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MAXIMUM AND MINIMUM PRINCIPLES FOR NONLINEAR TRANSPORT EQUATIONS ON DISCRETE-SPACE DOMAINS

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Abstract. We consider nonlinear scalar transport equations on the domain with discrete space and continuous time. As a motivation we derive a conservation law on these domains. In the main part of the paper we prove maximum and minimum principles that are later applied to obtain an a priori bound which is applied in the proof of existence of solution and its uniqueness. Further, we study several consequences of these principles such as boundedness of solutions, sign preservation, uniform stability and comparison theorem which deals with lower and upper solutions.

1. Introduction

The transport equation is one of the simplest nonlinear partial differential equations. Its importance follows from the fact that it describes traveling waves and that it forms the basis for study of hyperbolic equations of second order. The reader can see, e.g., [11] for details about transport PDE.

We study transport equations on the domain with discrete space and continuous time. This is a combination of difference and differential equations. As an application of these models we can mention semidiscrete numerical methods of Rothe or Galerkin (see [10, 16]). We consider nonlinear equations that arise from conservation laws. Linear equations that combine continuous, discrete and time-scale variables are studied in [20]. In that paper authors present some interesting relations between equations of this type and stochastic processes of Poisson–Bernoulli type.

In recent years so called dynamical systems on lattices have been studied extensively. In [6, 7, 12] authors deal with these related problems and focus on PDEs of reaction–diffusion type on finite space lattices. Their results can be helpful, e.g., in the modelling of binary alloys (see [7]).

Moreover, in the last few years the analysis of equations on infinite lattices has attracted some researchers. We can refer to [2, 3, 4, 21] for the introduction to these problems. These papers are concerned mainly with existence of traveling waves in discrete reaction–diffusion equations and their properties. The reader is invited to

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see [21] where the main ideas and principles of this field are presented. Problems in [2, 3, 4, 21] are solved often by topological methods using fixed point theorems, degree theory, comparison principles and lower and upper solutions.

Our analysis can contribute to this mathematical area. Our problem can be understood as an equation on infinite lattice. With the help of maximum and minimum principles we derive new comparison theorem that deals with ordering of lower and upper solutions.

In general, we study simpler problems than reaction-diffusion equations but on the other hand, our work can be interesting for another reason as well. It can be useful just from the point of view of maximum and minimum principles. These principles are strong tools in the theory of differential equations. They have many applications and important consequences. We can mention, e.g., a priori bounds that can be applied in proofs of existence and uniqueness of solution, oscillation results. For the review about these topics in ODEs and PDEs see [14] or more recent book [15]. In discrete problems these principles have rich behavior. The reader is invited to see papers [13, 17, 18, 19] or survey book about partial difference equations [5] for further details. Consequently, we want to explore if the transport equation where we combine continuous and discrete approach has some fruitful properties as in these works.

The structure of our paper is as follows. First, we motivate our study, derive a conservation law in discrete space and formulate our main problem in Section 2. In Section 3 we prove maximum and minimum principles for the nonlinear equation by the so-called *stairs method*. Then we deal with existence and uniqueness of solution in Section 4 and with other consequences in Section 5. In Section 6, we study a related nonlinear problem. At the end of the paper, in Section 7, we present some open problems and directions of future research.

We denote the intervals $[0,+\infty)$ and $(0,+\infty)$ by $\mathbb{R}_0^+$ and $\mathbb{R}^+$ respectively. Partial derivative of $u(x,t)$ w.r.t. t is denoted by $u_t(x,t)$ and partial difference w.r.t. x by

$$\nabla_x u(x,t) = u(x,t) - u(x-1,t).$$

2. Conservation law and nonlinear transport equation

As a motivation we derive the conservation law in discrete space. It leads to partial equations on discrete-space domain. Corresponding continuous conservation laws are presented, e.g., in [11].

We consider one dimensional discrete space. We simulate it by integers. Further, we suppose the density $u = u(x,t)$ which changes continuously in time and which is distributed in discrete space. The magnitude u can express, e.g., the concentration of mass or population, energy etc.

We denote by φ the flux of u. The flux $\varphi(i,t)$, $i \in \mathbb{Z}$, $t \in \mathbb{R}_0^+$, quantifies the amount of u that passes between positions $x = i$ and $x = i+1$ in time t. Further, $f = f(x,t)$ is the source function.

Therefore, consider an arbitrary space segment between $x = i$ and $x = j$ when $i < j$. The time change of total amount in that space segment between $x = i$ and $x = j$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{x=i}^{j} u(x,t) = \varphi(i-1,t) - \varphi(j,t) + \sum_{x=i}^{j} f(x,t). \tag{2.1}$$

We call (2.1) the conservation law in global form. Let us modify (2.1) as follows

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{x=i}^{j}u(x,t) = -\big[\varphi(j,t)-\varphi(j-1,t)+\varphi(j-1,t)-\dots-\varphi(i,t)+\varphi(i,t)\\ -\varphi(i-1,t)\big]+\sum_{x=i}^{j}f(x,t),\end{aligned}$$

and finally, we obtain

$$\sum_{x=i}^{j}\big[u_t(x,t)+\nabla_x\varphi(x,t)-f(x,t)\big]=0.$$

The space segment is arbitrary and thus, the following conservation law in local form has to hold necessarily

$$u_t(x,t)+\nabla_x\varphi(x,t)=f(x,t). \tag{2.2}$$

We study the case of

$$\varphi(x,t)=F(x,t,u(x,t)) \quad \text{when} \quad F:\mathbb{Z}\times\mathbb{R}_0^+\times\mathbb{R}\to\mathbb{R}.$$

This leads to the nonlinear transport equation with discrete space. Therefore, we deal with the following initial-boundary value problem (I-BVP):

$$\begin{gathered}u_t(x,t)+\nabla_xF(x,t,u(x,t))=f(x,t), \quad x\in\mathbb{Z},\ x>a\in\mathbb{Z},\ t\in\mathbb{R}^+,\\ u(x,0)=\phi(x), \quad \phi:\mathbb{Z}\to\mathbb{R},\\ u(a,t)=\xi(t), \quad \xi\in C(\mathbb{R}_0^+)\cap C^1(\mathbb{R}^+),\end{gathered} \tag{2.3}$$

where $F:\mathbb{Z}\times\mathbb{R}_0^+\times\mathbb{R}\to\mathbb{R}$ and $f:\mathbb{Z}\times\mathbb{R}_0^+\to\mathbb{R}$. We prove maximum and minimum principles for lower and upper solutions.

Definition 2.1. The function $v(x,t)$ is called a *lower solution* of (2.3) if

$$\begin{gathered}v_t(x,t)+\nabla_xF(x,t,v(x,t))\le f(x,t), \quad x\in\mathbb{Z},\ x>a\in\mathbb{Z},\ t\in\mathbb{R}^+,\\ v(x,0)\le\phi(x), \quad x\in\mathbb{Z},\ x>a\in\mathbb{Z},\\ v(a,t)\le\xi(t), \quad t\in\mathbb{R}_0^+.\end{gathered}$$

The function $w(x,t)$ is an *upper solution* of (2.3) if

$$\begin{gathered}w_t(x,t)+\nabla_xF(x,t,w(x,t))\ge f(x,t), \quad x\in\mathbb{Z},\ x>a\in\mathbb{Z},\ t\in\mathbb{R}^+,\\ w(x,0)\ge\phi(x), \quad x\in\mathbb{Z},\ x>a\in\mathbb{Z},\\ w(a,t)\ge\xi(t), \quad t\in\mathbb{R}_0^+.\end{gathered}$$

3. Maximum and minimum principles

In this section we derive main tools of our study, the maximum and minimum principles. Let us mention that if we consider problem (2.3) with more general difference

$$\nabla_x^{(\mu)}u(x,t)=\frac{u(x,t)-u(x-\mu,t)}{\mu}$$

with arbitrary step $\mu>0$ we can prove following results in the similar way. Hence, for the sake of simplicity we suppose only difference with unitary step $\nabla_xu(x,t)$. Next technical lemma helps us in the proof of maximum principle.

Lemma 3.1. *Let* $F : \mathbb{Z} \times \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ *satisfy*

(A1) $F(\chi, \tau, \omega)$ *is increasing in* χ*, i.e., for all* $\chi_1 < \chi_2$ *there is*
$$F(\chi_1, \tau, \omega) \le F(\chi_2, \tau, \omega),$$

(A2) $F(\chi, \tau, \omega)$ *is strictly increasing in* ω*, i.e., for all* $\omega_1 < \omega_2$ *there is*
$$F(\chi, \tau, \omega_1) < F(\chi, \tau, \omega_2).$$

Then the following holds:
$$\text{if } F(\chi_1, \tau, \omega_1) \le F(\chi_2, \tau, \omega_2) \text{ then } \chi_1 \le \chi_2 \text{ or } \omega_1 \le \omega_2, \tag{3.1}$$
$$\text{if } F(\chi_1, \tau, \omega_1) < F(\chi_2, \tau, \omega_2) \text{ then } \chi_1 < \chi_2 \text{ or } \omega_1 < \omega_2. \tag{3.2}$$

Proof. We show only (3.1). The proof of (3.2) is similar. Let us suppose by contradiction that $\chi_1 > \chi_2$ and $\omega_1 > \omega_2$. Then we have
$$F(\chi_2, \tau, \omega_2) \overset{\text{(A1)}}{\le} F(\chi_1, \tau, \omega_2) \overset{\text{(A2)}}{<} F(\chi_1, \tau, \omega_1),$$
a contradiction with the assumption of $F(\chi_1, \tau, \omega_1) \le F(\chi_2, \tau, \omega_2)$. □

Theorem 3.2 (Maximum principle)**.** *Assume that* $F(\chi, \tau, \omega)$ *satisfies* (A1) *and* (A2) *and* $f(\chi, \tau) \le 0$ *for all* $\chi \in \mathbb{Z}, \chi > a$*,* $\tau \in \mathbb{R}^+$*. Let* $u(x,t)$ *be a lower solution of* (2.3)*. Then*
$$u(x,t) \le \sup_{\substack{x \in \mathbb{Z}, x \ge a \\ t \in \mathbb{R}_0^+}} \{\phi(x), \xi(t)\}$$
holds for all $x \in \mathbb{Z}$*,* $x \ge a$*, and for all* $t \in \mathbb{R}_0^+$*.*

Proof. We prove the statement by the so-called *stairs method*. The idea of our proof is shown on Figure 1. First, we denote
$$M := \sup_{\substack{x \in \mathbb{Z}, x \ge a \\ t \in \mathbb{R}_0^+}} \{\phi(x), \xi(t)\}.$$
Assume by contradiction that there exist $x_0 \in \mathbb{Z}$, $x_0 > a$, and $t_0 \in \mathbb{R}^+$ such that
$$u(x_0, t_0) > M. \tag{3.3}$$
Now from assumptions (A2), (3.3) and from the fact that $u(x,t)$ is a lower solution we obtain
$$u_t(x_0, t_0) \le F(x_0 - 1, t_0, u(x_0 - 1, t_0)) - F(x_0, t_0, u(x_0, t_0)), \tag{3.4}$$
$$u_t(x_0, t_0) < F(x_0 - 1, t_0, u(x_0 - 1, t_0)) - F(x_0, t_0, M). \tag{3.5}$$
Now there are two possibilities.

(1) If $F(x_0 - 1, t_0, u(x_0 - 1, t_0)) > F(x_0, t_0, M)$ then from (3.2) in Lemma 3.1 we get $u(x_0 - 1, t_0) > M$. Hence, in this case we define
$$x_1 = x_0 - 1 \quad \text{and} \quad t_1 = t_0.$$

(2) The second possibility is that $F(x_0 - 1, t_0, u(x_0 - 1, t_0)) \le F(x_0, t_0, M)$ holds. From (3.5) there is $u_t(x_0, t_0) < 0$. Therefore, the function $u(x_0, t)$ is strictly decreasing in $t = t_0$ and we can define
$$\overline{t_0} = \inf\{\tau = [0, t_0] : u(x_0, t) \text{ is strictly decreasing on the interval } (\tau, t_0)\}.$$

If $\overline{t_0} = 0$ then we have a contradiction with the definition of M via the initial condition $\phi(x)$. If $\overline{t_0} > 0$ then there is necessarily $u_t(x_0, \overline{t_0}) = 0$ and from (3.4) we obtain

$$F(x_0, \overline{t_0}, u(x_0, \overline{t_0})) \leq F(x_0 - 1, \overline{t_0}, u(x_0 - 1, \overline{t_0})).$$

Then (3.1) in Lemma 3.1 implies $u(x_0, \overline{t_0}) \leq u(x_0 - 1, \overline{t_0})$ which gives

$$M < u(x_0, t_0) < u(x_0, \overline{t_0}) \leq u(x_0 - 1, \overline{t_0}).$$

Consequently, in this case we define

$$x_1 = x_0 - 1 \quad \text{and} \quad t_1 = \overline{t_0}.$$

Finally, we have $u(x_1, t_1) > M$. If we continue iteratively then after at most $x_0 - a$ steps we get a contradiction with definition of M. $\square$

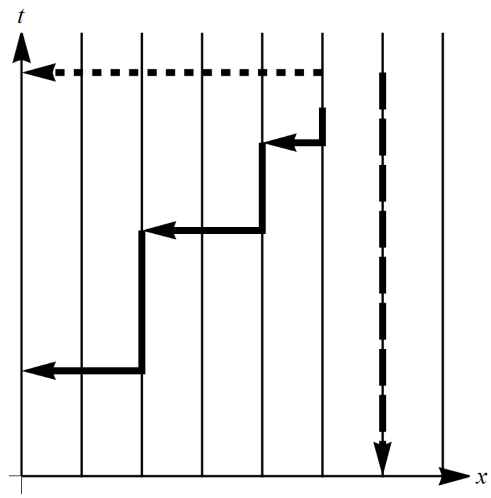

FIGURE 1. The idea of the stairs method. The dotted line shows the situation when only possibility (1) occurs which yields a contradiction via the boundary condition $\xi(t)$. The bold line shows the combination of possibilities (1) and (2) and a contradiction via the boundary condition $\xi(t)$ again. The dashed line shows the situation when we get a contradiction via the initial condition $\phi(x)$ in possibility (2).

Next we have the minimum principle which can be proved by a stairs method similarly to the one in Theorem 3.2.

Theorem 3.3 (Minimum principle). *Assume that $F(\chi, \tau, \omega)$ satisfies* (A2) *and*

(A3) *$F(\chi, \tau, \omega)$ is decreasing in χ,*

and $f(\chi, \tau) \geq 0$ for all $\chi \in \mathbb{Z}, \chi > a$, $\tau \in \mathbb{R}^+$. Let $u(x,t)$ be an upper solution of (2.3)*. Then*

$$\inf_{\substack{x \in \mathbb{Z},\, x \geq a \\ t \in \mathbb{R}_0^+}} \{\phi(x), \xi(t)\} \quad \leq \quad u(x,t)$$

holds for all $x \in \mathbb{Z}$, $x \geq a$, and for all $t \in \mathbb{R}_0^+$.

4. Existence and uniqueness of solution

In this section we use maximum and minimum principles as a priori bounds to prove the existence and uniqueness of solution of (2.3). The proof is based on induction and further, we use the following lemma about global solution of IVP for ordinary differential equation.

Lemma 4.1 ([9, Corollary 8.64]). *Consider the following IVP for ordinary differential equation*

$$\begin{gathered} u'(t) = g(t, u(t)), \quad g : I \times \mathbb{R}^n \to \mathbb{R}^n, \\ u(t_0) = u_0, \quad u_0 \in \mathbb{R}^n, \end{gathered} \tag{4.1}$$

when $I \subset \mathbb{R}$ is an interval. Assume that $h : \mathbb{R}_0^+ \to \mathbb{R}^+$ is continuous and there is a $v_0 \in \mathbb{R}_0^+$ such that

$$\int_{v_0}^{+\infty} \frac{\mathrm{d}s}{h(s)} = +\infty.$$

Let the function $g : [t_0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and let

$$\|g(\tau, \omega)\| \le h(\|\omega\|)$$

hold for all $(\tau, \omega) \in [t_0, +\infty) \times \mathbb{R}^n$. Then for all $u_0 \in \mathbb{R}^n$ with $\|u_0\| \le v_0$ all solutions of (4.1) exist on $[t_0, +\infty)$.

Theorem 4.2 (Existence and uniqueness). *Suppose that:*

- (A4) *$\phi(x)$, $\xi(t)$ are bounded; i.e., there exist $K > 0$ such that for all $x \in \mathbb{Z}, x \ge a$, and for all $t \in \mathbb{R}_0^+$ $|\phi(x)| \le K$ and $|\xi(t)| \le K$ hold,*
- (A5) *$f(\chi, \tau) = 0$ identically,*

the function $F = F(\tau, \omega)$ is independent of χ, satisfies (A2) *and*

- (A6) *$F(\tau, \omega)$ is continuous w.r.t. τ on $\mathbb{R}_0^+$,*
- (A7) *$F(\tau, \omega)$ is locally Lipschitz continuous w.r.t. ω on $\mathbb{R}_0^+ \times \mathbb{R}$, i.e., for all $\tau_0 \in \mathbb{R}_0^+$ and for all $\omega_0 \in \mathbb{R}$ there exists a rectangle*
$$\mathcal{R}(\tau_0, \omega_0) = \big\{(\tau, \omega) \in \mathbb{R}_0^+ \times \mathbb{R} : \quad 0 \le \tau - \tau_0 \le a, |\omega - \omega_0| \le b\big\}$$
and $L = L(\tau_0, \omega_0) > 0$ such that for all $(\tau, \omega_1), (\tau, \omega_2) \in \mathcal{R}(\tau_0, \omega_0)$ there is
$$|F(\tau, \omega_1) - F(\tau, \omega_2)| \le L|\omega_1 - \omega_2|,$$
- (A8) *$F(\tau, \omega)$ is sublinear w.r.t. ω, i.e., there exist $A, B > 0$ such that for all $\tau \in \mathbb{R}_0^+$ and for all $\omega \in \mathbb{R}$ there is*
$$|F(\tau, \omega)| \le A|\omega| + B.$$

Then (2.3) *possesses a unique solution $u(x, t)$ which is defined for all $x \in \mathbb{Z}$, $x \ge a$, and $t \in \mathbb{R}_0^+$.*

Proof. We prove the statement by induction on $x \in \mathbb{Z}$, $x \ge a$.

(1) For $x = a$ we put $u(a, t) = \xi(t)$.

(2) Let us have a solution $u(\overline{x}, t)$ which is unique and defined for all $\overline{x} \in \mathbb{Z}$, $a \le \overline{x} < x$, on $\mathbb{R}_0^+$. Then for fixed x we get from (2.3) the following IVP for ordinary differential equation

$$\begin{gathered} u_t(x, t) = F(x-1, t, u(x-1, t)) - F(x, t, u(x, t)), \\ u(x, 0) = \phi(x), \quad \phi(x) \in \mathbb{R}, \end{gathered} \tag{4.2}$$

where $F(x-1, t, u(x-1, t))$ is a given function of t from the induction hypothesis.

• Assumptions (A6), (A7) and Picard-Lindelöf's theorem (see [9, Theorem 8.13]) imply the existence and uniqueness of a local solution $u(x,t)$ of (4.2) on some small interval $[0,\delta]$, $\delta>0$.

• We can make the estimate

$$
\begin{aligned}
&|F(x-1,t,u(x-1,t))-F(x,t,u(x,t))|\\
&\le |F(x-1,t,u(x-1,t))|+|F(x,t,u(x,t))|\\
&\overset{\text{(A8)}}{\le} A|u(x-1,t)|+A|u(x,t)|+2B\\
&\overset{\text{Th. 3.2+Th. 3.3+(A4)}}{\le} A|u(x,t)|+AK+2B.
\end{aligned}
$$

If we define $g(t,u)=F(x-1,t,u(x-1,t))-F(x,t,u)$, $h(s)=As+AK+2B$ and $v_0=|\phi(x)|$ then assumptions of Lemma 4.1 are satisfied. Therefore, the local solution $u(x,t)$ can be extended to the whole $\mathbb{R}_0^+$.

• Finally, we have to check if there is no other solution from some time $t_0>0$ which disjoins from $u(x,t)$ in t_0. Hence, suppose by contradiction that there is a $t_0>0$ such that there exist two solutions $u_1(x,t)$ and $u_2(x,t)$ of (4.2) with $u_1(x,t)=u_2(x,t)$ on $[0,t_0]$ and $u_1(x,t)\ne u_2(x,t)$ on $(t_0,t_0+\epsilon)$, $\epsilon>0$. Let us denote $u_{t_0}=u_1(x,t_0)$ and investigate the solvability of the IVP

$$
\begin{gathered}
u_t(x,t)=F(x-1,t,u(x-1,t))-F(x,t,u(x,t)),\quad t>t_0,\\
u(x,t_0)=u_{t_0}.
\end{gathered}
\tag{4.3}
$$

The right-hand side of equation in (4.3) is unique by induction hypotheses. Functions $u_1(x,t)$, $u_2(x,t)$ solve (4.3) on $[t_0,t_0+\epsilon)$. But assumptions of Picard–Lindelf's theorem are also satisfied for (4.3) thanks to (A6), (A7) and consequently, there cannot be two distinct solutions. This is a contradiction which finishes the proof. □

Remark 4.3. If we omit the assumption (A7) of local Lipschitz continuity of $F(\tau,\omega)$ in Theorem 4.2 then the uniqueness is not guaranteed and we get only the existence result by the same procedure with the help of Cauchy–Peano's theorem (see [9, Theorem 8.27]) instead of Picard–Lindelöf's theorem.

We present the following example for an illustration what functions $F(\chi,\tau,\omega)$ can be considered in Theorem 4.2.

Example 4.4. Assumptions of Theorem 4.2 are satisfied, e.g., for following functions $F(\tau,\omega)$:

- $F(\tau,\omega)=k(\tau)\omega$ when $k(\tau)>0$ (linear equation),
- $F(\tau,\omega)=k(\tau)\arctan\omega$ when $k(\tau)>0$.

For the following function F we have only existence guaranteed (cf. Remark 4.3):

- $F(\tau,\omega)=\begin{cases}-\sqrt[3]{-\omega}, & \text{for } \omega<0,\\ \sqrt[3]{\omega}, & \text{for } \omega\ge 0.\end{cases}$

5. Consequences of maximum and minimum principles

In this section we study well-known consequences of maximum and minimum principles. Corresponding results for classical differential equations can be found in [14]. The next two corollaries follow immediately from Theorems 3.2 and 3.3.

Corollary 5.1 (Boundedness of solutions)**.** *Let $F = F(\tau,\omega)$ satisfy Assumption* (A2)*, $f(\chi,\tau) = 0$ identically, $\phi(x)$ and $\xi(t)$ be bounded and $u(x,t)$ be a solution of* (2.3)*. Then $u(x,t)$ is bounded.*

Corollary 5.2 (Sign preservation)**.** *Let $F = F(\chi,\tau,\omega)$ satisfy* (A2) *and* (A3)*, $f(\chi,\tau)$ be nonnegative, $\phi(x)$ and $\xi(t)$ be nonnegative and $u(x,t)$ be a solution of* (2.3)*. Then $u(x,t)$ is nonnegative.*

Last application of maximum and minimum principles from Theorems 3.2 and 3.3 is the uniform stability of solutions of the linear problem and its consequences. Thus, let us consider the linear problem

$$
\begin{gathered}
u_t(x,t) + \nabla_x \left[k(t)u(x,t)\right] = 0, \quad x \in \mathbb{Z}, \quad x > a \in \mathbb{Z},\ t \in \mathbb{R}^+, \\
u(x,0) = \phi(x), \quad \phi : \mathbb{Z} \to \mathbb{R}, \\
u(a,t) = \xi(t), \quad \xi \in C(\mathbb{R}_0^+) \cap C^1(\mathbb{R}^+),
\end{gathered}
\tag{5.1}
$$

where $k(t) > 0$.

Corollary 5.3 (Uniform stability)**.** *Let $u_1(x,t)$ be a solution of* (5.1) *with initial-boundary conditions $\phi_1(x)$ and $\xi_1(x)$. Let $u_2(x,t)$ be a solution of* (5.1) *with initial-boundary conditions $\phi_2(x)$ and $\xi_2(x)$. Then*

$$
\sup_{\substack{x \in \mathbb{Z}, x \geq a \\ t \in \mathbb{R}_0^+}} |u_1(x,t) - u_2(x,t)| \leq \sup_{\substack{x \in \mathbb{Z}, x \geq a \\ t \in \mathbb{R}_0^+}} \{|\phi_1(x) - \phi_2(x)|, |\xi_1(t) - \xi_2(t)|\}
\tag{5.2}
$$

holds.

Proof. Define function $v(x,t) = u_1(x,t) - u_2(x,t)$. Then $v(x,t)$ solves I-BVP (5.1) with the initial-boundary conditions $\phi_1(x) - \phi_2(x)$ and $\xi_1(t) - \xi_2(t)$. Assumptions of the maximum principle in Theorem 3.2 are satisfied and hence, we obtain

$$
\begin{aligned}
u_1(x,t) - u_2(x,t) = v(x,t) &\leq \sup_{\substack{x \in \mathbb{Z}, x \geq a \\ t \in \mathbb{R}_0^+}} \{\phi_1(x) - \phi_2(x), \xi_1(t) - \xi_2(t)\} \\
&\leq \sup_{\substack{x \in \mathbb{Z}, x \geq a \\ t \in \mathbb{R}_0^+}} \{|\phi_1(x) - \phi_2(x)|, |\xi_1(t) - \xi_2(t)|\}.
\end{aligned}
\tag{5.3}
$$

Similarly, assumptions of the minimum principle in Theorem 3.3 are satisfied and therefore, there is

$$
\begin{aligned}
u_1(x,t) - u_2(x,t) = v(x,t) &\geq \inf_{\substack{x \in \mathbb{Z}, x \geq a \\ t \in \mathbb{R}_0^+}} \{\phi_1(x) - \phi_2(x), \xi_1(t) - \xi_2(t)\} \\
&\geq - \sup_{\substack{x \in \mathbb{Z}, x \geq a \\ t \in \mathbb{R}_0^+}} \{|\phi_1(x) - \phi_2(x)|, |\xi_1(t) - \xi_2(t)|\}.
\end{aligned}
\tag{5.4}
$$

Finally, inequalities in (5.3) and (5.4) yield (5.2). □

Corollary 5.3 directly implies the following claim.

Corollary 5.4. *Let $\{u_n\}_{n=1}^{+\infty}$ be a sequence of solutions $u_n(x,t)$ of* (5.1) *with the initial-boundary conditions $\phi_n(x)$ and $\xi_n(t)$ such that*

$$\phi_n(x) \rightrightarrows \phi(x) \text{ for } x \in \mathbb{Z}, x \geq a, \quad \text{and} \quad \xi_n(t) \rightrightarrows \xi(t) \text{ for } t \in \mathbb{R}_0^+.$$

Assume that $u(x,t)$ is a solution of (5.1) *with the initial-boundary conditions $\phi(x)$ and $\xi(t)$. Then*

$$u_n(x,t) \rightrightarrows u(x,t) \quad \text{for} \quad x \in \mathbb{Z}, x \geq a, \quad \text{and} \quad t \in \mathbb{R}_0^+.$$

6. Similar problem with space difference inside nonlinearity

In this section we analyze a similar problem as (2.3). We consider the following I-BVP where the nonlinear function F depends on difference of $u(x,t)$:

$$\begin{gathered} u_t(x,t) + F(x,t,\nabla_x u(x,t)) = f(x,t), \quad x \in \mathbb{Z},\ x > a \in \mathbb{Z},\ t \in \mathbb{R}^+, \\ u(x,0) = \phi(x), \quad \phi : \mathbb{Z} \to \mathbb{R}, \\ u(a,t) = \xi(t), \quad \xi \in C(\mathbb{R}_0^+) \cap C^1(\mathbb{R}^+). \end{gathered} \tag{6.1}$$

Remark 6.1. We define lower and upper solutions of (6.1) similarly as in Definition 2.1.

The following two theorems are the maximum and minimum principles for (6.1). We let proofs to the reader because we can prove them by stairs method again.

Theorem 6.2 (Maximum principle)**.** *Assume that $F(\chi,\tau,\omega)$ satisfies*

(A9) *for all $\chi \in \mathbb{Z}, \chi > a$, and for all $\tau \in \mathbb{R}^+$, there is*

$$F(\chi,\tau,\omega) \begin{cases} > 0, & \text{for } \omega > 0, \\ < 0, & \text{for } \omega < 0, \\ = 0, & \text{for } \omega = 0, \end{cases}$$

and $f(\chi,\tau) \leq 0$ for all $\chi \in \mathbb{Z}, \chi > a$, $\tau \in \mathbb{R}^+$. Let $u(x,t)$ be a lower solution of (6.1)*. Then*

$$u(x,t) \quad \leq \sup_{\substack{x \in \mathbb{Z}, x \geq a \\ t \in \mathbb{R}_0^+}} \{\phi(x), \xi(t)\}$$

holds for all $\chi \in \mathbb{Z}$, $\chi \geq a$, and for all $\tau \in \mathbb{R}^+$.

Theorem 6.3 (Minimum principle)**.** *Assume that $F(\chi,\tau,\omega)$ satisfies* (A9) *and $f(\chi,\tau) \geq 0$ for all $x \in \mathbb{Z}$, $x > a$, and for all $t \in \mathbb{R}_0^+$. Let $u(x,t)$ be an upper solution of* (6.1)*. Then*

$$\inf_{\substack{x \in \mathbb{Z}, x \geq a \\ t \in \mathbb{R}_0^+}} \{\phi(x), \xi(t)\} \quad \leq \quad u(x,t)$$

holds for all $x \in \mathbb{Z}$, $x \geq a$, and for all $t \in \mathbb{R}_0^+$.

Now, we introduce analogue results for (6.1) as in Sections 4 and 5. We omit proofs again because they are also similar as for (2.3).

Theorem 6.4 (Existence and uniqueness)**.** *Suppose that* (A4), (A5) *hold, function $F(\chi,\tau,\omega)$ satisfies* (A6)–(A9)*. Then* (6.1) *possesses a unique solution $u(x,t)$ which is defined for all $x \in \mathbb{Z}$, $x \geq a$, and $t \in \mathbb{R}_0^+$.*

Corollary 6.5 (Boundedness of solutions)**.** *Let $F(\chi,\tau,\omega)$ satisfy Assumption* (A9)*, $f(\chi,\tau)=0$ identically, $\phi(x)$ and $\xi(t)$ be bounded and $u(x,t)$ be a solution of* (6.1)*. Then $u(x,t)$ is bounded.*

Corollary 6.6 (Sign preservation)**.** *Let $F(\chi,\tau,\omega)$ satisfy* (A9)*, $f(\chi,\tau)$ be nonnegative, $\phi(x)$ and $\xi(t)$ be nonnegative and $u(x,t)$ be a solution of* (6.1)*. Then $u(x,t)$ is nonnegative.*

Finally, in contrast to previous sections about the problem (2.3), we are able to prove following assertions about nonlinear problem (6.1).

Corollary 6.7 (Uniform stability)**.** *Consider a function $F(\chi,\tau,\omega)$ for which the partial derivative $F_\omega(\chi,\tau,\omega)$ is a continuous and positive function. Let $u_1(x,t)$ be a solution of* (6.1) *with initial-boundary conditions $\phi_1(x)$ and $\xi_1(t)$. Let $u_2(x,t)$ be a solution of* (6.1) *with initial-boundary conditions $\phi_2(x)$ and $\xi_2(t)$. Then*

$$\sup_{\substack{x\in\mathbb{Z},x\geq a\\ t\in\mathbb{R}_0^+}}|u_1(x,t)-u_2(x,t)|\leq\sup_{\substack{x\in\mathbb{Z},x\geq a\\ t\in\mathbb{R}_0^+}}\{|\phi_1(x)-\phi_2(x)|,|\xi_1(t)-\xi_2(t)|\}$$

holds.

Proof. We prove the statement with the help of maximum and minimum principles from Theorems 6.2 and 6.3. Thanks to the assumption that $u_1(x,t)$ and $u_2(x,t)$ are solutions we get the equality

$$(u_1)_t(x,t)+F(x,t,\nabla_x u_1(x,t))-(u_2)_t(x,t)-F(x,t,\nabla_x u_2(x,t))=0.$$

Applying the mean value theorem we can rewrite it to the form

$$(u_1)_t(x,t)-(u_2)_t(x,t)+F_\omega(x,t,\theta(x,t))\nabla_x(u_1(x,t)-u_2(x,t))=0,$$

where $\theta(x,t)=\alpha\nabla_x u_1(x,t)+(1-\alpha)\nabla_x u_2(x,t)$, $\alpha\in[0,1]$. Let us define an auxiliary function $v(x,t)=u_1(x,t)-u_2(x,t)$. Consequently, $v(x,t)$ solves

$$\begin{gathered}v_t(x,t)+F_\omega(x,t,\theta(x,t))\nabla_x v(x,t)=0,\\ v(x,0)=\phi_1(x)-\phi_2(x),\\ v(a,t)=\xi_1(t)-\xi_2(t),\end{gathered}$$

when the assumptions of Theorems 6.2 and 6.3 are satisfied. Thus, from Theorem 6.2 we obtain

$$\begin{aligned}u_1(x,t)-u_2(x,t)=v(x,t)&\leq\sup_{\substack{x\in\mathbb{Z},x\geq a\\ t\in\mathbb{R}_0^+}}\{\phi_1(x)-\phi_2(x),\xi_1(t)-\xi_2(t)\}\\ &\leq\sup_{\substack{x\in\mathbb{Z},x\geq a\\ t\in\mathbb{R}_0^+}}\{|\phi_1(x)-\phi_2(x)|,|\xi_1(t)-\xi_2(t)|\}.\end{aligned}$$

Similarly, from Theorem 6.3, there is

$$\begin{aligned} u_1(x,t) - u_2(x,t) = v(x,t) &\geq \inf_{\substack{x\in\mathbb{Z}, x\geq a\\ t\in\mathbb{R}_0^+}} \{\phi_1(x)-\phi_2(x), \xi_1(t)-\xi_2(t)\} \\ &\geq - \sup_{\substack{x\in\mathbb{Z}, x\geq a\\ t\in\mathbb{R}_0^+}} \{|\phi_1(x)-\phi_2(x)|, |\xi_1(t)-\xi_2(t)|\} \end{aligned}$$

which completes the proof. □

Corollary 6.8. *Consider a function $F(\chi,\tau,\omega)$ for which the partial derivative $F_\omega(\chi,\tau,\omega)$ is a continuous and positive function. Let $\{u_n\}_{n=1}^{+\infty}$ be a sequence of solutions $u_n(x,t)$ of* (6.1) *with the initial-boundary conditions $\phi_n(x)$ and $\xi_n(t)$ for that*

$$\phi_n(x) \rightrightarrows \phi(x) \text{ for } x\in\mathbb{Z}, x\geq a, \quad \text{and} \quad \xi_n(t) \rightrightarrows \xi(t) \text{ for } t\in\mathbb{R}_0^+.$$

Assume that $u(x,t)$ is a solution of (6.1) *with the initial-boundary conditions $\phi(x)$ and $\xi(t)$. Then*

$$u_n(x,t) \rightrightarrows u(x,t) \quad \text{for } x\in\mathbb{Z},\ x\geq a,\ t\in\mathbb{R}_0^+.$$

Corollary 6.9 (Comparison theorem)**.** *Consider a function $F(\chi,\tau,\omega)$ for which the partial derivative $F_\omega(\chi,\tau,\omega)$ is continuous and positive function. Suppose, there exists a solution $u(x,t)$ of* (6.1)*. Moreover, let $v(x,t)$ be a lower solution and $w(x,t)$ be an upper solution of* (6.1)*. Then*

$$v(x,t) \leq u(x,t) \leq w(x,t)$$

is necessarily satisfied for all $x\in\mathbb{Z}$, $x\geq a$, and for all $t\in\mathbb{R}_0^+$.

Proof. We define two auxiliary functions $\overline{v}(x,t) = u(x,t) - v(x,t)$ and $\overline{w}(x,t) = w(x,t) - u(x,t)$ and investigate their sign.

(1) First, we study the function $\overline{v}(x,t)$. Because $v(x,t)$ is a lower solution we get

$$0 \leq u_t(x,t) + F(x,t,\nabla_x u(x,t)) - v_t(x,t) - F(x,t,\nabla_x v(x,t)).$$

Thanks to assumptions on F we can use the mean value theorem and we can continue with our estimate,

$$\begin{aligned} 0 &\leq u_t(x,t) + F(x,t,\nabla_x u(x,t)) - v_t(x,t) - F(x,t,\nabla_x v(x,t)) \\ &= [u(x,t)-v(x,t)]_t + F_\omega(x,t,\theta(x,t))\,[\nabla_x u(x,t) - \nabla_x v(x,t)] \\ &= \overline{v}_t(x,t) + F_\omega(x,t,\theta(x,t))\nabla_x \overline{v}(x,t). \end{aligned}$$

for some $\theta(x,t) = \alpha\nabla_x u(x,t) + (1-\alpha)\nabla_x v(x,t)$, $\alpha\in[0,1]$. For initial and boundary conditions we have

$$\begin{gathered} \overline{v}(x,0) = u(x,0) - v(x,0) \geq 0, \\ \overline{v}(a,t) = u(a,t) - v(a,t) \geq 0. \end{gathered}$$

Thus, assumptions of Theorem 6.3 are satisfied for $\overline{v}(x,t)$ which implies

$$\overline{v}(x,t) \geq 0, \quad \text{i.e.,} \quad v(x,t) \leq u(x,t).$$

(2) For the function $\overline{w}(x,t)$ it is similar. By the same procedure we get

$$\overline{w}(x,t) \geq 0, \quad \text{i.e.,} \quad u(x,t) \leq w(x,t).$$

□

Remark 6.10. *If we would like to prove the similar assertions for* (2.3) *by the same procedure then proofs would fail after using the mean value theorem. In that case, the backward difference operator* ∇_x *would be applied on the partial derivative* $F_\omega(x, t, \theta(x, t))$*. Hence, we would not be able to satisfy assumptions of Theorems 3.2 and 3.3 because we would not know the behavior of the function* $\theta(x, t)$*.*

7. Concluding remarks

In this paper we present some maximum and minimum principles for transport equations with discrete space and continuous time and derive several applications. But there are still many open questions left.

First, we can try to find another maximum principles with distinct or weaker assumptions or we can try to derive another properties of solutions of (2.3) and (6.1). Next, we should say that, although, we consider nonlinear function F as a function $F(\chi, \tau, \omega)$ in our problems, in many cases we have to assume that F is not a function of χ. Therefore, we can try to improve it and find better conditions.

We study only initial–boundary value problems as well. We can ask what will change if we consider an initial value problem on the whole $\mathbb{Z}$. One can show that in that case we cannot prove maximum or minimum principles in the same way by *stairs method* as Theorem 3.2. Moreover, we cannot use mathematical induction to prove the existence of solution of IVP because we have not where to start.

Further, we could try to generalize our results for more general time and space structures as in [17, 18, 19] (in these papers dynamic equations on time-scales are studied, for more information about time-scale calculus see [1, 8]).

In this paper we analyze equations with one space variable and hence, we can state the question what happens if we consider more space variables as on finite-dimensional lattice dynamical systems in [6, 7, 12].

Another natural generalization is to study evolutionary equations of higher order, e.g., diffusion or wave-type equations on discrete-space domains as in [2, 3, 4, 21].

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References

[1] M. Bohner, A. Peterson; *Dynamic Equations on Time Scales: An Introduction with Applications*, Boston: Birkhuser, 2001.
[2] X. Chen et. al.; *Traveling Waves in Discrete Periodic Media for Bistable Dynamics*, Arch. Rational Mech. Anal. **189** (2008), pp. 189-236.
[3] X. Chen, J.-S. Guo; *Existence and Asymptotic Stability of Traveling Waves of Discrete Quasilinear Monostable Equations*, J. Differential Eq. **184** (2002), pp. 549-569.
[4] X. Chen, J.-S. Guo; *Uniqueness and existence of traveling waves for discrete quasilinear monostable dynamics*, Math. Ann. **326** (2003), pp. 123-146.
[5] S. S. Cheng; *Partial Difference Equations*, London: Taylor and Francis, 2003.
[6] S.-N. Chow, J. Mallet-Paret, W. Shen; *Traveling Waves in Lattice Dynamical Systems*, J. Differential Eq. **149** (1998), pp. 248-291.
[7] S.-N. Chow et al.; *Dynamical Systems*, Berlin: Springer, 2003.

[8] S. Hilger; *Analysis on measure chains – A unified approach to continuous and discrete calculus*, Results Math. **18**(1-2) (1990), pp. 18-56.
[9] W. G. Kelley, A. C. Peterson; *The Theory of Differential Equations: Classical and Qualitative*, New York: Springer, 2010.
[10] R. J. Le Veque; *Finite Volume Methods for Hyperbolic Problems* (Cambridge Texts in Applied Mathematics), Cambridge: Cambridge University Press, 2002.
[11] J. D. Logan; *An Introduction to Nonlinear Partial Differential Equations*, New Yersey: John Wiley, 2008.
[12] J. Mallet-Paret; *The global structure of traveling waves in spatial discrete dynamical systems*, J. Dynam. Differential Eq. **11**(1) (1999), pp. 49-127.
[13] J. Mawhin, H. B. Thompson, E. Tonkes; *Uniqueness for boundary value problems for second order finite difference equations*, J. Difference Eq. Appl. **10**(8) (2004), pp. 749-757.
[14] M. H. Protter, H. F. Weinberger; *Maximum Principles in Differential Equations*, New York: Springer-Verlag, 1984.
[15] P. Pucci, J. Serrin; *The Maximum Principle*, Basel: Birkhuser, 2007.
[16] E. Rothe; *Zweidimensionale parabolische randwertaufgaben als grenzfall eindimensionaler randwertaufgaben*, Math. Ann. 102 (1930), pp. 650670.
[17] P. Stehlík; *Maximum principles for elliptic dynamic equations*, Math. Comp. Mod. **51**(9-10) (2010), pp. 1193-1201.
[18] P. Stehlík, B. Thompson; *Applications of maximum principles to dynamic equations on time scales*, J. Difference Eq. Appl. **16**(4) (2010), pp. 373-388.
[19] P. Stehlík, B. Thompson; *Maximum principles for second order dynamic equations on time scales*, J. Math. Anal. Appl. **331** (2007), pp. 913-926.
[20] P. Stehlík, J. Volek; *Transport equation on semidiscrete domains and Poisson-Bernoulli processes*, J. Difference Eq. Appl. **19**(3) (2013), pp. 439-456.
[21] B. Zinner; *Existence of Traveling Wavefront Solutions for the Discrete Nagumo Equation*, J. Differential Eq. **96** (1992), pp. 1-27.

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