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EXISTENCE OF SOLUTIONS FOR AN n-DIMENSIONAL OPERATOR EQUATION AND APPLICATIONS TO BVPS

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Abstract. By applying the Guo-Lakshmikantham fixed point theorem on high dimensional cones, sufficient conditions are given to guarantee the existence of positive solutions of a system of equations of the form

$$x_i(t)=\sum_{k=1}^{n}\sum_{j=1}^{n}\gamma_{ij}(t)w_{ijk}(\Lambda_{ijk}[x_k])+(F_ix)(t),\quad t\in[0,1],\quad i=1,\dots,n.$$

Applications are given to three boundary value problems: A 3-dimensional 3+3+3 order boundary value problem with mixed nonlocal boundary conditions, a 2-dimensional 2+4 order nonlocal boundary value problem discussed in [14], and a 2-dimensional 2+2 order nonlocal boundary value problem discussed in [35]. In the latter case we provide some fairly simpler conditions according to those imposed in [35].

1. Introduction

In most of the cases, where systems of boundary value problems are discussed and make use of Krasnosel'skii' s fixed point theorem (see [23], reformulated by Guo-Lakshmikantham [6]), the authors construct an auxiliary scalar equation and then use a cone in the real valued functions space. See, for example [8, 9, 10, 25, 36, 39] and the references therein. Here, motivated from some ideas applied to 2-dimensional systems in, e.g., [14, 26, 30, 35], we suggest the use of a high-dimensional cone to provide sufficient conditions for the existence of positive solutions of an operator equation of the form

$$x(t)=(Rx)(t)+(Fx)(t),\quad t\in[0,1]=:I,\tag{1.1}$$

lying in a cone of the space $\tilde{C}_n(I):=C(I,\mathbb{R})^n\simeq C(I,\mathbb{R}^n)$, where F is a compact operator acting on $\tilde{C}_n(I)$ and taking values therein.

Equation (1.1) can be thought of as a perturbation of the compact operator equation $x=Fx$. And, if the perturbation R is a contraction, then Krasnosel'skii's fixed point theorem (see, e.g., [22]) may provide sufficient conditions for the existence of solutions (lying into a pre-specified closed convex set). In this case the right-hand side of (1.1) maps a (nonempty) closed, convex, set into itself. A more

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general version of Krasnosel'skii's fixed point theorem can be found elsewhere in [19].

In this article we assume that the perturbation R is a (not necessarily contraction) function and it has the coordinate-separated form

$$(Rx)_i(t) := \sum_{k=1}^{n}\sum_{j=1}^{n}\gamma_{ij}(t)w_{ijk}(\Lambda_{ijk}[x_k]), \quad t\in I,\ i=1,\dots,n, \tag{1.2}$$

where, for all indices $i,j,k,\in\{1,2,\dots,n\}$ the item $\Lambda_{ijk}[\cdot]$ is a linear functional acting on the coordinate x_k of $x:=(x_n,x_2,\dots x_n)$. (Detailed conditions will be given in the text.)

A system of the form (1.1)-(1.2) is generated by a great number of boundary value problems. In [12] Infante et al., investigate the pair of the differential equations

$$\begin{gathered} u''(t)+g_1(t)f_1(t,u(t),v(t))=0, \quad t\in(0,1)\\ v^{(4)}(t)=g_2(t)f_2(t,u(t),v(t)), \quad t\in(0,1), \end{gathered}$$

associated with the boundary conditions

$$\begin{gathered} u(0)=\beta_{11}[u], \quad u(1)=\delta_{12}[v],\\ v(0)=\beta_{21}[v], \quad v''(0)=0, \quad v(1)=0, \quad v''(1)+\delta_{22}[u]=0, \end{gathered}$$

where β_{ij} and δ_{ij} are linear functionals defined by means of Riemann - Stieltjies integrals as follows:

$$\begin{aligned} \beta_{ij}[w]&=\int_0^1 w(s)dB_{ij}(s), \\ \delta_{ij}[w]&=\int_0^1 w(s)dC_{ij}(s). \end{aligned} \tag{1.3}$$

This system leads to the pair of integral equations of the form

$$\begin{aligned} u(t)&=\sum_{i=1,2}\gamma_{1i}(t)\Big(H_{1i}(\beta_{1i}[u])+L_{1i}(\delta_{1i}[v])\Big)+\int_0^1 k_1(t,s)g_1(s)f_1(s,u(s),v(s))ds,\\ v(t)&=\sum_{i=1,2}\gamma_{2i}(t)\Big(L_{2i}(\delta_{2i}[u])+H_{2i}(\beta_{2i}[v])\Big)+\int_0^1 k_2(t,s)g_2(s)f_2(s,u(s),v(s))ds, \end{aligned} \tag{1.4}$$

discussed, mainly, in [14]. The authors, in order to get their results do use of an idea applied by Infante in [11] and the classical fixed point index theory. These forms include as special cases several multi-point and integral conditions, assumed elsewhere, as, e.g., in [1, 2, 3, 4, 5, 12, 15, 16, 17, 18, 24, 31, 38].

A 2-dimensional second order differential system with Dirichlet boundary conditions (first-type) is studied by Xiyou Cheng at al. [3] and by Bingmei Liu et al. [24], while the same equation with mixed boundary conditions is studied, e.g., by Ling Hu et al. in [10]. The 2-dimensional Sturm-Liouville problem for a second order ordinary differential equation discussed by Henderson et al. in [7] and Yang in [35] leads to a system of the form (1.4), but with zero the first summation terms in the right side. Thus, only, the Hammerstein integral parts appear. See, also, Zhilin Yang [37]. The works due to Pietramala [28] and D. Franco et al. [13] refer to perturbed Hammerstein type integral equations. Some 2-dimensional $n+m$-order multi-point singular boundary value problems with mixed type boundary conditions are discussed by Hua Su et al. in [30]. The case of p-Laplacian, investigated, e.g, by Baofang Liu et al. in [26] for systems and by Karakostas in [20, 21], for 1-dimensional equations, is not covered by our situation, since in those cases the corresponding operators are expressed implicitly and, therefore, the perturbation R is not expressed coordinate separated.

In this article we shall apply the Guo-Lakshmikantham fixed point theorem on cones in $\tilde{C}_n(I)$. For the (classical) case of 1-dimensional cone (namely, cones in $\tilde{C}_1(I) = C(I, \mathbb{R})$), we refer, first, to the Hammerstein-type integral equation

$$u(t) = \gamma(t)\alpha[u] + \int_0^1 k(t,s)g(s)f(s,u(s))ds,$$

which is generated by a great number of local and non-local boundary value problems, and it is investigated by several authors as, e.g., by Webb [32] and Webb et al. in [34, 33]. Here, $\alpha[u]$ means a linear functional of the form (1.3). Also, we refer to Henderson et al. in [8] who studied a system of the form

$$\begin{aligned}
u(t) &= \int_0^T G_1(t,s)f(s,v(s))ds, \quad t \in [0,T] \\
v(t) &= \int_0^T G_2(t,s)g(s,u(s))ds, \quad t \in [0,T]
\end{aligned}$$

generated by a 2-dimensional second order boundary value problem with Liouville-type boundary conditions. Due to the form of the system, the authors of [8] prefer (quite naturally) to use a one dimensional equation and then to seek for sufficient conditions which guarantee the existence of positive fixed points of the operator

$$(\mathcal{A}u)(t) = \int_0^T G_1(t,s)f\big(s, \int_0^T G_2(s,\tau)g(\tau,u(\tau))d\tau\big)ds.$$

See, also, the references in [8]. The same idea was already used for ordinary differential equations, e.g., in [29, 39], while for functional differential equations, e.g., in [9] and the references therein.

In section 4 we shall apply our general existence results to the 3-dimensional system of third order differential equations of the form

$$u_i''' + X_i(u) = 0, \quad i = 1,2,3, \tag{1.5}$$

with $u := (u_1, u_2, u_3)$, associated with the mixed nonlocal boundary conditions

$$\begin{aligned}
u_i(0) &= \lambda \sum_{k=1}^n A_{ik}[u_k], \\
u_i'(1) &= \lambda \sum_{k=1}^n B_{ik}[u_k], \\
u_i''(0) &= \lambda \sum_{k=1}^n \Gamma_{ij}[u_k], .
\end{aligned} \tag{1.6}$$

for $i = 1,2,3$.

Another example, which we shall discuss, is the system of second-order nonlocal boundary value problem

$$\begin{aligned} -u'' &= f(t,u,v), \\ -v'' &= g(t,u,v), \\ u(0) &= v(0) = 0, \\ u(1) &= H_1\Big(\int_0^1 u(s)d\alpha(s)\Big), \\ v(1) &= H_2\Big(\int_0^1 v(s)d\beta(s)\Big), \end{aligned} \tag{1.7}$$

investigated in [35]. We show that, under rather mild conditions (which differ from those in [35]), at least one positive solution exists.

We close the paper by showing that the existence results of [14] can be obtained by applying our general theorem.

2. Some preliminaries

Following a classical procedure, we look for conditions guaranteeing the existence of a fixed point of the operator equation

$$x = Tx,$$

where T is the operator defined by

$$(Tx)_i(t) = \sum_{k=1}^n \sum_{j=1}^n \gamma_{ij}(t) w_{ijk}(\Lambda_{ijk}[x_k]) + (F_i x)(t), \quad t \in I,\ i = 1, \dots, n. \tag{2.1}$$

The domain of T is the space $\tilde{C}_n(I)$ endowed with the norm $|||x||| := \max_i \|x_i\|_\infty$, where $\|\cdot\|_\infty$ stands for the sup-norm in the space $C(I,\mathbb{R})$.

The main tools, which we shall use, lie on the following well known results of the fixed point index, see, e.g., [6, 23].

Theorem 2.1. *Let E be a Banach space, K a cone in E, and $\Omega(K)$ a bounded open subset of K with $0 \in \Omega(K)$. Suppose that $S : \overline{\Omega(K)} \to K$ is a completely continuous operator. If*

$$Su \neq \mu u, \quad \forall u \in \partial\Omega(K), \quad \mu \geq 1,$$

then the fixed point index

$$i(S, \Omega(K), K) = 1.$$

Theorem 2.2. *Let E be Banach space, K a cone in E and $\Omega(K)$ a bounded open subset of K. Suppose that $S : \overline{\Omega(K)} \to K$ is a completely continuous operator. If there exists $u_0 \in K \setminus \{0\}$ such that*

$$u - Su \neq \mu u_0, \quad \forall u \in \partial\Omega(K), \quad \mu \geq 0,$$

then the fixed point index

$$i(S, \Omega(K), K) = 0.$$

An obvious combination of Theorems 2.1 and 2.2 imply the existence of a (nonzero) fixed point in the cone.

Before presenting our results, we want to recall some facts from the Perron-Frobenius matrix theory concerning positive matrices. In particular we borrow some results from [27].

Let $\langle\cdot,\cdot\rangle$ be the known inner product in $\mathbb{R}^n$ and let $\geq$ be the strict coordinate-wise partial order in $\mathbb{R}^n$. Extending the notation, for a square matrix A, the symbol $A \geq 0$ (resp. $A > 0$) means that each entry of A is nonnegative (resp. positive). Also, A^T stands for the transpose of A, A^{-1} for the inverse of A and $\rho(A)$ is used for the spectral radius of A, namely the quantity

$$\rho(A) := \max\{|\lambda| : \lambda \in \mathbb{C},\ \det(\lambda I_{n\times n} - A) = 0\}.$$

An $n \times n$ matrix A that can be expressed in the form

$$A = sI_{n\times n} - B,$$

where $B = (b_{ij})$, with $b_{ij} > 0$, $1 \leq i, j \leq n$, and $s > \rho(B)$, is called an M-matrix. Obviously, an M-matrix is nonsingular.

[27, Theorem 1] provides forty conditions which are equivalent to the fact that the matrix with non-positive off-diagonal entries is an M-matrix.

Theorem 2.3. *Each of the following conditions is equivalent to the statement: A is an M-matrix.*

- (F15) *A is inverse-positive. That is, A^{-1} exists and $A^{-1} > 0$.*
- (F16) *A is monotone. That is,*

$$Ax \geq 0 \implies x \geq 0, \quad \textit{for all} \quad x \in \mathbb{R}^n.$$

- (N39) *A has all positive diagonal elements, and there exists a positive diagonal matrix D such that AD is strictly diagonally dominant. That is it satisfies the condition*

$$a_{ii}d_i > \sum_{j\neq i} |a_{ij}|d_j,$$

for $i = 1, 2, \dots, n$.

3. Main results

We start by setting our main conditions:

- (C1) All the functions w_{ijk} map $[0, +\infty)$ into itself, continuously.
- (C2) There exist $n \times n$-square nonnegative matrices $(a_{ij}), (b_{ij})$ and for each $k = 1, 2, \dots, n$, a matrix (η_{ijk}) such that

$$\begin{gathered} a_{ij} = 0 \implies b_{ij} = 0, \\ a_{ij}\xi \leq w_{iji}(\xi) \leq b_{ij}\xi, \quad \xi \geq 0, \\ k \neq i \implies w_{ijk}(\xi) \leq \eta_{ijk}\xi, \quad \xi \geq 0. \end{gathered}$$

- (C3) For all indices i, j, k the function Λ_{ijk} is linear and it maps the space $C^+(I) = C(I, \mathbb{R}^+)$ into $\mathbb{R}^+$, continuously.
- (C4) For each i the function F_i maps $\tilde{C}_n(I)$ into $C(I, \mathbb{R})$ and it is completely continuous.
- (C5) For each $i = 1, 2, \dots, n$, there exist continuous functions $U_i : C^n(I) \to [0, +\infty)$, such that

$$t \in I \text{ and } x \geq 0 \implies (F_i x)(t) \leq U_i(x).$$

(C6) There exists $c > 0$ and, for each $i = 1, 2, \dots, n$, there exist nontrivial intervals $[\alpha_i, \beta_i] \subseteq I$, such that

$$t \in [\alpha_i, \beta_i] \text{ and } x \geq 0 \implies (F_i x)(t) \geq cU_i(x).$$

(C7) For each i, j, the function γ_{ij} maps the interval I into $\mathbb{R}^+$, it is continuous and there exists $\sigma_{ij} \in (0, 1]$, such that

$$\sigma_{ij} \|\gamma_{ij}\|_\infty \leq \gamma_{ij}(t), \quad t \in [\alpha_i, \beta_i].$$

Put

$$d_{ij} := \begin{cases} a_{ij}/b_{ij}, & \text{if } b_{ij} \geq a_{ij} > 0 \\ 1, & \text{if } b_{ij} = a_{ij} = 0, \end{cases}$$

and $\zeta_i := \min\{c, \min_j \sigma_{ij} d_{ij}\}$, which, obviously, satisfies

$$\sigma_{ij} a_{ij} \geq \zeta_i b_{ij},$$

for all $i, j = 1, 2, \dots, n$.

Now, for each $i = 1, 2, \dots, n$, define the cone

$$K_i := \{u \in C^+(I) : u(t) \geq \zeta_i \|u\|_\infty, \quad t \in [\alpha_i, \beta_i]\}.$$

Then, the cartesian product

$$K := \times_i K_i$$

is a (vector) cone in $\tilde{C}_n(I)$.

For any fixed $\rho > 0$, define the cone section

$$K_\rho := \{x \in K : |||x||| < \rho\}.$$

We shall show the following result.

Lemma 3.1. *Under the previous conditions, the operator T defined by* (2.1) *maps the cone K into itself and it is completely continuous.*

Proof. Take any $x \in K$. Then $x_i \in K_i$ and so we have on the one hand

$$\|(Tx)_i\|_\infty \leq \sum_{k=1}^n \sum_{j=1}^n \|\gamma_{ij}\|_\infty b_{ij} \Lambda_{ijk}[x_k] + U_i(x),$$

and on the other hand, for all $t \in [\alpha_i, \beta_i]$,

$$\begin{aligned}(Tx)_i(t) &\geq \sum_{k=1}^n \sum_{j=1}^n \sigma_{ij} \|\gamma_{ij}\|_\infty a_{ij} \Lambda_{ijk}[x_k] + cU_i(x) \\ &\geq \zeta_i \Big[\sum_{k=1}^n \sum_{j=1}^n \|\gamma_{ij}\|_\infty b_{ij} \Lambda_{ijk}[x_k] + U_i(x) \Big] \\ &\geq \zeta_i \|(Tx)_i\|_\infty.\end{aligned}$$

The latter says that $TK \subseteq K$.

The complete continuity property of the operator T follows, easily, from conditions (C1)–(C4). $\square$

Next, for any fixed $\rho > 0$, define the set

$$V_\rho := \{x \in K : \sup_i \min_{t \in [\alpha_i, \beta_i]} x_i(t) < \rho\}.$$

Obviously, it satisfies the relation

$$K_\rho \subset V_\rho \subset K_{\rho/\zeta}, \tag{3.1}$$

where $\zeta := \min_i \zeta_i$. Set

$$p_{ijk} := \Lambda_{kik}[\gamma_{kj}]b_{kj},$$

and consider the $n \times n$ square matrix $P_k := (p_{ijk})$. Let

$$z_{im} := \sum_{k\neq m}^{n} \sum_{j=1}^{n} \Lambda_{mim}[\gamma_{mj}]\eta_{mjk}\Lambda_{mjk}[1] + \Lambda_{mim}[1]\Theta_\rho, \tag{3.2}$$

where

$$\Theta_\rho := \max_i \sup_{|||x|||=\rho} \frac{U_i(x)}{\rho}.$$

Also, we let the n-dimensional vectors

$$z_m := (z_{1m}, z_{2m}, \dots, z_{nm})^T,$$
$$d_i := (\|\gamma_{i1}\|_\infty b_{i1}, \|\gamma_{i2}\|_\infty b_{i2}, \dots, \|\gamma_{in}\|_\infty b_{in})^T$$

as well as the quantities

$$M_{i\rho} := \sum_{k\neq i}^{n} \sum_{j=1}^{n} \|\gamma_{ij}\|_\infty \eta_{ijk}\Lambda_{ijk}[1] + \Theta_\rho, \quad i = 1, 2, \dots, n.$$

Lemma 3.2. *Assume that for each $k = 1, 2, \dots, n$, the item $I_{n\times n} - P_k$ is an M-matrix and, moreover, the inequality*

$$\langle d_k, (I_{n\times n} - P_k)^{-1} z_k\rangle + M_{k\rho} < 1, \tag{3.3}$$

holds, for some $\rho > 0$ and all $k = 1, 2, \dots, n$. Then the operator T defined in (2.1) *satisfies the relation*

$$i_K(T, K_\rho) = 1.$$

Proof. To show the result we shall apply Theorem 2.1, namely we shall show that

$$\mu x \neq Tx,$$

for all $x \in \partial K_\rho$ and any $\mu \geq 1$. Indeed, let us assume that there is $\mu \geq 1$ with

$$\mu x = Tx,$$

for some $x \in \partial K_\rho$. Then, there is a coordinate x_{i_0} of x satisfying

$$\|x_{i_0}\| = \rho \quad \text{and} \quad \|x_j\| \leq \rho,$$

for all indices j.

From (3.2) we have

$$\begin{aligned} x_{i_0}(t) \leq \mu x_{i_0}(t) &= \sum_{k=1}^{n} \sum_{j=1}^{n} \gamma_{i_0 j}(t) w_{i_0 jk}(\Lambda_{i_0 jk}[x_k]) + (F_{i_0}x)(t) \\ &\leq \sum_{j=1}^{n} \gamma_{i_0 j}(t) b_{i_0 j}\Lambda_{i_0 j i_0}[x_{i_0}] + \sum_{k\neq i_0}^{n} \sum_{j=1}^{n} \gamma_{i_0 j}(t)\eta_{i_0 jk}\Lambda_{i_0 jk}[x_k] + (F_{i_0}x)(t). \end{aligned} \tag{3.4}$$

From the positivity of the functionals $\Lambda_{i_0ii_0}$ it follows that

$$\begin{aligned}
\Lambda_{i_0ii_0}[x_{i_0}] &\leq \sum_{j=1}^{n} \Lambda_{i_0ii_0}[\gamma_{i_0j}] b_{i_0j} \Lambda_{i_0ji_0}[x_{i_0}] \\
&\quad + \sum_{k\neq i_0}^{n} \sum_{j=1}^{n} \Lambda_{i_0ii_0}[\gamma_{i_0j}] \eta_{i_0jk} \Lambda_{i_0jk}[x_k] + \Lambda_{i_0ii_0}[F_{i_0}x]. \\
&\leq \sum_{j=1}^{n} \Lambda_{i_0ii_0}[\gamma_{i_0j}] b_{i_0j} \Lambda_{i_0ji_0}[x_{i_0}] \qquad (3.5)\\
&\quad + \rho\Big(\sum_{k\neq i_0}^{n} \sum_{j=1}^{n} \Lambda_{i_0ii_0}[\gamma_{i_0j}] \eta_{i_0jk} \Lambda_{i_0jk}[1] + \Lambda_{i_0ii_0}[1]\Theta_{\rho}\Big) \\
&= \sum_{j=1}^{n} \Lambda_{i_0ii_0}[\gamma_{i_0j}] b_{i_0j} \Lambda_{i_0ji_0}[x_{i_0}] + \rho z_{ii_0}.
\end{aligned}$$

Letting

$$v_{jk} := \Lambda_{kjk}[x_k], \quad v_k := (v_{1k}, v_{2k}, \dots, v_{nk})^T,$$

we obtain the system of vector inequalities

$$v_{i_0} \leq P_{i_0} v_{i_0} + \rho z_{i_0}.$$

Therefore we have

$$(I_{n\times n} - P_{i_0}) v_{i_0} \leq \rho z_{i_0}. \tag{3.6}$$

From our assumption and Theorem 2.3 we know that the matrix $I_{n\times n} - P_{i_0}$ is inverse-positive and monotone. Thus from (3.6), we obtain

$$v_{i_0} \leq \rho (I_{n\times n} - P_{i_0})^{-1} z_{i_0}. \tag{3.7}$$

Now, from (3.4) we obtain

$$\begin{aligned}
x_{i_0}(t) &\leq \sum_{j=1}^{n} \gamma_{i_0j}(t) b_{i_0j} \Lambda_{i_0ji_0}[x_{i_0}] + \sum_{k\neq i_0}^{n} \sum_{j=1}^{n} \gamma_{i_0j}(t) \eta_{i_0jk} \Lambda_{i_0jk}[x_k] + (F_{i_0}x)(t) \\
&\leq \sum_{j=1}^{n} \|\gamma_{i_0j}\|_\infty b_{i_0j} v_j + \rho\Big[\sum_{k\neq i_0}^{n} \sum_{j=1}^{n} \|\gamma_{i_0j}\|_\infty \eta_{i_0jk} \Lambda_{i_0jk}[1] + \Theta_\rho\Big] \\
&= \langle d_{i_0}, v_{i_0} \rangle + \rho M_{i_0\rho}.
\end{aligned}$$

Therefore, due to (3.7) we have

$$x_{i_0}(t) \leq \rho \langle d_{i_0}, (I_{n\times n} - P_{i_0})^{-1} z_{i_0} \rangle + \rho M_{i_0\rho}. \tag{3.8}$$

From here it follows that

$$1 \leq \langle d_{i_0}, (I_{n\times n} - P_{i_0})^{-1} z_{i_0} \rangle + M_{i_0\rho},$$

which contradicts to (3.3). This completes the proof. $\square$

To proceed, for $i = 1, 2, \dots, n$, we define the sets

$$E_i(\rho) := \{x = (x_1, x_2, \dots, x_n) : 0 \leq x_j \leq \frac{\rho}{\zeta}, \quad j \neq i, \quad \rho \leq x_i \leq \frac{\rho}{\zeta}\},$$

the real number

$$\theta_\rho := \min_i \inf_{x\in E_i(\rho)} \frac{U_i(x)}{\rho},$$

and the n-dimensional vectors

$$\nu_i := (\Lambda_{i1i}[1], \Lambda_{i2i}[1], \dots, \Lambda_{ini}[1])^T, \quad i = 1, 2, \dots, n,$$
$$h_i := \zeta_i(\|\gamma_{i1}\|_\infty a_{i1}, \|\gamma_{i2}\|_\infty a_{i2}, \dots, \|\gamma_{in}\|_\infty a_{in})^T, \quad i = 1, 2, \dots, n.$$

Lemma 3.3. *Assume that there is some $\rho > 0$ such that, for each $i = 1, 2, \dots, n$, it holds*

$$\theta_\rho c\big[\langle h_i, (I_{n\times n} - P_i)^{-1}\nu_i\rangle + 1\big] > 1. \tag{3.9}$$

Then the operator T defined in (2.1) *satisfies the relation*

$$i_K(T, V_\rho) = 0.$$

Proof. The result will follow if we show that the conditions of Theorem 2.2 are satisfied. So, let e be the n-vector $(1, 1, \dots, 1)^T$. Clearly, this is an element of the product cone K. We shall show that

$$x \neq Tx + \mu e,$$

for all $x \in \partial V_\rho$ and any $\mu \geq 0$, Indeed, let us assume that there is a $\mu \geq 0$ with $x = Tx + \mu e$, for some $x \in \partial V_\rho$. Therefore, we can assume that for some coordinate x_{i_0} of x it holds

$$\min_{t\in[\alpha_{i_0},\beta_{i_0}]} x_{i_0}(t) = \rho$$

and

$$0 \leq x_j(t) \leq \frac{\rho}{\zeta},$$

for all indices $j \neq i_0$ and all $t \in [\alpha_j, \beta_j]$.

Next, for all $t \in I$, from (2.1), we have

$$\begin{aligned} x_{i_0}(t) &= \sum_{k=1}^{n}\sum_{j=1}^{n} \gamma_{i_0 j}(t) w_{i_0 jk}(\Lambda_{i_0 jk}[x_k]) + (F_{i_0}x)(t) + \mu \\ &\geq \sum_{j=1}^{n} \gamma_{i_0 j}(t) a_{i_0 j}\Lambda_{i_0 j i_0}[x_{i_0}] + (F_{i_0}x)(t) + \mu, \end{aligned} \tag{3.10}$$

and therefore, for all indices $i = 1, 2, \dots, n$, it holds

$$\Lambda_{i_0 i i_0}[x_{i_0}] \geq \sum_{j=1}^{n} \Lambda_{i_0 i i_0}[\gamma_{i_0 j}] a_{i_0 j}\Lambda_{i_0 j i_0}[x_{i_0}] + \Lambda_{i_0 i i_0}[F_{i_0}x] + \mu\Lambda_{i_0 i i_0}[1].$$

Letting, as previously, $v_{jk} := \Lambda_{kjk}[x_k]$ and $v_k := (v_{1k}, v_{2k}, \dots, v_{nk})^T$, we obtain the vector-inequality

$$v_{i_0} \geq P_{i_0}v_{i_0} + \big(\rho\theta_\rho c + \mu\big)\nu_{i_0} \geq P_{i_0}v_{i_0} + \rho\theta_\rho c\nu_{i_0}.$$

Since $I_{n\times n} - P_{i_0}$ is an M-matrix, by Theorem 2.3, it is inversely positive, thus we have

$$v_{i_0} \geq \rho\theta_\rho c(I_{n\times n} - P_{i_0})^{-1}\nu_{i_0}. \tag{3.11}$$

From (C4), (C6) and inequality (3.10), for all $t \in [\alpha_{i_0}, \beta_{i_0}]$, we obtain

$$x_{i_0}(t) \geq \sum_{j=1}^{n} \zeta_{i_0}\|\gamma_{i_0 j}\|_\infty a_{i_0 j} v_{j i_0} + c\rho\theta_\rho + \mu$$

namely it holds

$$x_{i_0}(t) \geq \langle h_{i_0}, v_{i_0}\rangle + c\rho\theta_\rho + \mu.$$

Thus, from (3.11) and our hypothesis we obtain

$$\rho = \min_{t\in[\alpha_{i_0},\beta_{i_0}]} x_{i_0}(t) \geq c\rho\theta_\rho\Big[\langle h_{i_0}, (I_{n\times n} - P_{i_0})^{-1}\nu_{i_0}\rangle + 1\Big] + \mu > \rho + \mu,$$

because of (3.9). This is a contradiction and the proof is complete. □

Now we can, easily, combine the results of Lemmas 2.1 and 2.2 to obtain the main result of this paper, which stands as follows:

Theorem 3.4 (Existence results)**.** *Assume that conditions* (C1),..., (C5) *are satisfied and, for each $k = 1, 2, \dots, n$, the item $I_{n\times n} - P_k$ is an M-matrix. If there exist real numbers $\rho_1, \rho_2 \in (0, +\infty)$ with*

$$\frac{\rho_2}{\zeta} < \rho_1$$

satisfying relations (3.3) *and* (3.9)*, then the operator* (2.1) *has at least one fixed point in $\{x \in K : \frac{\rho_2}{\zeta} \leq |||x||| \leq \rho_1\}$.*

4. Some applications

Application 1. Consider the third-order ordinary differential equation (1.5) associated with the conditions (1.6), where A_{ik}, B_{ik}, Γ_{ik} are positive bounded linear functionals defined on the space $C(I, \mathbb{R}^+)$, with $B_{ik} \geq \Gamma_{ik}$, for all $i, k = 1, 2, 3$. It is not hard to see that the problem is equivalent to the integral equation

$$u = Tu,$$

with the operator $T : \tilde{C}_3(I) \to \tilde{C}_3(I)$ defined by

$$(Tu)_i(t) = \sum_{k=1}^{n}\sum_{j=1}^{n} \gamma_{ij}(t) w_{ijk}(\Lambda_{ijk}[u_k]) + \int_0^t \frac{(t-s)^2}{2} X_i(u(s))ds, \quad t \in I,$$

where $\gamma_{i1}(t) := \frac{t^2}{2}$, $\gamma_{i2}(t) := t$, $\gamma_{i3}(t) := 1$, $t \in I$,

$$\Lambda_{i1k}[x] := \lambda\Gamma_{ik}[x],$$
$$\Lambda_{i2k}[x] := \lambda(B_{ik} - \Gamma_{ik})[x], \quad x \in C(I, \mathbb{R}^+)$$
$$\Lambda_{i3k}[x] := \lambda A_{ik}[x]$$

and

$$w_{ijk}(s) := s, \quad s \in \mathbb{R},$$

for all indices $i, j, k = 1, 2, 3$.

We make the following assumption:

(A1) For each $i = 1, 2, 3$, there exist reals q_i, p_i, such that

$$0 < q_i \leq X_i(x) \leq p_i,$$

for all $x := (x_1, x_2, x_3) \geq 0$.

We shall prove the following result.

Theorem 4.1. *Under condition* (A1)*, there exist λ_0 and $R_1 > R_2 > 0$, such that, given any $\lambda \in (0, \lambda_0)$, the relation* (3.3) *holds for all $\rho > R_1$ while, the relation* (3.9) *holds, for all $0 < \rho < R_2$.*

Proof. First of all we observe that condition (C2) is satisfied with

$$a_{ij} = b_{ij} = \eta_{ijk} = 1, \quad i, j, k = 1, 2, 3,$$

and condition (C6) holds by choosing $U_i(x) := p_i$ and $c := \min_i q_i/p_i$. Also we have

$$\|\gamma_{i1}\|_\infty = \frac{1}{2}, \quad \|\gamma_{i2}\|_\infty = \|\gamma_{i3}\|_\infty = 1.$$

Now, fix any $\rho > 0$. Then we have

$$\Theta_\rho = \max_i \sup_{|||x|||=\rho} \frac{U_i(x)}{\rho} = \max_i \frac{p_i}{\rho}.$$

Also, it is easy to see that the vector z_i is the value of the vector function Ψ_i given by $\Psi_i(\cdot) := \lambda\Delta_i(\cdot)$ where

$$\Delta_i(\cdot) := (\Gamma_{ii}[\cdot], B_{ii}[\cdot] - \Gamma_{ii}[\cdot], A_{i1}[\cdot])^T$$

at the point

$$\vartheta_i(\rho, \lambda)(\cdot) := \Theta_\rho + \lambda \sum_{k\neq i} \big(A_{ik}[1]\gamma_{i3}(\cdot) + B_{ik}[1]\gamma_{i2}(\cdot) + \Gamma_{ik}[1](\gamma_{i1}(\cdot) - \gamma_{i2}(\cdot))\big).$$

Also, the vector d_i is equal to $(\frac{1}{2}, 1, 1)$, for each $i = 1, 2, 3$, and, finally, the constant $M_{i\rho}$, which corresponds to λ, is given by

$$M_{i\rho}(\lambda) = \lambda \sum_{k\neq i} \big(A_{ik}[1] + B_{ik}[1] + \frac{1}{2}\Gamma_{ik}[1]\big) + \Theta_\rho.$$

Next, choose λ_1 such that for each $k = 1, 2, 3$ and for all $\lambda \in (0, \lambda_1)$ it holds

$$1 > \lambda A_{kk}[\phi], \quad 1 + \lambda\Gamma_{kk}[\phi] > \lambda B_{kk}[\phi], \quad 1 > \lambda\Gamma_{kk}[\phi] \tag{4.1}$$

where

$$\phi(t) := 1 + t + \frac{t^2}{2}, \quad t \in I.$$

Under these assumptions, we can easily see that the matrix P_k with entries p_{ijk} is defined by

$$P_k := \lambda Q_k,$$

where Q_k has entries q_{ijk} given by

$$q_{1jk} := \Gamma_{kk}[\gamma_{kj}], \quad q_{2jk} := (B_{kk} - \Gamma_{kk})[\gamma_{kj}], \quad q_{3jk} := A_{kk}[\gamma_{kj}].$$

Due to (4.1) we can see that it holds

$$1 - p_{iik} > \sum_{j\neq i} p_{ijk},$$

for all indices $i, j, k = 1, 2, 3$. Hence, according to [27, property (N_{39})], the item $I_{3\times3} - P_k$ is an M-matrix.

Now, the left quantity in relation (3.3) is given by

$$g_k(\rho, \lambda) := \lambda\langle(\frac{1}{2}, 1, 1), (I_{3\times3} - \lambda Q_k)^{-1}\Delta_k(\vartheta_k(\rho, \lambda))\rangle + M_{k\rho}(\lambda),$$

which, obviously, depends continuously on the parameter $(\rho, \lambda) \in (0, +\infty)\times(0, \lambda_1))$. Since, obviously, we have

$$\lim_{(\rho,\lambda)\to(+\infty,0^+)} g_k(\rho, \lambda) = 0,$$

it follows that there exists $(R_1, \lambda_2) \in (0, +\infty) \times (0, \lambda_1))$ such that

$$g_k(\rho, \lambda) < 1, \quad k = 1, 2, 3,$$

for all $\rho > R_1$ and $\lambda \in (0, \lambda_2)$. This shows that (3.3) is satisfied for all $k = 1, 2, 3$ and such ρ and λ.

Next, define $\alpha := \min_i \sqrt{q_i/p_i}$ and let $\beta := 1$. By setting $\alpha_i = \alpha$ and $\beta_i = \beta$, $i = 1, 2, 3$, we see that condition $(C7)$ is satisfied with

$$\zeta_i = \alpha^2 = \zeta, \quad i = 1, 2, 3.$$

Hence the vectors ν_i and h_i are given by

$$\nu_i = (\Gamma_{ii}[1], B_{ii}[1] - \Gamma_{ii}[1], A_{ii}[1])^T = \Delta_i[1],$$
$$h_i = \alpha^2(\frac{1}{2}, 1, 1)^T,$$

while the quantity θ_ρ is given by

$$\theta_\rho = \min_i \inf_{|\|x\||=\rho} \frac{U_i(x)}{\rho} = \min_i \frac{p_i}{\rho} =: \frac{1}{\rho}\tilde{\theta}.$$

Now, the left quantity in relation (3.9) is given by

$$f_i(\rho, \lambda) := \frac{1}{\rho} V_i(\lambda),$$

where

$$V_i(\lambda) := c\tilde{\theta}\Big(\alpha^2\big[\langle(\frac{1}{2}, 1, 1), \frac{q_i}{p_i}(I_{3\times 3} - \lambda Q_i)^{-1}\nu_i\rangle\big] + 1\Big).$$

Obviously, the latter depends continuously on the parameter $\lambda \in (0, \lambda_1)$ and moreover it satisfies

$$\lim_{\lambda \to 0^+} V_i(\lambda) = c\tilde{\theta}\Big(\alpha^2\big[\frac{1}{2}\Gamma_{ii}[1] + B_{ii}[1] - \Gamma_{ii}[1] + A_{ii}[1]\big] + 1\Big).$$

The quantity inside the parenthesis is strictly positive. Thus, there exists $(R_2, \lambda_0) \in (0, R_1) \times (0, \lambda_2)$ such that

$$f_i(\rho, \lambda) > 1, \quad i = 1, 2, 3,$$

for all $\rho < R_2$ and $\lambda \in (0, \lambda_0)$. This shows that (3.9) is, also, satisfied for all i. $\square$

Thus we obtain the following existence result.

Theorem 4.2. *Under the conditions of Theorem 4.1 there exists $\lambda_0 > 0$ such that, for all $\lambda \in (0, \lambda_0)$, the problem* (1.5)-(1.6) *admits a positive solution.*

Proof. Fix $\lambda < \lambda_0$. Then choose ρ_1, ρ_2 such that $0 < \rho_2 < R_2 < \zeta R_1 < \zeta \rho_1$ and apply Theorem 3.4. $\square$

Application 2. As we said in the introduction, in [35] the author studies the system of second-order nonlocal boundary-value problem (1.6), where α and β are increasing non-constant functions defined on $[0, 1]$ with $\alpha(0) = 0 = \beta(0)$ and $f, g \in C([0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ and $H_i \in C(\mathbb{R}^+, \mathbb{R}^+)$, $(i = 1, 2)$. Here the integrals are in the Riemann-Stieltjies sense. Setting the problem (1.6) in the form of (1.1)-(1.2), we obtain the system of integral equations

$$\begin{aligned} u(t) &= \int_0^1 K(t,s)f(s,u(s),v(s))ds + H_1\Big(\int_0^1 u(\tau)d\alpha(\tau)\Big)t, \\ v(t) &= \int_0^1 K(t,s)g(s,u(s),v(s))ds + H_2\Big(\int_0^1 v(\tau)d\beta(\tau)\Big)t, \end{aligned} \tag{4.2}$$

where $K(t,s)$ is the Green's function

$$K(t,s) := \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases} \tag{4.3}$$

However, we can assume that the kernel $K(t,s)$ can be a general kernel and not necessarily of the previous form. Then we assume the following conditions:

(C1') There exist a continuous function $\Phi : I \to \mathbb{R}^+$, a positive real number c and an interval $[\alpha,\beta] \subset (0,1)$, such that

$$\begin{gathered} K(t,s) \le \Phi(s), \quad (t,s) \in I \times I, \\ K(t,s) \ge c\Phi(s), \quad (t,s) \in [\alpha,\beta] \times I. \end{gathered}$$

This condition is satisfied by choosing, for instance, $\alpha = 1/3$, $\beta = 2/3$, $c = 1/3$ and $\Phi(s) := s(1-s)$.

(C2') There exist positive real numbers $\tilde{a}_i, \tilde{b}_i$, $i = 1,2$, such that

$$\begin{gathered} \tilde{b}_1 \int_0^1 s d\alpha(s) < 1, \quad \tilde{b}_2 \int_0^1 s d\beta(s) < 1, \\ \tilde{a}_i \xi \le H_i(\xi) \le \tilde{b}_i \xi, \quad i = 1,2, \end{gathered}$$

for all $\xi \ge 0$.

Comparing system (4.2) with (1.1)-(1.2), we have

$$\begin{gathered} \gamma_{ij}(t) = t, \quad i,j = 1,2, \\ w_{111}(z) = H_1(z), \quad w_{222}(z) = H_2(z), \\ w_{112}(z) = w_{121}(z) = w_{122}(z) = w_{211}(z) = w_{212}(z) = w_{211}(z) = 0, \\ \Lambda_{111}(z) = \int_0^1 z(s)d\alpha(s), \quad \Lambda_{222}(z) = \int_0^1 z(s)d\beta(s), \\ \Lambda_{112} = \Lambda_{121} = \Lambda_{122} = \Lambda_{211} = \Lambda_{212} = \Lambda_{211} = 0. \end{gathered}$$

Define

$$\begin{gathered} U_1(u,v) := \int_0^1 \Phi(s)f(s,u(s),v(s))ds, \\ U_2(u,v) := \int_0^1 \Phi(s)g(s,u(s),v(s))ds \end{gathered}$$

and, for each $\rho > 0$, let

$$\Theta_\rho := \frac{1}{\rho} \max_{i=1,2} \sup_{|||(x_1,x_2)|||=\rho} U_i(x_1,x_2),$$

Then we obtain

$$a_{ii} = \tilde{a}_i, \quad b_{ii} := \tilde{b}_i, \quad i = 1,2$$

and

$$a_{12} = a_{21} = b_{12} = b_{21} = 0.$$

Also, we have $\sigma_{ij} = \alpha$, $i, j = 1, 2$,

$$P_1 = \begin{bmatrix} \tilde{b}_1 \int_0^1 s d\alpha(s) & 0 \\ 0 & 0 \end{bmatrix} \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{b}_2 \int_0^1 s d\beta(s) \end{bmatrix},$$

$$z_{11} = \tilde{b}_1 \Theta_\rho \alpha(1), \quad z_{21} = 0 = z_{12}, \quad z_{22} = b_{22} \Theta_\rho \beta(1),$$

$$d_1 = \begin{bmatrix} \tilde{b}_1 \\ 0 \end{bmatrix}, \quad d_2 = \begin{bmatrix} 0 \\ \tilde{b}_2 \end{bmatrix}, \quad M_{1\rho} = M_{2\rho} = \Theta_\rho.$$

Finally, we obtain $\sigma_{ij} = \alpha$, $i, j = 1, 2$,

$$E_1(\rho) := \{(x_1, x_2) : 0 \le x_2 \le \frac{\rho}{\alpha},\ \rho \le x_1 \le \frac{\rho}{\alpha}\},$$

$$E_2(\rho) := \{(x_1, x_2) : 0 \le x_1 \le \frac{\rho}{\alpha},\ \rho \le x_2 \le \frac{\rho}{\alpha}\},$$

$$\theta_\rho := \frac{1}{\rho} \min_{i=1,2} \inf_{x \in E_i(x)} U_i(x),$$

$$\nu_1 = \begin{bmatrix} \tilde{b}_1 \int_0^1 s d\alpha(s) \\ 0 \end{bmatrix}, \quad \nu_2 = \begin{bmatrix} 0 \\ \tilde{b}_2 \int_0^1 s d\beta(s) \end{bmatrix},$$

$$\zeta_1 := \min\{c, \frac{\alpha \tilde{a}_1}{\tilde{b}_1}\}, \quad \zeta_2 := \min\{c, \frac{\alpha \tilde{a}_2}{\tilde{b}_2}\},$$

$$\zeta := \min\{\zeta_1, \zeta_2\}, \quad h_1 := \zeta_1 \begin{bmatrix} \tilde{a}_1 \\ 0 \end{bmatrix}, \quad h_2 := \zeta_2 \begin{bmatrix} 0 \\ \tilde{a}_2 \end{bmatrix}.$$

After these denotations we can formulate the following theorem.

Theorem 4.3. *Let* $\rho_1, \quad \rho_2 > 0$ *be such that* $\rho_2 \zeta < \rho_1$, *and*

$$\Theta_{\rho_1} \Big[1 + \frac{\tilde{b}_1^2 \alpha(1)}{1 - \tilde{b}_1 \int_0^1 s d\alpha(s)} \Big] < 1, \tag{4.4}$$

$$\Theta_{\rho_1} \Big[1 + \frac{\tilde{b}_2^2 \beta(1)}{1 - \tilde{b}_2 \int_0^1 s d\beta(s)} \Big] < 1, \tag{4.5}$$

$$c\theta_{\rho_2} \Big[\frac{\zeta_1 \tilde{a}_1 \tilde{b}_1 \int_0^1 s d\alpha(s)}{1 - \tilde{b}_1 \int_0^1 s d\alpha(s)} + 1 \Big] > 1, \tag{4.6}$$

$$c\theta_{\rho_2} \Big[\frac{\zeta_2 \tilde{a}_2 \tilde{b}_2 \int_0^1 s d\beta(s)}{1 - \tilde{b}_2 \int_0^1 s d\beta(s)} + 1 \Big] > 1. \tag{4.7}$$

Then the system of equations (4.2) *admits at least one positive solution.*

Proof. The proof follows from Theorem 3.4, once we observe that (4.4) and (4.5) are relations (3.3) with ρ_1 instead of ρ, while (4.6) and (4.7) are relations (3.9) with ρ_2 instead of ρ. □

Application 3. Next consider the system of equations (1.4). It is easy to see that this system takes the form (1.1)-(1.2), when $n = 2$, γ_{ij} are the same functions,

$$w_{1j1} = H_{1j}, \quad w_{1j2} = L_{1j}, \quad w_{2j1} = L_{2j}, \quad w_{2j2} = H_{2j},$$

$$\Lambda_{1j1} = \beta_{1j}, \quad \Lambda_{1j2} = \delta_{1j}, \quad \Lambda_{2j1} = \delta_{2j}, \quad \Lambda_{2j2} = \beta_{2j},$$

$$b_{ij} = h_{ij2}, \quad a_{ij} = h_{ij1},$$
$$\eta_{1j2} = l_{1j2}, \quad \eta_{2j1} = l_{2j2}, \quad \sigma_{ij} = c_{ij}.$$

Also, here we have $x_1 = u, x_2 = v$, as well as

$$(Fx)_i(t) = \int_0^1 k_i(t,s)g_i(s)f_i(s, x_1(s), x_2(s))ds, \quad i = 1, 2,$$

where k_1, k_2 satisfy the inequalities of the form

$$k_i(t,s) \leq \Phi_i(s), \quad t \in I, \quad \text{a.e. } s \in I,$$

and

$$c_i\Phi_i(s) \leq k_i(t,s), \quad t \in [a_i, b_i], \quad \text{a.e. } s \in I,$$

for some subinterval $[a_i, b_i]$ of I. Hence conditions (C5), (C6) are satisfied with

$$U_i(x) := \int_0^1 \Phi_i(s)g_i(s)f_i(s, x_1(s), x_2(s))ds.$$

It is not hard to see that for $k = 1, 2$, the matrix P_k is the same with D_k in [14] and, under the conditions on D_k stated in [14], the matrix $I_{2\times 2} - P_k$ is inverse-positive, thus it is an M-matrix. Then our conditions are the same with those of [14] and the existence results in [14, Theorem 2.7 (S1)] follow from theorem 3.4.

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