0^1(\Omega)$. □

Then we obtain the following result.

Lemma 2.6. *For each $0 < \lambda < \min\{\lambda_1, \lambda_2\}$, the functional J_λ has a minimizer u_λ^+ in S_λ^+ and it satisfies:*

(i) $J_\lambda(u_\lambda^+) = \alpha_\lambda^+ = \inf_{u \in S_\lambda^+} J_\lambda(u)$;
(ii) u_λ^+ *is a solution of* (1.1);
(iii) $J_\lambda(u_\lambda^+) \to 0$ *as* $\lambda \to 0$.
(iv) $\lim_{\lambda \to 0} \|u_\lambda^+\| = 0$.

Proof. (i)–(iii) are consequences in [10, Theorem 1.1]. Now we show (iv). By (i)–(iii), we have

$$0 = \lim_{\lambda \to 0} J_\lambda(u_\lambda^+) = \lim_{\lambda \to 0} \Big(\frac{1}{N} \int_\Omega |\nabla u_\lambda^+|^2 dx - (\frac{1}{q} - \frac{1}{2^*}) \lambda \int_\Omega (u_\lambda^+)^q dx \Big). \tag{2.9}$$

Since J_λ is coercive and bounded below on S_λ, $\int_\Omega |\nabla u_\lambda^+|^2 dx$ is bounded and so that

$$\lim_{\lambda \to 0} \lambda \int_\Omega (u_\lambda^+)^q dx = 0. \tag{2.10}$$

Hence, from (2.9) and (2.10) we complete the proof. □

3. Concentration behavior

In this Section, we will recall and prove some Lemmas which are crucial in the proof of the main theorem. Firstly, we denote $c_\lambda := \frac{1}{N}S^{N/2} + \alpha_\lambda^+$ and consider the filtration of the manifold S_λ^- as follows:

$$S_\lambda^-(c_\lambda) := \{u \in S_\lambda^-; J_\lambda(u) \le c_\lambda\}.$$

In Section 4, we will prove that (1.1) admits at least $\mathrm{cat}(\Omega)$ solutions in this set. Then we need the following Lemmas.

Lemma 3.1. *Let $\{u_k\} \subset H_0^1(\Omega)$ be a nonnegative function sequence with $|u_k|_{2^*} = 1$ and $\int_\Omega |\nabla u_k|^2 dx \to S$. Then there exists a sequence $(y_k, \lambda_k) \in \mathbb{R}^N \times \mathbb{R}^+$ such that*

$$\upsilon_k(x) := \lambda_k^{\frac{N-2}{2}} u_k(\lambda_k x + y_k)$$

contains a convergent subsequence denoted again by $\{\upsilon_k\}$ such that $\upsilon_k \to \upsilon$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ with $\upsilon(x) > 0$ in $\mathbb{R}^N$. Moreover, we have $\lambda_k \to 0$ and $y_k \to y \in \overline{\Omega}$.

For a proof of the above lemma, see Willem [19].

Lemma 3.2. *Suppose that X is a Hilbert manifold and $F \in C^1(X, \mathbb{R})$. Assume that for $c_0 \in \mathbb{R}$ and $k \in \mathbb{N}$:*

(i) *$F(x)$ satisfies the $(PS)_c$ condition for $c \le c_0$,*
(ii) *$\mathrm{cat}(\{x \in X; F(x) \le c_0\}) \ge k$.*

Then $F(x)$ has at least k critical points in $\{x \in X; F(x) \le c_0\}$.

For a proof of the above lemma, see See [1, Theorem 2.3].

Up to translations, we may assume that $0 \in \Omega$. Moreover, in what follows, we fix $r > 0$ such that $B_r = \{x \in \mathbb{R}^N; |x| < r\} \subset \Omega$ and the sets

$$\Omega_r^+ := \{x \in \mathbb{R}^N; \mathrm{dist}(x, \Omega) < r\}, \quad \Omega_r^- := \{x \in \Omega; \mathrm{dist}(x, \Omega) > r\}$$

are both homotopically equivalent to Ω. Now we define the continuous map $\Phi : S_\lambda^- \to \mathbb{R}^N$ by setting

$$\Phi(u) := \frac{\int_\Omega x(u_+)^{2^*} dx}{\int_\Omega (u_+)^{2^*} dx}.$$

Lemma 3.3. *There exists $\lambda_3 > 0$ such that if $\lambda \in (0, \lambda_3)$ and $u \in S_\lambda^-(c_\lambda)$, then $\Phi(u) \in \Omega_r^+$.*

Proof. By way of contradiction, let $\{\lambda_k\}$ and $\{u_k\}$ be such that $\lambda_k \to 0$, $u_k \in S_{\lambda_k}^-(c_{\lambda_k})$ and $\Phi(u_k) \notin \Omega_r^+$. From (1.3), we have that $\{u_k\}$ is bounded in $H_0^1(\Omega)$ and $\lambda_k \int_\Omega (u_k)_+^q dx \to 0$. Thus, by Lemma 2.6 (iii) we have

$$\lim_{k\to\infty} J_{\lambda_k}(u_k) = \lim_{k\to\infty} \frac{1}{N} \int_\Omega |\nabla u_k|^2 dx = \lim_{k\to\infty} \frac{1}{N} \int_\Omega (u_k)_+^{2^*} dx \le \frac{1}{N} S^{N/2}. \tag{3.1}$$

Defining $\omega_k = u_k/|(u_k)_+|_{2^*}$, we see that $|(\omega_k)_+|_{2^*} = 1$. By (3.1) and the definition of S, we obtain

$$\lim_{k\to\infty} \int_\Omega |\nabla \omega_k|^2 dx = \lim_{k\to\infty} \int_\Omega |\nabla (\omega_k)_+|^2 dx = S.$$

Furthermore, the functions $\widetilde{\omega}_k = (\omega_k)_+$ satisfy

$$|\widetilde{\omega}_k|_{2^*} = 1, \quad \int_\Omega |\nabla \widetilde{\omega}_k|^2 dx \to S. \tag{3.2}$$

By Lemma 3.1, there is $\{\varepsilon_k\}$ in $\mathbb{R}^+$ and $\{y_k\}$ in $\mathbb{R}^N$, such that $\varepsilon_k \to 0$, $y_k \to y \in \overline{\Omega}$ and $\upsilon_k(x) = \varepsilon_k^{\frac{N-2}{N}} \widetilde{\omega}_k(\varepsilon_k x + y_k) \to \upsilon$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ with $\upsilon(x) > 0$ in $\mathbb{R}^N$.

Considering $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\varphi(x) = x$ in Ω, we infer

$$\Phi(u_k) = \frac{\int_\Omega x(u_k)_+^{2^*}\,dx}{\int_\Omega (u_k)_+^{2^*}\,dx} = \int_{\mathbb{R}^N} \varphi(x)(\widetilde{\omega}_k)^{2^*}\,dx = \int_{\mathbb{R}^N} \varphi(\varepsilon_k x + y_k)(\upsilon_k(x))^{2^*}\,dx. \quad (3.3)$$

Moreover, by Lebesgue Theorem,

$$\int_{\mathbb{R}^N} \varphi(\varepsilon_k x + y_k)(\upsilon_k(x))^{2^*}\,dx \to y \in \overline{\Omega},$$

so that $\lim_{k\to\infty} \Phi(u_k) = y \in \overline{\Omega}$, in contradiction with $\Phi(u_k) \notin \Omega_r^+$. □

It is well known that S is attained when $\Omega = \mathbb{R}^N$ by the functions

$$y_\varepsilon(x) = \frac{[N(N-2)\varepsilon^2]^{(N-2)/4}}{(\varepsilon^2 + |x|^2)^{(N-2)/2}}.$$

for any $\varepsilon > 0$. Moreover, the functions $y_\varepsilon(x)$ are the only positive radial solutions of

$$-\Delta u = |u|^{2^*-2}u$$

in $\mathbb{R}^N$. Hence,

$$S\Big(\int_{\mathbb{R}^N} |y_\varepsilon|^{2^*}\,dx\Big)^{2/2^*} = \int_{\mathbb{R}^N} |\nabla y_\varepsilon|^2 dx = \int_{\mathbb{R}^N} |y_\varepsilon|^{2^*}\,dx = S^{N/2}.$$

Let $0 \le \phi(x) \le 1$ be a function in $C_0^\infty(\Omega)$ defined as

$$\phi(x) = \begin{cases} 1, & \text{if } |x| \le r/4, \\ 0, & \text{if } |x| \ge r/2. \end{cases}$$

Assume

$$\upsilon_\varepsilon(x) = \phi(x) y_\varepsilon(x).$$

The argument in [14] gives

$$\int_\Omega |\nabla \upsilon_\varepsilon|^2 dx = S^{N/2} + O(\varepsilon^{N-2}), \quad \int_\Omega |\upsilon_\varepsilon|^{2^*}\,dx = S^{N/2} + O(\varepsilon^N). \quad (3.4)$$

Moreover, we have the following result.

Lemma 3.4. *There exist $\varepsilon_0, \sigma(\varepsilon) > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ and $\sigma \in (0, \sigma(\varepsilon))$, we have*

$$\sup_{t\ge 0} J_\lambda(u_\lambda^+ + t\upsilon_\varepsilon(x-y)) < c_\lambda - \sigma \quad \textit{uniformly in } y \in \Omega_r^-,$$

where u_λ^+ is a local minimum in Lemma 2.6. Furthermore, there exists $t^-_{(\lambda,\varepsilon,y)} > 0$ such that

$$u_\lambda^+ + t^-_{(\lambda,\varepsilon,y)}\upsilon_\varepsilon(x-y) \in S_\lambda^-(c_\lambda - \sigma), \quad \Phi(u_\lambda^+ + t^-_{(\lambda,\varepsilon,y)}\upsilon_\varepsilon(x-y)) \in \Omega_r^+.$$

Proof. From Lemma 2.6 and the definition of Ω_r^-, we can define

$$c_0 := \inf_{M_r} u_\lambda^+ > 0, \quad (3.5)$$

where $M_r := \{x \in \Omega; \operatorname{dist}(x, \Omega_r^-) \le \frac{r}{2}\}$. Since

$$
\begin{aligned}
&J_\lambda(u_\lambda^+ + t\upsilon_\varepsilon(x-y)) \\
&= \frac{1}{2}\int_\Omega |\nabla(u_\lambda^+ + t\upsilon_\varepsilon(x-y))|^2 dx - \frac{\lambda}{q}\int_\Omega |u_\lambda^+ + t\upsilon_\varepsilon(x-y)|^q dx \\
&\quad - \frac{1}{2^*}\int_\Omega |u_\lambda^+ + t\upsilon_\varepsilon(x-y)|^{2^*} dx \\
&= \frac{1}{2}\int_\Omega |\nabla u_\lambda^+|^2 dx + \frac{t^2}{2}\int_\Omega |\nabla \upsilon_\varepsilon|^2 dx + \langle u_\lambda^+, t\upsilon_\varepsilon(x-y)\rangle \\
&\quad - \frac{\lambda}{q}\int_\Omega |u_\lambda^+ + t\upsilon_\varepsilon(x-y)|^q dx - \frac{1}{2^*}\int_\Omega |u_\lambda^+ + t\upsilon_\varepsilon(x-y)|^{2^*} dx.
\end{aligned}
\tag{3.6}
$$

Note (3.5) and a useful estimate obtained by Brezis and Nirenberg (see [7, (17) and (21)]) shows that

$$
\begin{aligned}
&\int_\Omega |u_\lambda^+ + t\upsilon_\varepsilon(x-y)|^{2^*} dx \\
&= \int_\Omega |u_\lambda^+|^{2^*} dx + t^{2^*}\int_\Omega |\upsilon_\varepsilon|^{2^*} dx + 2^* t\int_\Omega (u_\lambda^+)^{2^*-1}\upsilon_\varepsilon(x-y) dx \\
&\quad + 2^* t^{2^*-1}\int_\Omega (\upsilon_\varepsilon(x-y))^{2^*-1} u_\lambda^+ dx + o(\varepsilon^{\frac{N-2}{2}}),
\end{aligned}
$$

uniformly in $y \in \Omega_r^-$.

Substituting in (3.6) and by Lemma 2.6, (3.4), (3.5), we obtain

$$
\begin{aligned}
&J_\lambda(u_\lambda^+ + t\upsilon_\varepsilon(x-y)) \\
&= \frac{1}{2}\int_\Omega |\nabla u_\lambda^+|^2 dx + \frac{t^2}{2} S^{\frac{N}{2}} + t\langle u_\lambda^+, \upsilon_\varepsilon(x-y)\rangle \\
&\quad - \frac{1}{2^*}\int_\Omega |u_\lambda^+|^{2^*} dx - \frac{t^{2^*}}{2^*} S^{\frac{N}{2}} - t\int_\Omega (u_\lambda^+)^{2^*-1}\upsilon_\varepsilon(x-y) dx \\
&\quad - t^{2^*-1}\int_\Omega (\upsilon_\varepsilon(x-y))^{2^*-1} u_\lambda^+ dx - \frac{\lambda}{q}\int_\Omega |u_\lambda^+ + t\upsilon_\varepsilon(x-y)|^q dx + o(\varepsilon^{\frac{N-2}{2}}) \\
&= J_\lambda(u_\lambda^+) + \frac{t^2}{2} S^{\frac{N}{2}} - \frac{t^{2^*}}{2^*} S^{\frac{N}{2}} - t^{2^*-1}\int_\Omega (\upsilon_\varepsilon(x-y))^{2^*-1} u_\lambda^+ dx \\
&\quad - \frac{\lambda}{q}\int_\Omega |u_\lambda^+ + t\upsilon_\varepsilon(x-y)|^q dx + \frac{\lambda}{q}\int_\Omega |u_\lambda^+|^q dx \\
&\quad + t\lambda\int_\Omega (u_\lambda^+)^{q-1}\upsilon_\varepsilon(x-y) dx + o(\varepsilon^{\frac{N-2}{2}}) \\
&= \alpha_\lambda^+ + \frac{t^2}{2} S^{\frac{N}{2}} - \frac{t^{2^*}}{2^*} S^{\frac{N}{2}} - t^{2^*-1}\int_\Omega (\upsilon_\varepsilon(x-y))^{2^*-1} u_\lambda^+ dx \\
&\quad - \lambda\int_\Omega \Big\{\int_0^{t\upsilon_\varepsilon(x-y)} [(u_\lambda^+ + s)^{q-1} - (u_\lambda^+)^{q-1}] ds\Big\} dx + o(\varepsilon^{\frac{N-2}{2}}) \\
&\le \alpha_\lambda^+ + \frac{t^2}{2} S^{\frac{N}{2}} - \frac{t^{2^*}}{2^*} S^{\frac{N}{2}} - t^{2^*-1}\int_\Omega (\upsilon_\varepsilon(x-y))^{2^*-1} u_\lambda^+ dx + o(\varepsilon^{\frac{N-2}{2}})
\end{aligned}
$$

for all $y \in \Omega_r^-$.

Applying (3.5) and the fact that $\int_\Omega (\upsilon_\varepsilon(x-y))^{2^*-1}dx = O(\varepsilon^{\frac{N-2}{2}})$, also note the compactness of Ω_r^-, we conclude that there exist $\varepsilon_0, \sigma(\varepsilon) > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ and $\sigma \in (0, \sigma(\varepsilon))$,

$$\sup_{t\geq 0} J_\lambda(u_\lambda^+ + t\upsilon_\varepsilon(x-y)) < \frac{1}{N}S^{N/2} + \alpha_\lambda^+ - \sigma \quad \text{uniformly in } y \in \Omega_r^-. \tag{3.7}$$

Next we will prove that there exists $t^-_{(\lambda,\varepsilon,y)} > 0$ such that $u_\lambda^+ + t^-_{(\lambda,\varepsilon,y)}\upsilon_\varepsilon(x-y) \in S_\lambda^-$ for each $y \in \Omega_r^-$. Let

$$U_1 = \left\{u \in H_0^1(\Omega)\backslash\{0\}; \frac{1}{\|u\|}t^-\left(\frac{u}{\|u\|}\right) > 1\right\} \cup \{0\};$$
$$U_1 = \left\{u \in H_0^1(\Omega)\backslash\{0\}; \frac{1}{\|u\|}t^-\left(\frac{u}{\|u\|}\right) < 1\right\}.$$

Then S_λ^- disconnects $H_0^1(\Omega)$ into two connected components U_1 and U_2. Moreover, $H_0^1(\Omega)\backslash S_\lambda^- = U_1 \cup U_2$. For each $u \in S_\lambda^+$, we have

$$1 < t_{\max} < t^-(u).$$

Since $t^-(u) = \frac{1}{\|u\|}t^-\left(\frac{u}{\|u\|}\right)$, then $S_\lambda^+ \subset U_1$. In particular, $u_\lambda^+ \in U_1$. We claim that we can find a constant $c > 0$ such that

$$0 < t^-\left(\frac{u_\lambda^+ + t\upsilon_\varepsilon(x-y)}{\|u_\lambda^+ + t\upsilon_\varepsilon(x-y)\|}\right) < c \quad \text{for each } t \geq 0 \text{ and } y \in \Omega_r^-.$$

Otherwise, there exists a sequence $\{t_k\}$ such that $t_k \to \infty$ and

$$t^-\left(\frac{u_\lambda^+ + t_k\upsilon_\varepsilon(x-y)}{\|u_\lambda^+ + t_k\upsilon_\varepsilon(x-y)\|}\right) \to \infty \quad \text{as } k \to \infty.$$

Let

$$\upsilon_k = \frac{u_\lambda^+ + t_k\upsilon_\varepsilon(x-y)}{\|u_\lambda^+ + t_k\upsilon_\varepsilon(x-y)\|}.$$

Since $t^-(\upsilon_k)\upsilon_k \in S_\lambda^- \subset S_\lambda$ and by the Lesbesgue dominated convergence theorem,

$$\begin{aligned}
\int_\Omega |\upsilon_k|^{2^*}dx &= \frac{1}{\|u_\lambda^+ + t_k\upsilon_\varepsilon(x-y)\|^{2^*}}\int_\Omega |u_\lambda^+ + t_k\upsilon_\varepsilon(x-y)|^{2^*}dx \\
&= \frac{1}{\|\frac{u_\lambda^+}{t_k} + \upsilon_\varepsilon(x-y)\|^{2^*}}\int_\Omega |\frac{u_\lambda^+}{t_k} + \upsilon_\varepsilon(x-y)|^{2^*}dx \\
&\to \frac{\int_\Omega |\upsilon_\varepsilon|^{2^*}dx}{\|\upsilon_\varepsilon\|^{2^*}} \quad \text{as } k \to \infty,
\end{aligned}$$

we have

$$\begin{aligned}
J_\lambda(t^-(\upsilon_k)\upsilon_k) &= \frac{1}{2}[t^-(\upsilon_k)]^2 - \lambda\frac{[t^-(\upsilon_k)]^q}{q}\int_\Omega |\upsilon_k|^q dx \\
&\quad - \frac{[t^-(\upsilon_k)]^{2^*}}{2^*}\int_\Omega |\upsilon_k|^{2^*}dx \to -\infty \quad \text{as } k \to \infty.
\end{aligned}$$

This contradicts that J_λ is bounded below on S_λ and the claim is proved. Let

$$t_\lambda = \frac{|c^2 - \|u_\lambda^+\|^2|^{\frac{1}{2}}}{\|\upsilon_\varepsilon\|} + 1,$$

then

$$\begin{aligned}\|u_\lambda^+ + t_\lambda \upsilon_\varepsilon(x-y)\|^2 &= \|u_\lambda^+\|^2 + t_\lambda^2\|\upsilon_\varepsilon\|^2 + 2t_\lambda\langle u_\lambda^+, \upsilon_\varepsilon(x-y)\rangle \\ &> \|u_\lambda^+\|^2 + |c^2 - \|u_\lambda^+\|^2| + 2t_\lambda\int_\Omega u_\lambda^+\upsilon_\varepsilon(x-y)dx \\ &> c^2 > \big[t^-\big(\frac{u_\lambda^+ + t_\lambda\upsilon_\varepsilon(x-y)}{\|u_\lambda^+ + t_\lambda\upsilon_\varepsilon(x-y)\|}\big)\big]^2,\end{aligned}$$

that is $u_\lambda^+ + t_\lambda\upsilon_\varepsilon(x-y) \in U_2$.

Thus there exists $0 < t^-_{(\lambda,\varepsilon,y)} < t_\lambda$ such that $u_\lambda^+ + t^-_{(\lambda,\varepsilon,y)}\upsilon_\varepsilon(x-y) \in S_\lambda^-$. Moreover, by (3.7) and Lemma 3.3, we obtain $\Phi(u_\lambda^+ + t^-_{(\lambda,\varepsilon,y)}\upsilon_\varepsilon(x-y)) \in \Omega_r^+$ for each $y \in \Omega_r^-$. □

From Lemma 3.4, we can define the map $\gamma : \Omega_r^- \to S_\lambda^-(c_\lambda - \sigma)$ defined by

$$\gamma(y)(x) := u_\lambda^+(x) + t^-_{(\lambda,\varepsilon,y)}\upsilon_\varepsilon(x-y).$$

Furthermore, by Lemma 2.4 (ii) and Lemma 2.6 (iv), we can define the map $\Phi_\lambda : S_\lambda^- \to \mathbb{R}^N$ by setting

$$\Phi_\lambda(u) := \frac{\int_\Omega x(u-u_\lambda^+)_+^{2^*}\,dx}{\int_\Omega (u-u_\lambda^+)_+^{2^*}\,dx}.$$

Then for each $y \in \Omega_r^-$, note $\upsilon_\varepsilon(x)$ is radial, we have

$$(\Phi_\lambda \circ \gamma)(y) = y.$$

Next we define the map $H_\lambda : [0,1] \times S_\lambda^-(c_\lambda - \sigma) \to \mathbb{R}^N$ by

$$H_\lambda(t,u) = t\Phi_\lambda(u) + (1-t)\Phi_\lambda(u).$$

Lemma 3.5. *For $\varepsilon \in (0,\varepsilon_0)$, there exists $0 < \lambda_0 \le \min\{\lambda_1,\lambda_2,\lambda_3,\sigma(\varepsilon)\}$ such that if $\lambda,\sigma \in (0,\lambda_0)$,*

$$H_\lambda([0,1] \times S_\lambda^-(c_\lambda - \sigma)) \subset \Omega_r^+.$$

Proof. Suppose by contradiction that there exist $t_k \in [0,1]$, $\lambda_k, \sigma_k, \to 0$, and $u_k \in S^-_{\lambda_k}(c_{\lambda_k} - \sigma_k)$ such that

$$H_{\lambda_k}(t_k,u_k) \notin \Omega_r^+ \quad \text{for all } k.$$

Furthermore, we can assume that $t_k \to t_0 \in [0,1]$. Then by Lemma 2.6 (iv) and argue as in the proof of Lemma 3.3, we have

$$H_{\lambda_k}(t_k,u_k) \to y \in \overline{\Omega}, \quad \text{as } k \to \infty,$$

which is a contradiction. □

4. Proof of Theorem 1.1

We begin with the following Lemma.

Lemma 4.1. *If u is a critical point of J_λ on S_λ^-, then it is a critical point of J_λ in $H_0^1(\Omega)$.*

Proof. Assume $u \in S_\lambda^-$, then $\langle J_\lambda'(u), u\rangle = 0$. On the other hand,

$$J_\lambda'(u) = \theta\psi_\lambda'(u) \tag{4.1}$$

for some $\theta \in \mathbb{R}$, where ψ_λ is defined in (2.1). We remark that $u \in S_\lambda^-$, and so $\langle \psi_\lambda'(u), u\rangle < 0$. Thus by (4.1)

$$0 = \theta\langle \psi_\lambda'(u), u\rangle,$$

which implies that $\theta = 0$, consequently $J_\lambda'(u) = 0$. □

Below we denote by $J_{S_\lambda^-}$ the restriction of J_λ on S_λ^-.

Lemma 4.2. *Any sequence $\{u_k\} \subset S_\lambda^-$ such that $J_{S_\lambda^-}(u_k) \to \beta \in (-\infty, \frac{1}{N}S^{N/2} + \alpha_\lambda^+)$ and $J'_{S_\lambda^-}(u_k) \to 0$ contains a convergent subsequence for all $\lambda \in (0, \lambda_0)$.*

Proof. By hypothesis there exists a sequence $\{\theta_k\} \subset \mathbb{R}$ such that

$$J_\lambda'(u_k) = \theta_k\psi_\lambda'(u_k) + o(1).$$

Recall that $u_k \in S_\lambda^-$ and so

$$\langle \psi_\lambda'(u_k), u_k\rangle < 0.$$

If $\langle \psi_\lambda'(u_k), u_k\rangle \to 0$, we from (2.2) and (2.3) obtain that there are two positive numbers c_1, c_2 independent of u_k and λ such that

$$\int_\Omega |\nabla u_k|^2 dx \le c_1\Big(\int_\Omega |\nabla u_k|^2 dx\Big)^{2^*/2} + o(1),$$
$$\int_\Omega |\nabla u_k|^2 dx \le \lambda c_2\Big(\int_\Omega |\nabla u_k|^2 dx\Big)^{q/2} + o(1)$$

or

$$\int_\Omega |\nabla u_k|^2 dx \ge c_1^{-\frac{2}{2^*-2}} + o(1), \quad \int_\Omega |\nabla u_k|^2 dx \le (\lambda c_2)^{\frac{2}{2-q}} + o(1).$$

If λ is sufficiently small, this is impossible. Thus we may assume that $\langle \psi_\lambda'(u_k), u_k\rangle \to l < 0$. Since $\langle J_\lambda'(u_k), u_k\rangle = 0$, we conclude that $\theta_k \to 0$ and, consequently, $J_\lambda'(u_k) \to 0$. Using this information we have

$$J_\lambda(u_k) \to \beta \in (-\infty, \frac{1}{N}S^{N/2} + \alpha_\lambda^+), \quad J_\lambda'(u_k) \to 0,$$

so by Lemma 2.5 the proof is complete. □

Lemma 4.3. *If $\lambda, \sigma \in (0, \lambda_0)$, then*

$$\mathrm{cat}(S_\lambda^-(c_\lambda - \sigma)) \ge \mathrm{cat}(\Omega).$$

Proof. Suppose that

$$S_\lambda^-(c_\lambda - \sigma) = A_1 \cup \dots \cup A_n,$$

where A_j, $j = 1, \dots, n$, is closed and contractible in $S_\lambda^-(c_\lambda - \sigma)$, i.e., there exists $h_j \in C([0,1] \times A_j, S_\lambda^-(c_\lambda - \sigma))$ such that

$$h_j(0, u) = u, \quad h_j(1, u) = \omega \quad \text{for all } u \in A_j,$$

where $\omega \in A_j$ is fixed. Consider $B_j := \gamma^{-1}(A_j)$, $1 \le j \le n$. The sets B_j are closed and

$$\Omega_r^- = B_1 \cup \dots \cup B_n.$$

Note Lemma 3.5, we define the deformation $g_j : [0,1] \times B_j \to \Omega_r^+$ by setting

$$g_j(t, y) := H_\lambda(t, h_j(t, \gamma(y))).$$

for $\lambda \in (0, \lambda_0)$. Note that

$$g_j(0, y) := H_\lambda(0, h_j(0, \gamma(y))) = y \quad \text{for all } y \in B_j$$

and

$$g_j(1, y) := H_\lambda(1, h_j(1, \gamma(y))) = \Phi(\omega) \in \Omega_r^+.$$

Thus the sets B_j are contractible in Ω_r^+. It follows that

$$\operatorname{cat}(\Omega) = \operatorname{cat}_{\Omega_r^+}(\Omega_r^-) \leq n.$$

□

Proof of Theorem 1.1. Applying Lemmas 2.5 and 4.2, $J_{S_\lambda^-}$ satisfies the $(PS)_\beta$ condition for all $\beta \in (-\infty, \frac{1}{N}S^{N/2} + \alpha_\lambda^+)$. Then, by Lemmas 3.2 and 4.3, $J_{S_\lambda^-}$ contains at least $\operatorname{cat}(\Omega)$ critical points in $S_\lambda^-(c_\lambda - \sigma)$. Hence, from Lemma 4.1, J_λ has at least $\operatorname{cat}(\Omega)$ critical points in S_λ^-. Moreover, by Lemma 2.6 and $S_\lambda^+ \cap S_\lambda^- = \emptyset$ we complete the proof. □

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