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TRAVELING WAVE SOLUTIONS OF NONLOCAL DELAY REACTION-DIFFUSION EQUATIONS WITHOUT LOCAL QUASIMONOTONICITY

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Abstract. This article concerns the traveling wave solutions of nonlocal delay reaction-diffusion equations without local quasimonotonicity. The existence of traveling wave solutions is obtained by constructing upper-lower solutions and passing to a limit function. The nonexistence of traveling wave solutions is also established by the theory of asymptotic spreading. The results are applied to a food limit model with nonlocal delays, which completes and improves some known results.

1. Introduction

Reaction-diffusion systems with nonlocal delays are important models reflecting the random walk as well as the history behavior of individuals in population dynamics, and provide more precise description in some evolutionary processes. This kind of model was earlier proposed by Britton [3, 4] in population dynamics, and we refer to Gourley et al. [9], Gourley and Wu [10] for more biological background and literature results of reaction-diffusion systems with nonlocal delays. A typical example of reaction-diffusion equations with nonlocal delays takes the form as follows

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) + u(x,t) g\Big(u(x,t), \int_0^\infty \int_{\mathbb{R}} u(x-y,t-s) J(y,s)\, dy\, ds\Big), \quad (1.1)$$

in which $x \in \mathbb{R}, t > 0$, $u(x,t)$ denotes the population density in population dynamics, $g : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, and $J(y,s) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ is a probability function formulating the random walk of individuals in history, and is the so-called kernel function in literature.

In particular, the traveling wave solutions of (1.1) have been widely studied. For some special forms of J, the existence of traveling wave solutions was obtained by employing linear chain techniques and geometric singular perturbation theory, see [2, 8, 25]. Wang et al. [31] developed the monotone iteration in [34] and established an abstract scheme to prove the existence of traveling wave solutions of nonlocal delayed reaction-diffusion systems admitting proper monotone conditions, and the results were applied to a food limit model in [30, 37]. Ou and Wu [23] proved the

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persistence of traveling wave solutions with respect to the small (average) delays. In particular, if an equation is (local) quasimonotone (i.e., $g(u,v)$ is monotone increasing in v near the unstable steady state), then the existence of traveling wave solutions can be obtained by the monotonicity of semiflows (see Smith [26]) or by constructing auxiliary monotone equations, see [12, 17, 18, 28, 29, 36]. Besides the existence of traveling wave solutions, another important topic is the stability of traveling wave solutions, and much attention has been paid to it by different methods including squeezing technique, spectral theory and energy method, see [14, 15, 19, 20, 21, 22, 27, 32, 33] and the references cited therein. Moreover, some other results on spatial-temporal propagation of (1.1) can be found in Zhao [38].

In this paper, we shall consider the minimal wave speed of traveling wave solutions of (1.1) if $g(u,v)$ is monotone decreasing in v, and (1.1) does not satisfy the monotone conditions in the known results. In particular, let

$$g(u,v) = r\Big[\frac{1-u-av}{1+du+dav}\Big], \tag{1.2}$$

in which $r > 0$, $d \geq 0$, $a \geq 0$ are constants. Then (1.1) with (1.2) is the food limit model in [7, 30, 37], and the authors obtained the existence of traveling wave solutions for several special J if the (average) time delay is small enough. For more results with special J and d in (1.1) with (1.2), we also refer to [5, 6, 10, 11, 16]. In particular, if (1.1) with (1.2) takes the discrete delay and $d = 0$, then Lin [13] and Pan [24] investigated the asymptotic speed of spreading, which implies the persistence of asymptotic speed of spreading.

In what follows, we shall further develop the corresponding theory of traveling wave solutions such that we can obtain the minimal wave speed of (1.1), which at least contains (1.1) with (1.2) as an example and completes some well known results. The existence and nonexistence of traveling wave solutions are proved by the idea in Lin and Ruan [16], which implies the minimal wave speed of traveling wave solutions of (1.1) is the same as that in

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) + u(x,t) g\big(u(x,t), u(x,t)\big)$$

with some additional assumptions. These results indicate that even if the (large) delay leads to the failure of local quasimonotonicity, it is also possible to obtain the persistence of traveling wave solutions with respect to the (large) delay.

The rest of this paper is organized as follows. In Section 2, we list some preliminaries including notation and the theory of asymptotic spreading. By Schauder's fixed point theorem, the existence of traveling wave solutions is established in Section 3. The minimal wave speed is obtained in Section 4 by passing to a limit function and applying the theory of asymptotic spreading. Finally, the traveling wave solutions of (1.1) with (1.2) are studied in the last section.

2. Preliminaries

In this article, we define

$$C(\mathbb{R},\mathbb{R}) = \{u : \mathbb{R} \to \mathbb{R} : u \text{ is uniformly continuous and bounded}\}.$$

Then C is a Banach space equipped with the standard supremum norm. When $a < b$ is true, denote

$$C_{[a,b]} = \{u \in C : a \leq u \leq b\}.$$

If $u \in C^2(\mathbb{R},\mathbb{R})$, then $u \in C$, $u' \in C$, $u'' \in C$. For $\mu > 0$, define

$$B_\mu(\mathbb{R},\mathbb{R}) = \big\{u \in C(\mathbb{R},\mathbb{R}) : \sup_{t\in\mathbb{R}} |u(t)|e^{-\mu|t|} < \infty\big\},$$

then $B_\mu(\mathbb{R},\mathbb{R})$ is a Banach space when it is equipped with the norm $|\cdot|_\mu$ defined by

$$|u|_\mu = \sup_{t\in\mathbb{R}} |u(t)|e^{-\mu|t|} \quad \text{for } u \in B_\mu(\mathbb{R},\mathbb{R}).$$

For (1.1), we give the following assumptions:

(A1) $g(0,0) > 0$, $g(0,1) > 0$ and $g(1,0) = 0$;
(A2) $g(u,v)$ is strictly monotone decreasing and Lipschitz continuous in $u, v \in [0,\infty)$, we also suppose that $L > 0$ is the Lipschitz constant and $g(u,v) \to -\infty$ if $u + v \to \infty$;
(A3) there exists $E \in (0,1)$ such that $g(E,E) = 0$;
(A4) $J(y,s) = J(-y,s) \geq 0, y \in \mathbb{R}, s \geq 0, \int_0^\infty \int_{\mathbb{R}} J(y,s)\,dy\,ds = 1$;
(A5) for some $\lambda_0 > \sqrt{g(0,0)}$,

$$\int_0^\infty \int_{\mathbb{R}} J(y,s)e^{\lambda y + (\lambda^2 + g(0,0))s}\,dy\,ds < \infty \text{ for all } \lambda \in (0,\lambda_0);$$

(A6) if $1 \geq E_1 \geq E_2 > 0$ such that

$$g(E_1,E_2) \geq 0, g(E_2,E_1) \leq 0,$$

then $E_1 = E_2 = E$.

Clearly, (1.1) with (1.2) satisfies (A1)-(A3) if $d = 0$ and (A6) is true if $d \geq 0, a \in (0,1)$. Although (1.2) does not satisfy (A2), we will illustrate that our results remain true for (1.1) with (1.2) by introducing an auxiliary equation in the last section. Therefore, our results can be applied to (1.1) with (1.2) by adding proper conditions satisfied by J.

Definition 2.1. A traveling wave solution of (1.1) is a special solution with the form $u(x,t) = \phi(x + ct)$, in which $c > 0$ is the wave speed and $\phi \in C^2(\mathbb{R},\mathbb{R})$ is the wave profile that propagates in $\mathbb{R}$.

Then ϕ, c must satisfy

$$c\phi'(\xi) = \phi''(\xi) + \phi(\xi)g\Big(\phi(\xi), \int_0^\infty \int_{\mathbb{R}} \phi(\xi - y - cs)J(y,s)\,dy\,ds\Big). \tag{2.1}$$

To reflect transition processes between different states, we also require

$$\lim_{\xi\to-\infty} \phi(\xi) = 0, \quad \lim_{\xi\to\infty} \phi(\xi) = E. \tag{2.2}$$

Then a traveling wave solution satisfying (2.1)-(2.2) can reflect the successful biological invasion in the population dynamics.

For all $v \in [0,1]$, let $\beta > 0$ be a constant such that

$$\beta u + ug(u,v)$$

is monotone increasing in $u \in [0,1]$. If $\phi(\xi) \in C_{[0,1]}$, we define

$$H(\phi)(\xi) = \phi(\xi) + \phi(\xi)g\Big(\phi(\xi), \int_0^\infty \int_{\mathbb{R}} \phi(\xi - y - cs)J(y,s)\,dy\,ds\Big)$$

and $F(\phi)(\xi)$ as follows

$$F(\phi)(\xi)=\frac{1}{\lambda_2(c)-\lambda_1(c)}\int_{-\infty}^{\infty}\min\{e^{\lambda_1(c)(\xi-s)},e^{\lambda_2(c)(\xi-s)}\}H(\phi)(s)ds,$$

in which

$$\lambda_1(c)=\frac{c-\sqrt{c^2+4\beta}}{2},\quad \lambda_2(c)=\frac{c+\sqrt{c^2+4\beta}}{2}.$$

Then a fixed point of F in $C_{[0,1]}$ is a solution to (2.1).

Consider the initial value problem

$$\begin{gathered}\frac{\partial w(x,t)}{\partial t}=\Delta w(x,t)+w(x,t)g(w(x,t),\delta),\\ w(x,0)=\varphi(x)\in C_{[0,1]}\end{gathered}\tag{2.3}$$

with $\delta\in[0,1]$, then the following result is true by Aronson and Weinberger [1], Ye et al. [35].

Lemma 2.2. *Equation* (2.3) *admits a unique solution such that $u(\cdot,t)\in C_{[0,1]}$ for all $t>0$. If $z(\cdot,t)\in C$ with $t>0$ such that*

$$\begin{gathered}\frac{\partial z(x,t)}{\partial t}\geq(\leq)\Delta z(x,t)+w(x,t)g(w(x,t),\delta),\\ z(x,0)\geq(\leq)\varphi(x),\end{gathered}$$

then $z(x,t)\geq(\leq)w(x,t)$ for all $x\in\mathbb{R},t>0$. Moreover, if $\varphi(x)$ admits a nonempty support, then $w(x,t)$ satisfies

$$\liminf_{t\to\infty}\inf_{|x|<ct}w(x,t)=\limsup_{t\to\infty}\sup_{|x|<ct}w(x,t)=\kappa$$

with any $c<c'=:2\sqrt{g(0,\delta)}$ and unique $\kappa\in(0,1]$ such that $g(\kappa,\delta)=0$. In particular, if $\varphi(x)$ admits a nonempty compact support, then $\limsup_{t\to\infty}\sup_{|x|>ct}w(x,t)=0$ with any $c>c'$.

3. Existence of traveling wave solutions

In this section, we shall prove the existence of traveling wave solutions of (1.1), which is motivated by Lin and Ruan [16]. For $c>c^*=:2\sqrt{g(0,0)}$, define

$$\begin{gathered}\gamma_1(c)=\frac{c-\sqrt{c^2-4g(0,0)}}{2},\quad \gamma_2(c)=\frac{c+\sqrt{c^2-4g(0,0)}}{2},\\ \overline{\phi}(\xi)=\min\{e^{\gamma_1(c)\xi},1\},\quad \underline{\phi}(\xi)=\max\{e^{\gamma_1(c)\xi}-qe^{\eta\gamma_1(c)\xi},0\}\end{gathered}$$

with $1<\eta<\min\{2,\gamma_2(c)/\gamma_1(c)\}$ and $q>1$.

Lemma 3.1. *Assume that $c>c^*$ and* (A1)–(A5) *hold. If*

$$q=1-\frac{g(0,0)(1+2L\int_0^\infty\int_{\mathbb{R}}J(y,s)e^{\gamma_1(c)y+(\gamma_1^2(c)+g(0,0))s}\,dy\,ds)}{(\eta\gamma_1(c))^2-c\eta\gamma_1(c)+g(0,0)},$$

then for $\xi\neq0$ and $\xi\neq\frac{\ln q}{(1-\eta)\gamma_1(c)}$, we have

$$\begin{gathered}c\overline{\phi}'(\xi)\geq\overline{\phi}''(\xi)+\overline{\phi}(\xi)g(\overline{\phi}(\xi),\int_0^\infty\int_{\mathbb{R}}\underline{\phi}(\xi-y-cs)J(y,s)\,dy\,ds),\\ c\underline{\phi}'(\xi)\leq\underline{\phi}''(\xi)+\underline{\phi}(\xi)g(\underline{\phi}(\xi),\int_0^\infty\int_{\mathbb{R}}\overline{\phi}(\xi-y-cs)J(y,s)\,dy\,ds).\end{gathered}\tag{3.1}$$

The proof of the above lemma is trivial and we omit it here.

Lemma 3.2. *Assume that $c > c^*$ and* (A1)–(A5) *hold. Let*
$$\Gamma = \{\phi \in C : \underline{\phi}(\xi) \le \phi(\xi) \le \overline{\phi}(\xi), \xi \in \mathbb{R}\}.$$
Then Γ is convex and nonempty. Moreover, for any $\mu > 0$, it is bounded and closed with respect to the norm $|\cdot|_\mu$. In particular, $F : \Gamma \to \Gamma$.

Proof. The properties of Γ in Theorem 3.2 are clear and we omit the proof here. Now it suffices to verify $F : \Gamma \to \Gamma$. By (A2) and the definition of β, H admits the following nice conclusions
$$\begin{aligned}
&\beta\overline{\phi}(\xi) + \overline{\phi}(\xi)g(\overline{\phi}(\xi), \int_0^\infty \int_{\mathbb{R}} \underline{\phi}(\xi - y - cs)J(y, s)\,dy\,ds)\\
&\ge \beta\phi(\xi) + \phi(\xi)g(\phi(\xi), \int_0^\infty \int_{\mathbb{R}} \underline{\phi}(\xi - y - cs)J(y, s)\,dy\,ds)\\
&\ge \beta\phi(\xi) + \phi(\xi)g(\phi(\xi), \int_0^\infty \int_{\mathbb{R}} \phi(\xi - y - cs)J(y, s)\,dy\,ds)\\
&= H(\phi)(\xi)\\
&\ge \beta\phi(\xi) + \phi(\xi)g(\phi(\xi), \int_0^\infty \int_{\mathbb{R}} \overline{\phi}(\xi - y - cs)J(y, s)\,dy\,ds)\\
&\ge \beta\underline{\phi}(\xi) + \underline{\phi}(\xi)g(\underline{\phi}(\xi), \int_0^\infty \int_{\mathbb{R}} \overline{\phi}(\xi - y - cs)J(y, s)\,dy\,ds)
\end{aligned}$$
for any $\phi \in \Gamma$, $\xi \in \mathbb{R}$.

If $\xi \ne 0$, then
$$\begin{aligned}
F(\phi)(\xi) &= \frac{1}{\lambda_2 - \lambda_1}\Big[\int_{-\infty}^{\xi} e^{\lambda_1(\xi - s)} + \int_{\xi}^{\infty} e^{\lambda_2(\xi - s)}\Big]H(\phi)(s)ds\\
&= \frac{1}{\lambda_2 - \lambda_1}\Big[\int_{-\infty}^{0} + \int_{0}^{\infty}\Big]\min\{e^{\lambda_1(\xi - s)}, e^{\lambda_2(\xi - s)}\}H(\phi)(s)ds\\
&\le \frac{1}{\lambda_2 - \lambda_1}\Big[\int_{-\infty}^{0} + \int_{0}^{\infty}\Big]\min\{e^{\lambda_1(\xi - s)}, e^{\lambda_2(\xi - s)}\}\\
&\quad \times \Big(\beta\underline{\phi}(s) + \underline{\phi}(s)g(\underline{\phi}(s), \int_0^\infty \int_{\mathbb{R}} \overline{\phi}(s - y - cz)J(y, z)dydz)\Big)ds\\
&\le \frac{1}{\lambda_2 - \lambda_1}\Big[\int_{-\infty}^{0} + \int_{0}^{\infty}\Big]\min\{e^{\lambda_1(\xi - s)}, e^{\lambda_2(\xi - s)}\}\\
&\quad \times \Big(\beta\overline{\phi}(s) + c\overline{\phi}'(s) - \overline{\phi}''(s)\Big)ds\\
&= \underline{\phi}(\xi) + \frac{1}{\lambda_2 - \lambda_1}\Big[\min\{e^{\lambda_2\xi}, e^{\lambda_1\xi}\}(\overline{\phi}'(0+) - \overline{\phi}'(0-))\Big]\\
&\le \overline{\phi}(\xi)
\end{aligned}$$
by (3.1). Since $F(\phi)(\xi), \underline{\phi}(\xi)$ are continuous for all $\xi \in \mathbb{R}$, then
$$F(\phi)(\xi) \le \overline{\phi}(\xi), \quad \xi \in \mathbb{R}.$$
Similarly, we have
$$F(\phi)(\xi) \ge \underline{\phi}(\xi), \quad \xi \in \mathbb{R}.$$
The proof is complete. □

Lemma 3.3. *Assume that $c > c^*$ and (A1)–(A5) hold. If $c\mu < \beta$ and $\mu \in (0, \sqrt{g(0,0)})$, then $F : \Gamma \to \Gamma$ is complete continuous in the sense of $|\cdot|_\mu$.*

The proof of the complete continuity is independent of the monotone condition, and we omit it here. For the complete discussion, we refer to Lin et al. [15, Theorem 2.4] and Ma [17, Theorem 1.1].

Theorem 3.4. *Assume that* (A1)–(A5) *hold. Then for each $c > c^*$,* (2.1) *has a positive solution $\phi(\xi)$ such that*

$$0 < \phi(\xi) < 1, \quad \lim_{\xi\to-\infty} \phi(\xi) = 0, \quad 1 \ge \limsup_{\xi\to\infty} \phi(\xi) \ge \liminf_{\xi\to\infty} \phi(\xi) > 0. \tag{3.2}$$

Further suppose that (A6) *holds, then $\phi(\xi)$ satisfies* (2.2).

Proof. Using Schauder's fixed point theorem, the existence of $\phi(\xi)$ is confirmed. Moreover,

$$0 < \phi(\xi) < 1, \quad \lim_{\xi\to-\infty} \phi(\xi) e^{-\gamma_1(c)\xi} = 1$$

are clear by the asymptotic behavior of $\underline{\phi}(\xi)$ and $\overline{\phi}(\xi)$. Note that $\phi(\xi) = u(x,t)$ is a special solution to (2.1), then

$$\begin{gathered}
\frac{\partial u(x,t)}{\partial t} \ge \Delta u(x,t) + u(x,t) g(u(x,t), 1), \\
\frac{\partial u(x,t)}{\partial t} \le \Delta u(x,t) + u(x,t) g(u(x,t), 0), \\
u(x,0) = \phi(x) > 0.
\end{gathered} \tag{3.3}$$

Combining Lemma 2.2 with (3.3), we see that

$$0 < \liminf_{t\to\infty} u(0,t) \le \limsup_{t\to\infty} u(0,t) \le 1,$$

which completes the proof of (3.2). Let

$$E_1 = \limsup_{\xi\to\infty} \phi(\xi), \quad E_2 = \liminf_{\xi\to\infty} \phi(\xi),$$

then $0 < E_2 \le E_1 \le 1$. Using the dominated convergence theorem in F when $\xi \to \infty$, we obtain

$$g(E_1, E_2) \ge 0, g(E_2, E_1) \le 0,$$

and (2.2) is true by (A6). The proof is complete. □

4. Minimal wave speed

By what we have done, we have obtained the existence of traveling wave solutions of (1.1) if $c > c^*$. In this section, we shall confirm the existence of traveling wave solutions of (1.1) if $c = c^*$ and the nonexistence of traveling wave solutions of (1.1) if $c < c^*$ by the idea in Lin and Ruan [16]. To continue our discussion, we first present the following nice property of any bounded solutions of (2.1).

Lemma 4.1. *Assume that $\phi(\xi)$ is a bounded solution of* (2.1) *or a bounded fixed point of F. Then $\phi(\xi) \in C_{[0,1]}$ holds and $\phi'(\xi)$ is uniformly bounded for $\xi \in \mathbb{R}, c \in (c^*, 4c^*]$.*

The above result is evident and we omit its verification.

Theorem 4.2. *Assume that* (A1)–(A5) *hold. If $c = c^*$, then* (2.1) *has a positive solution $\phi(\xi)$ satisfying* (3.2)*. Further suppose that* (A6) *holds, then $\phi(\xi)$ also satisfies* (2.2)*.*

Proof. Let $\{c_n\}$ be a strictly decreasing sequence satisfying

$$c_n \to c^*, \quad n \to \infty, \quad c^* < c_n \le 2c^*, \quad n \in \mathbb{N}.$$

Then for each c_n, F with $c = c_n$ has a fixed point $\phi_n(\xi)$ such that (3.2) is true. Since $\phi_n(\xi)$ is invariant in the sense of phase shift, we assume that

$$\phi_n(0) = \epsilon, \quad \phi_n(\xi) < \epsilon, \quad \xi < 0 \quad \text{for } n \in \mathbb{N}$$

with $g(4\epsilon, 1) > 0$. Clearly, $\{\phi_n(\xi)\}$ are equicontinuity (see Lemma 4.1), then we can choose a subsequence of $\{\phi_n(\xi)\}$, still denoted by $\{\phi_n(\xi)\}$ such that $\{\phi_n(\xi)\}$ convergence to a function $\phi(\xi) \in C_{[0,1]}$, and the convergence is pointwise and locally uniform on any bounded interval of $\xi \in \mathbb{R}$. Moreover, if $c_n \to c^*$, then

$$\frac{\min\{e^{\lambda_1(c_n)(\xi-s)}, e^{\lambda_2(c_n)(\xi-s)}\}}{\lambda_2(c_n) - \lambda_1(c_n)} \to \frac{\min\{e^{\lambda_1(c^*)(\xi-s)}, e^{\lambda_2(c^*)(\xi-s)}\}}{\lambda_2(c^*) - \lambda_1(c^*)},$$

and the convergence is uniform in $\xi, s \in \mathbb{R}$. Applying the dominated convergence theorem in F with $c = c_n$, we see that $\phi(\xi)$ is a fixed point of F with $c = c^*$ and $\phi(\xi)$ is uniformly continuous in $\xi \in \mathbb{R}$. Therefore, (2.1) with $c = c^*$ has a solution $\phi(\xi)$ such that

$$\phi(0) = \epsilon, \quad \phi(\xi) \le \epsilon, \quad \xi < 0.$$

Since $\phi(0) > 0$, then the proof of limit behavior for $\xi \to \infty$ is similar to that in Theorem 3.4. If $\limsup_{\xi \to -\infty} \phi(\xi) > 0$, then there exist constants $\delta \in (0, \epsilon], \eta > 0$ and a sequence $\xi_m \to -\infty, m \to \infty$ such that

$$\phi(\xi_m) \to \delta, \quad \phi(\xi_m - x) > \delta/2, \quad m \in \mathbb{N}, \quad |x| \le \eta$$

by the uniform continuity. At the same time, Lemma 2.2 implies that $\phi(\xi_m) \ge 4\epsilon$ for $\xi_m < 0, m \in \mathbb{N}$, and a contradiction occurs. Therefore, we obtain (3.2), and the proof is complete. □

Remark 4.3. If $g(u, v)$ is monotone increasing in v, then the limit behavior can be proved by the monotonicity of traveling wave solutions, see Thieme and Zhao [28].

Theorem 4.4. *Assume that* (A1)–(A5) *hold. If $c < c^*$, then* (2.1) *has no positive solution $\phi(\xi)$ satisfying* (3.2)*.*

Proof. If the statement were false, then for some $c_1 < c^*$, (2.1) with $c = c_1$ has a positive solution $\phi(\xi)$ satisfying (3.2), which is bounded and uniformly continuous for $\xi \in \mathbb{R}$. Let $\epsilon > 0$ such that

$$\gamma^2 - c_1\gamma + g(0, 4\epsilon) = 0$$

has no real root. By (3.2), there exists $T < 0$ such that

$$\int_0^\infty \int_{\mathbb{R}} \phi(\xi - y - cs) J(y, s) \, dy \, ds < \epsilon, \quad \xi \le T.$$

Define

$$\delta = \liminf_{\xi > T} \phi(\xi).$$

Then $\delta > 0$ is well defined and there exists $M > 1$ such that

$$g(\phi(\xi), \int_0^\infty \int_{\mathbb{R}} \phi(\xi - y - cs)J(y,s)\,dy\,ds) \geq g(\phi(\xi), 1) \geq g(M\delta, \epsilon)$$

and so

$$c_1\phi'(\xi) \geq \phi''(\xi) + \phi(\xi)g(M\phi(\xi), \epsilon). \tag{4.1}$$

Let $c_2 > c_1$ such that

$$\gamma^2 - c_2\gamma + g(0, 2\epsilon) = 0$$

has no real root. Note that $u(x,t) = \phi(\xi)$ also satisfies (3.2), then Lemma 2.2 implies that

$$\liminf_{t\to\infty} \inf_{|x|=c_2 t} u(x,t) > \varepsilon$$

with $\varepsilon > 0$ such that $g(M\varepsilon, \epsilon) > 0$.

Let $-x = c_2 t$, then $t \to \infty$ indicates that $\xi \to -\infty$ and

$$\limsup_{t\to\infty} \sup_{-x=c_2 t} u(x,t) = 0,$$

which implies a contradiction. The proof is complete. $\square$

Remark 4.5. The proof of Theorem 4.4 is also independent of $g(0,1) > 0$.

5. Applications

In this part, we consider the traveling wave solutions of (1.1) with (1.2) by presenting the conclusion if J takes several special kernels in Zhao and Liu [37]. For (1.1) with (1.2), it is easy to check that a bounded positive traveling wave solution $u(x,t) = \phi(\xi)$ satisfying

$$0 \leq \phi(\xi) \leq 1, \quad \forall \xi \in \mathbb{R}.$$

Then it is equivalent to consider (1.1) with

$$g^*(u,v) = \begin{cases} g(u,v), & u, v \in [0,2], \\ \frac{r}{1+du+2ad}[1-u-av], & u \in [0,2],\ v > 2, \\ \frac{r}{1+2d+adv}[1-u-av], & u > 2,\ v \in [0,2], \\ \frac{r}{1+2d+2ad}[1-u-av], & u, v > 2. \end{cases}$$

Theorem 5.1. *Assume that $a \in [0,1)$ holds and one of the following seven statements are true:*

(K1) $\rho \in (0, 1/\sqrt{g(0,0)})$ *with* $J(y,s) = \frac{\delta(s)}{2\rho} e^{-\|y|/\rho}$;
(K2) *for any* $\tau > 0$ *with* $J(y,s) = \delta(y)\frac{s}{\tau^2} e^{-s/\tau}$;
(K3) *for any* $\tau > 0$ *with* $J(y,s) = \delta(y)\delta(s-\tau)$;
(K4) *for any* $\tau > 0$ *with* $J(y,s) = \delta(y)\frac{1}{\tau} e^{-s/\tau}$;
(K5) *for any* $\tau > 0$ *with* $J(y,s) = \frac{1}{\tau} e^{-s/\tau} \frac{1}{\sqrt{4\pi s}} e^{-y^2/(4s)}$;
(K6) *for any* $\tau > 0$ *with* $J(y,s) = \frac{s}{\tau^2} e^{-s/\tau} \frac{1}{\sqrt{4\pi s}} e^{-y^2/(4s)}$;
(K7) *for any* $\tau > 0$ *with* $J(y,s) = \delta(s-\tau) \frac{1}{\sqrt{4\pi s}} e^{-y^2/(4s)}$.

Then $2\sqrt{r}$ is the minimal wave speed of traveling wave solution $\phi(\xi)$ of (1.1) *with* (1.2), *which connects* 0 *with* $1/(1+a)$ *in the sense of*

$$\lim_{\xi\to-\infty} \phi(\xi) = 0, \quad \lim_{\xi\to\infty} \phi(\xi) = \frac{1}{1+a}.$$

Remark 5.2. For more kernel functions, we can obtain some conditions on the parameters such that $2\sqrt{r}$ is the minimal wave speed of traveling wave solutions of (1.1) with (1.2). It should be noted that we cannot prove the monotonicity of traveling wave solutions by the methods in this paper.

From Remark 4.5, we also have the following result.

Theorem 5.3. *Assume that $a \geq 0, d \geq 0$. Then, for any $c < 2\sqrt{r}$,* (1.1) *with* (1.2) *has not a bounded positive traveling wave solution $\phi(\xi)$ such that*

$$\lim_{\xi \to -\infty} \phi(\xi) = 0, \liminf_{\xi \to \infty} \phi(\xi) > 0.$$

Remark 5.4. Theorem 5.3 remains true for monotone and bounded traveling wave solutions, which completes the results in Gourley and Chaplain [7], Wang and Li [30] and Zhao and Liu [37].

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