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STABILITY OF SIMULTANEOUSLY TRIANGULARIZABLE SWITCHED SYSTEMS ON HYBRID DOMAINS

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Abstract. In this paper, we extend the results of [8, 15, 22] which provide sufficient conditions for the global exponential stability of switched systems under arbitrary switching via the existence of a common quadratic Lyapunov function. In particular, we extend the Lie algebraic results in [15] to switched systems with hybrid non-uniform discrete and continuous domains, a direct unifying generalization of switched systems on $\mathbb{R}$ and $\mathbb{Z}$, and extend the results in [8, 22] to a larger class of switched systems, namely those whose subsystem matrices are *simultaneously triangularizable*. In addition, we explore an easily checkable characterization of our required hypotheses for the theorems. Finally, conditions are provided under which there exists a stabilizing switching pattern for a collection of (not necessarily stable) linear systems that are simultaneously triangularizable and separate criteria are formed which imply the stability of the system under a given switching pattern given *a priori*.

1. Introduction

Stability of switched linear systems has been a topic of increasing discussion over the past decade, as evident in the recently published book [14], survey paper [16], and the references therein. Both switched systems and dynamic equations on time scales are of particular interest due to their numerable applications, as shown in [4, 10, 14, 20, 22]. Stability of switched systems under arbitrary switching on time scales can be determined by the identification of a single quadratic Lyapunov function applicable to all component systems [14, 22]. These *common quadratic Lyapunov functions* (CQLFs) have been used as a method for determining stability under arbitrary switching and are discussed in several papers encompassing time scale, continuous, and (uniform) discrete domains [4, 5, 13, 14, 15, 22, 24]. In this paper, we generalize the results of [15] to time scale (or *hybrid*) domains as well as extend and further illuminate the results of [18, 22] to include subsystem matrices which are not necessarily pairwise commutative. This work can also be seen as a natural sequel to [8], which considered the stability of switched systems comprised of subsystems represented by normal matrices, an extension to hybrid domains of results in [16, 25].

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After some preliminary definitions regarding time scale calculus and the stability of switched system in sections two and three, we derive one of our main results: the existence of a CQLF for a switched system comprised of subsystem matrices which are *simultaneously triangularizable*. Some checkable characterizations of simultaneous triangularizability are covered, and it is explained afterwards why this is a generalization to time scale domains of the Lie algebraic conditions in [15]. We then deduce conditions on the domain and the switching signal which imply stable behavior of switched systems which potentially contain unstable subsystems. Finally, we end the paper with a method for constructing time scales for which switched systems evolving under specified switching orders yield stable trajectories.

2. Time scale preliminaries

We gather here for convenience a few preliminaries regarding dynamic equations and time scale calculus. For a more in-depth survey of the topic, the reader is referred to [2].

Definition 2.1. A *time scale* $\mathbb{T}$ is a closed subset of $\mathbb{R}$. The *successor* of a point $t \in \mathbb{T}$ is given by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

and the *graininess* at a point $t \in \mathbb{T}$ is defined as

$$\mu(t) = \sigma(t) - t.$$

The *time scale* or *delta derivative* of a function $f(t) : \mathbb{T} \to \mathbb{R}$ is given by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)},$$

which is interpreted in the limit sense when $\mu(t) = 0$.

Notice that when $\mathbb{T} = \mathbb{R}$, $f^{\Delta}(t) = f'(t)$ and when $\mathbb{T} = \mathbb{Z}$, $f^{\Delta}(t) = \Delta f(t)$, the forward difference operator. In this sense, the time scale calculus is a direct unifying generalization of the theory on $\mathbb{R}$ and $\mathbb{Z}$.

Definition 2.2. For each point $t \in \mathbb{T}$, the set

$$\mathcal{H}(t) := \big\{z \in \mathbb{C} : |z + \frac{1}{\mu(t)}| < \frac{1}{\mu(t)}\big\}$$

is called the *Hilger circle at time t*.

Although the region described above is the interior of a circle in the complex plane the convention in the literature to refer is to it as the *Hilger circle*. When the set of time scale grainінesses is bounded above, the smallest Hilger circle (denoted $\mathcal{H}_{\min}$) is the Hilger circle associated with $\mu(t) = \mu_{\max}$. When $\mu(t) = 0$ we define $\mathcal{H}_0 := \mathbb{C}^-$, the open left-half complex plane.

Definition 2.3. A complex number λ is *regressive* if $\lambda \neq \frac{-1}{\mu(t)}$, *positively regressive* if $\lambda > \frac{-1}{\mu(t)}$, and *uniformly regressive* if there exists a neighborhood $B_{\varepsilon}(\lambda)$ for which $\frac{-1}{\mu(t)} \notin B_{\varepsilon}(\lambda)$ for all $t \in \mathbb{T}$. A matrix is (uniformly) regressive if all of its eigenvalues are (uniformly) regressive.

Definition 2.4. The *time scale exponential function*, which we denote by $e_{\lambda}(t, t_0)$, is the unique solution to the regressive, dynamic IVP

$$x^{\Delta} = \lambda x, \quad x(t_0) = 1. \tag{2.1}$$

An explicit formula for $e_\lambda(t,t_0)$ is available [2], but not needed here. Similarly, the unique solution to the regressive matrix IVP

$$x^\Delta = A(t)x, \quad x(t_0) = I, \tag{2.2}$$

is the *time scale transition matrix*, $\Phi_{A(t)}(t,t_0)$, which coincides with the *time scale matrix exponential*, $e_A(t,t_0)$, when $A(t) \equiv A$. These concepts are all rigorously treated in [2].

Definition 2.5. A uniformly regressive matrix $A(t)$ is called *Hilger stable* (or just *Hilger*) if $\operatorname{spec}(A(t)) \subset \mathcal{H}(t)$ for all $t \in \mathbb{T}$. If $A(t) \equiv A$, then this is equivalent to $\operatorname{spec}(A) \subset \mathcal{H}_{\min}$.

Throughout our analysis, the following function plays an important role in determining when matrices are Hilger.

Lemma 2.6. *Let $g(\lambda(t),\mu(t)) := 2\operatorname{Re}(\lambda(t)) + \mu(t)|\lambda(t)|^2$. Given an $n \times n$ matrix $A(t)$, $g(\lambda(t),\mu(t)) < 0$ for all $t \in \mathbb{T}$ and all $\lambda(t) \in \operatorname{spec}(A(t))$ if and only if $A(t)$ is Hilger.*

Proof. Let $A(t) \in \mathbb{R}^{n\times n}$, $\lambda_i(t) \in \operatorname{spec}(A(t))$, and $\mathbb{T}$ be given. Fix $t \in \mathbb{T}$. Notice that $g(\lambda_i(t),\mu(t)) < 0$ if and only if

$$\begin{aligned}
2\operatorname{Re}(\lambda_i(t)) + \mu(t)|\lambda_i(t)|^2 &< 0 \\
2\operatorname{Re}(\lambda_i(t)) + \mu(t)\big(\operatorname{Re}(\lambda_i(t))^2 + \operatorname{Im}(\lambda_i(t))^2\big) + \frac{1}{\mu(t)} &< \frac{1}{\mu(t)} \\
\frac{2}{\mu(t)}\operatorname{Re}(\lambda_i(t)) + \operatorname{Re}(\lambda_i(t))^2 + \operatorname{Im}(\lambda_i(t))^2 + \frac{1}{\mu(t)^2} &< \frac{1}{\mu(t)^2} \\
\Big(\operatorname{Re}(\lambda_i(t)) + \frac{1}{\mu(t)}\Big)^2 + \big(\operatorname{Im}(\lambda_i(t))^2 - 0\big)^2 &< \frac{1}{\mu(t)^2} \\
\big|\lambda_i(t) + \frac{1}{\mu(t)}\big| &< \frac{1}{\mu(t)}.
\end{aligned}$$

That is, $g(\lambda_i(t),\mu(t)) < 0$ if and only if $\lambda_i(t) \in \mathcal{H}(t)$. Thus $g(\lambda_i(t),\mu(t)) < 0$ for all $t \in \mathbb{T}$ and all $\lambda_i(t) \in \operatorname{spec}(A(t))$ if and only if $A(t)$ is Hilger. □

We finish this section by defining the concept of stability for a dynamic system and stating a useful characterization.

Definition 2.7. We say that a dynamic system $x^\Delta = Ax$ is *exponentially stable* if there exist $\gamma > 0$ and $\lambda > 0$ (with $-\lambda$ positively regressive) such that for any t_0 and $x(t_0)$, the corresponding solution satisfies

$$\|x(t)\| \le \|x(t_0)\|\gamma e_{-\lambda}(t,t_0).$$

Definition 2.8 ([19])**.** Given a time scale $\mathbb{T}$ which is unbounded above, define for arbitrary $t_0 \in \mathbb{T}$

$$\mathcal{S}_{\mathbb{C}}(\mathbb{T}) := \Big\{\lambda \in \mathbb{C} : \limsup_{T\to\infty} \frac{1}{T-t_0}\int_{t_0}^{T} \lim_{s\searrow\mu(t)} \frac{\log|1+s\lambda|}{s}\Delta t < 0\Big\}$$

and

$$\mathcal{S}_{\mathbb{R}}(\mathbb{T}) := \{\lambda \in \mathbb{R} : \forall T \in \mathbb{T}, \exists t \in \mathbb{T} \text{ with } t > T \text{ such that } 1 + \mu(t)\lambda = 0\},$$

where the integral given above is the time scale integral defined in [2]. Then the *region of exponential stability* for the time scale $\mathbb{T}$ is defined by

$$\mathcal{S}(\mathbb{T}) := \mathcal{S}_{\mathbb{C}}(\mathbb{T}) \cup \mathcal{S}_{\mathbb{R}}(\mathbb{T}).$$

Theorem 2.9 ([19]). *Let $\mathbb{T}$ be a time scale that is unbounded above and let $A \in \mathbb{R}^{n\times n}$ be regressive. Then the following holds:*

1. *If the system $x^\Delta = Ax$ is exponentially stable, then* $\operatorname{spec}(A) \subset \mathcal{S}_{\mathbb{C}}(\mathbb{T})$.
2. *If each eigenvalue of A is uniformly regressive, then $x^\Delta = Ax$ is exponentially stable.*

In [9] it is shown that the smallest Hilger circle $\mathcal{H}_{\min}$ is a subset of the region of exponential stability $\mathcal{S}(\mathbb{T})$. The relationship between Hilger circles and the region of exponential stability is shown in Figure 1. Notice that Hilger circles are not all required to be subsets of $\mathcal{S}(\mathbb{T})$, but $\mathcal{H}_{\min} \subset \mathcal{S}(\mathbb{T})$.

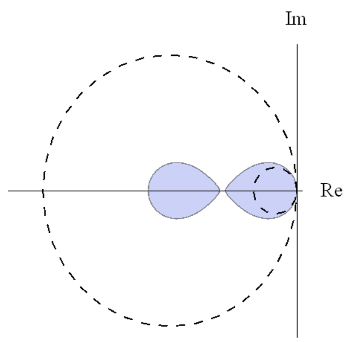

Figure 1. The region in the complex plane of exponential stability for a time scale comprised of two graininesses is shaded and the two associated Hilger circles are dashed

3. Summary of stability for switched systems

Definition 3.1. A *dynamic linear switched system under arbitrary switching* is a dynamic inclusion and initial condition of the form

$$x^\Delta \in \{A_i x\}_{i\in I}, \quad x(t_0) = x_0, \tag{3.1}$$

where $A_i \in \mathbb{R}^{n\times n}$ and I is an index set. When we wish to draw attention to a specific switching pattern, we will denote the switched system by

$$x^\Delta = A_{i(t)} x, \quad x(t_0) = x_0, \tag{3.2}$$

where $i(t) : \mathbb{T} \to I$ is a piecewise continuous *switching signal*. We say that $i(t)$ is *complete* if for every $j \in I$ there exists a $t \in \mathbb{T}$ such that $i(t) = j$.

Definition 3.2. The equilibrium $x(t) \equiv 0$ of (3.1) is *globally uniformly exponentially stable*, or GUES, if there exist a $\gamma > 0$ and a $\lambda > 0$ (with $-\lambda$ positively regressive) such that for any t_0 and $x(t_0)$, the corresponding solution of (3.1) $x(t)$ satisfies

$$\|x(t)\| \le \|x(t_0)\| \gamma e_{-\lambda}(t, t_0).$$

Stability for switched systems under arbitrary switching requires stronger conditions than the component systems being stable; this is evident in [14], where the author provides an example of a switched system over $\mathbb{R}$ with stable subsystems which produces unstable trajectories under a particular switching signal.

As noted in the introduction, one method for determining the stability of switched systems is through the identification of common quadratic Lyapunov functions (CQLFs). These functions have been studied extensively [4, 5, 11, 13, 14, 16, 24] and are defined now.

Definition 3.3. A *common quadratic Lyapunov function* (CQLF) associated with (3.1) is a function $V : \mathbb{R}^{n\times n} \to \mathbb{R}$ of the form

$$V_{P(t)}(x) := x^T P(t)x \quad P(t) = P^T(t) \succ 0,$$

such that $V^{\Delta}_{P(t)}(x) < 0$ for all nonzero $x \in \mathbb{R}^n$, where the derivative is taken along solutions to $x^{\Delta}(t) = A_i x(t)$ for each $i \in I$.

Using the product rule for the time scale derivative [2] and substituting in the system dynamics given by

$$x^{\sigma}(t) = (I + \mu(t)A_i)x(t),$$

one can easily derive the following useful form for V^{Δ}:

$$V^{\Delta}_{P(t)}(x) = x^T(A_i^T P(t) + P(t)A_i + \mu(t)A_i^T P(t)A_i + G_i^T(t)P^{\Delta}(t)G_i(t))x, \tag{3.3}$$

where $G_i(t) := (I_n + \mu(t)A_i)$. Thus, if (3.3) is negative for all $i \in I$ and all nonzero $x \in \mathbb{R}^n$, then $V_{P(t)}(x)$ is a CQLF.

Ramos [22] extended the results of Narendra and Balakrishnan [18] to time scale domains, showing that a sufficient condition on the matrices A_i to guarantee the existence of a CQLF is for the subsystem matrices to commute pairwise and have eigenvalues in the smallest Hilger circle. A main contribution of this paper is that we relax the pairwise commuting and stability hypotheses used in [4, 22], generalize the CQLF results in [15] to time scale domains, and expand the results in [8] to prove the existence of CQLFs for systems whose subsystem matrices are not normal.

In the case of continuous ($\mathbb{R}$, $\mu(t) \equiv 0$) or uniformly discrete ($\mathbb{Z}$, $\mu(t) \equiv 1$) domains, determining the existence of a CQLF has typically been achieved by solving the linear matrix equality

$$A_i^T P + PA_i + \mu(t)A_i^T PA_i = -M_i, \tag{3.4}$$

for the unknown P, given positive definite M_i. This equation is called the *time scale algebraic Lyapunov equation* (TSALE), and solutions to it are steady state solutions to the *time scale differential Lyapunov inequality* (TSDLI)

$$A_i^T P(t) + P(t)A_i + \mu(t)A_i^T P(t)A_i + G^T(t)P^{\Delta}(t)G^T(t) \prec 0, \tag{3.5}$$

as investigated in [5]. Solutions $P(t)$ to the TSDLI result in quadratic Lyapunov functions $V_{P(t)}(x) = x^T P(t)x$.

It was shown in [22] that the unique solution to the TSALE is time-varying when $\mu(t)$ varies with $t \in \mathbb{T}$, and therefore is not necessarily a solution to the TSDLI, as they are on $\mathbb{R}$ and $\mathbb{Z}$. As a result, the theory for quadratic Lyapunov functions on time scales has to be adapted to study the *time scale Lyapunov algebraic inequality* (TSALI), for which there do exist constant solutions. These constant solutions to the TSALI are also solutions to the TSDLI, and thus produce *bona fide* quadratic

Lyapunov functions. Constant solutions to the TSALI are investigated by examining when the associated time scale algebraic Lyapunov operator is negative definite. We denote this operator by

$$\mathcal{L}_a^{\mathbb{T}}(A, P, \mu(t)) := A^T P + PA + \mu(t) A^T PA. \tag{3.6}$$

Notice that the output of the operator $\mathcal{L}_a^{\mathbb{T}}$ is a symmetric, time-varying matrix which is dependent on the graininess $\mu(t)$ at each $t \in \mathbb{T}$. However, it suffices in many situations to study the time-invariant output of $\mathcal{L}_a^{\mathbb{T}}(A, P, \mu_{\max})$ due to the following lemma.

Lemma 3.4. *Let $\mathbb{T}$ be given and fix $A \in \mathbb{R}^{n\times n}$. If there exists a positive definite P_0 such that $\mathcal{L}_a^{\mathbb{T}}(A, P_0, \mu_{\max})$ is negative definite, then $\mathcal{L}_a^{\mathbb{T}}(A, P_0, \mu(t))$ is negative definite for all $\mu(t) \le \mu_{\max}$.*

Proof. Let $P_0 \succ 0$ and suppose $\mathcal{L}_a^{\mathbb{T}}(A, P_0, \mu_{\max})$ is negative definite. Then

$$\mu_{\max} A^T P_0 A \preceq \mu_{\max} \lambda_{\max}\{A^T P_0 A\} I$$

is a tight inequality, where $\lambda_{\max}\{A^T P_0 A\}$ is the largest eigenvalue of the Hermitian matrix $A^T P_0 A$. So

$$A^T P_0 + P_0 A + \mu_{\max} \lambda_{\max}\{A^T P_0 A\} I \prec 0.$$

Therefore,

$$\begin{aligned} \mathcal{L}_a^{\mathbb{T}}(A, P_0, \mu(t)) &\preceq A^T P_0 + P_0 A + \mu(t) \lambda_{\max}\{A^T P_0 A\} I \\ &\preceq A^T P_0 + P_0 A + \mu_{\max} \lambda_{\max}\{A^T P_0 A\} I \\ &\prec 0, \end{aligned}$$

for all $\mu(t) \le \mu_{\max}$, which proves the claim. □

4. Constructing CQLFs for dynamic linear switched systems under arbitrary switching

Before constructing CQLFs for arbitrary switched systems, we first detail how this construction takes place on a single, or "one switch," system. In doing so, we appeal to two theorems in matrix theory [12].

Theorem 4.1 (Schur). *Given $A \in \mathbb{C}^{n\times n}$ with eigenvalues $\{\lambda_i\}_{i=1}^n$ ordered in any manner, there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $UAU^* = T$ is upper triangular, with the eigenvalues ordered as specified down the diagonal.*

Theorem 4.2 (Sylvester's Criterion). *A matrix is positive definite if and only if its leading principal minors are all positive.*

Throughout the rest of this paper, $U \in \mathbb{C}^{n\times n}$ and $T \in \mathbb{C}^{n\times n}$ will denote unitary and upper triangular matrices respectively. Although quadratic Lyapunov functions have been constructed for systems comprised of a single Hilger matrix [22], this next result is important since the methods used here will be extended to the case of arbitrary switching between multiple subsystems, and the particular QLF constructed here has a special form.

Theorem 4.3 ([7]). *Let $\mathbb{T}$ be given. If $A = U^*TU \in \mathbb{R}^{n\times n}$ is Hilger, then there exists a quadratic Lyapunov function for the linear dynamic system $x^\Delta = Ax$ of the form $V_{U^*DU}(x) = x^T U^* DUx$, where D is a diagonal matrix.*

Proof. Let $\mathbb{T}$ be given and A be a real Hilger matrix. We will prove the result by constructing a positive definite diagonal matrix D with eigenvalues $\{p_k\}_{k=1}^n$ such that $\mathcal{L}_a^{\mathbb{T}}(T, D, \mu_{\max})$ is negative definite which, by Lemma 3.4, implies that $\mathcal{L}_a^{\mathbb{T}}(T, D, \mu(t))$ is negative definite as well for all $t \in \mathbb{T}$. Since similarity transformations preserve the spectrum of a matrix, we conclude that $\mathcal{L}_a^{\mathbb{T}}(A, U^*DU, \mu_{\max})$ is negative definite and $P = U^*DU$ is a QLF for the linear dynamic equation $x^\Delta = Ax$.

To accomplish the outline above, note that the i, j^{th} entry of $-\mathcal{L}_a^{\mathbb{T}}(T, D, \mu_{\max})$ is given by

$$\left[-\mathcal{L}_a^{\mathbb{T}}(T, D, \mu_{\max})\right]_{i,j} = -p_m \left(H(i-j)\bar{t}_{j,i} + H(j-i)t_{i,j}\right) - \mu_{\max} \sum_{k=1}^{m} p_k t_{k,i} \bar{t}_{k,j},$$

where $m := \min\{i, j\}$, $t_{i,j}$ is the i, j entry of T, and $H(\cdot)$ represents the Heaviside function

$$H(n) = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}.$$

Recall that $t_{i,i}$ are the eigenvalues of A, since they are the diagonal entries of T.

Appealing to Sylvester's Criterion, the eigenvalues $\{p_k\}_{k=1}^n$ of D can be chosen such that the leading principal minors of $-\mathcal{L}_a^{\mathbb{T}}(T, D, \mu_{\max})$ are all positive. The $1, 1$ entry of $-\mathcal{L}_a^{\mathbb{T}}(T, D, \mu_{\max})$ (or the first leading principal minor) is $-p_1 g(t_{1,1}, \mu_{\max})$, where $g(\cdot, \cdot)$ was defined in Lemma 2.6. We may arbitrarily select $p_1 > 0$, and since A is Hilger (or equivalently, $g(t_{i,i}, \mu_{\max}) < 0$ for all $i = 1, \dots, n$, by Lemma 2.6), it follows that the first leading principal minor of $-\mathcal{L}_a^{\mathbb{T}}(T, D, \mu_{\max})$ is positive.

We suppose now that the $(d-1)\times(d-1)$ leading principal minor of $-\mathcal{L}_a^{\mathbb{T}}(T, D, \mu_{\max})$ is positive, and show that the $d \times d$ leading principal minor can be made positive with a judicious choice of p_d. Laplace's determinant expansion is used on the leading $d \times d$ submatrix of $-\mathcal{L}_a^{\mathbb{T}}(T, D, \mu_{\max})$, which will be denoted $\mathcal{L}_{\text{sub}}^d$ in this proof. We adopt the notation $\mathcal{L}_{i,j}$ for the i, j^{th} entry of $-\mathcal{L}_a^{\mathbb{T}}(T, D, \mu_{\max})$, and represent the $\{d, j\}$ minor of $\mathcal{L}_{\text{sub}}^d$ by $M_{d,j}$; notice that $M_{d,d} = \det \mathcal{L}_{\text{sub}}^{d-1}$. Then

$$\begin{aligned}
\det \mathcal{L}_{\text{sub}}^d &= \sum_{j=1}^{d} (-1)^{d+j} \mathcal{L}_{d,j} M_{d,j} \\
&= \mathcal{L}_{d,d} M_{d,d} + \sum_{j=1}^{d-1} (-1)^{d+j} \mathcal{L}_{d,j} M_{d,j} \\
&= \Big(- p_d g(t_{d,d}, \mu_{\max}) - \mu_{\max} \sum_{k=1}^{d-1} p_k |t_{k,d}|^2 \Big) M_{d,d} + \sum_{j=1}^{d-1} (-1)^{d+j} \mathcal{L}_{d,j} M_{d,j} \\
&= -p_d g(t_{d,d}, \mu_{\max}) M_{d,d} - \mu_{\max} \sum_{k=1}^{d-1} p_k |t_{k,d}|^2 M_{d,d} + \sum_{j=1}^{d-1} (-1)^{d+j} \mathcal{L}_{d,j} M_{d,j}.
\end{aligned}$$

By the induction hypothesis, $M_{d,d} = \det \mathcal{L}_{\text{sub}}^{d-1} > 0$, and $g(t_{d,d}, \mu_{\max}) < 0$ since A is Hilger. As a result, $\det \mathcal{L}_{\text{sub}}^d > 0$ if and only if

$$p_d > \frac{\mu_{\max} \sum_{k=1}^{d-1} p_k |t_{k,d}|^2 M_{d,d} - (-1)^{d+k} \mathcal{L}_{d,k} M_{d,k}}{-M_{d,d} g(t_{d,d}, \mu_{\max})}.$$

Because the right-hand side will play a role in future proofs, define

$$J_d(A,\mu(t)) := \frac{\mu(t)\sum_{k=1}^{d-1} p_k |t_{k,d}|^2 M_{d,d} - (-1)^{d+k}\mathcal{L}_{d,k}M_{d,k}}{-M_{d,d}g(t_{d,d},\mu(t))}, \tag{4.1}$$

and (in general) choose each eigenvalue p_d of D so that $p_d > J_d(A(t),\mu(t))$ for all $t \in \mathbb{T}$. Since $A(t) \equiv A$ it is sufficient to choose $p_d > J_d(A,\mu_{\max})$ for each $p_d \in \operatorname{spec}(D)$.

By this construction, a $P = U^*DU \succ 0$ is obtained which shares a unitary factor with A and $V_P(x) = x^T Px$ is a quadratic Lyapunov function for the system $x^\Delta = Ax$. $\square$

The previous theorem can be naturally extended to prove stability of switched systems under arbitrary switching provided the set of subsystem matrices is compact and the systems can all be put into upper triangular form by the *same* unitary matrix U. We gather here for convenience some definitions and lemmas regarding sets of "simultaneously triangularizable" matrices.

Definition 4.4. A set of matrices $\{A_i\} \subset \mathbb{R}^{n\times n}$ is said to be *simultaneously triangularizable by M* if there exists a matrix M such that $MA_iM^{-1} = T_i$ is upper triangular for each i.

Lemma 4.5 ([21]). *If a set of matrices is simultaneously triangularizable by M, then there exists a unitary matrix U such that the set is simultaneously triangularizable by U.*

In [6], the authors give one of the primary characterizations for the simultaneous triangularizability of a set of matrices, based on the work of N.H. McCoy. By appealing to this theorem, we obtain tractable conditions which are easily checked given the subsystem matrices. Two preliminary definitions are needed first.

Definition 4.6. Let $A \in \mathbb{R}^{n\times n}$ be given. The j^{th} *subordinate principal submatrix* of A, denoted $S_j(A)$, is the principal submatrix of A resulting from the deletion of the first j many columns and rows.

This notation will be utilized in the following definition.

Definition 4.7. We introduce the terminology *mutually deflatable* to describe a set of $n \times n$ matrices $\{A_i\}_{i\in I}$ which satisfy the following:

(1) Each of the matrices A_i share an eigenvector, $v_1 \in \mathbb{R}^n$.

(2) Given the $n\times n$ unitary matrix U_1 formed by expanding v_1 to a normalized basis,[1] each of the first subordinate principal $(n-1)\times(n-1)$ submatrices of $U_1^{-1}A_iU_1$, denoted $S_1(U_1^{-1}A_iU_1)$, share an eigenvector, $v_2 \in \mathbb{R}^{n-1}$.

(3) Given the $(n-1)\times(n-1)$ unitary matrix U_2 formed by expanding v_2 to a normalized basis, each of the first subordinate principal $(n-2)\times(n-2)$ submatrices of $U_2^{-1}S_1(U_1^{-1}A_iU_1)U_2$ share an eigenvector, $v_3 \in \mathbb{R}^{n-2}$.

$(n-2)$ Given the 3×3 unitary matrix U_{n-2} formed by expanding v_{n-2} to a normalized basis, each of the first subordinate principal 2×2 submatrices of $U_{n-2}^{-1}S_1(U_{n-3}^{-1}\dots S_1(U_1^{-}1A_iU_1)\dots U_{n-3})U_{n-2}$ share an eigenvector, $v_{n-1} \in \mathbb{R}^2$.

[1] In $\mathbb{R}^{n\times n}$, this is done by finding $n-1$ many linearly independent vectors to v_1 (e.g., members of the standard basis), applying the Gram-Schmidt process, and creating a matrix whose columns are the normalized vectors.

Notice that it is fairly straightforward to verify whether a given set of N many $n \times n$ matrices is mutually deflatable in $N \cdot n$ many computations.

Theorem 4.8 ([6])**.** *Let $\{A_i\}_{i\in I} \subseteq \mathbb{R}^{n\times n}$ be a collection of mutually deflatable matrices. Then the matrices are simultaneously triangularizable.*

Theorem 4.8 justifies our exploration into sets of matrices that are simultaneously triangularizable, as it easy to determine if a set of matrices are mutually deflatable. We now extend Theorem 4.3 to the case of switched systems under arbitrary switching. For convention, we will state our theorems in terms of simultaneous triangularizability. For the rest of the paper, we will take the term "compact" to mean compact in the usual topology bestowed on $\mathbb{R}$ and $\mathbb{R}^{n\times n}$.

Theorem 4.9 ([7])**.** *Let $\{A_i\}_{i\in I} \subset \mathbb{R}^{n\times n}$ be a compact collection of simultaneously triangularizable matrices and $\mathbb{T}$ be a time scale. If each A_i is Hilger stable, then there exists a common quadratic Lyapunov function for the system* (3.1)*.*

Proof. Let $\{A_i\}_{i\in I}$ be a compact collection of Hilger matrices. Since each $A_i = U^*T_iU$ is triangularizable by the same unitary transformation, we construct a single $P = U^*DU$ such that $\mathcal{L}_a^{\mathbb{T}}(A_i, P, \mu_{\max})$ is negative definite for all $i \in I$. This is done, as before, by choosing the eigenvalues of D such that $\mathcal{L}_a^{\mathbb{T}}(T_i, D, \mu_{\max})$ is negative definite for all $i \in I$. Notice that the time scale must have a largest graininess $\mu_{\max}$ since the set of Hilger matrices is compact.

Since each A_i is Hilger, the first eigenvalue p_1 of the diagonal matrix D can be chosen arbitrarily positive, as in the proof of Theorem 4.3. However, in the induction step of this proof the successive p_i must now be chosen across multiple inequalities. Specifically, each eigenvalue must satisfy

$$p_d > \max_{i\in I} J_d(A_i, \mu_{\max}),$$

where $J_d(A_i, \mu_{\max})$ was defined in (4.1). The maximum is obtainable since the index set I is compact and the function $J_d(\cdot,\cdot)$ is continuous for each $1 \le d \le n$ over invertible A (as the composition of continuous functions over invertible A and μ). Then $V_P(x)$ with $P = U^*DU$ is a CQLF for the system (3.1). $\square$

A direct corollary to Theorem 4.9 generalizes to time scales the primary result of [18]; this corollary is due to results in [6], which reveals that sets of pairwise commutative matrices are also simultaneously triangularizable.

Corollary 4.10. *Let $\{A_i\}_{i\in I} \subset \mathbb{R}^{n\times n}$ be a compact collection of pairwise commutative matrices and $\mathbb{T}$ be a time scale. If each A_i is Hilger stable, then there exists a common quadratic Lyapunov function for the system* (3.1)*.*

This corollary also improves upon the major result in [22], in which the author found the time-varying, closed form solution to the TSALE (3.4). In that work, the author was investigating time-varying Lyapunov functions of the form $V_{P(t)} = x^TP(t)x$ and an additional hypothesis had to be satisfied, namely

$$\mathcal{L}_a^{\mathbb{T}}(A_i, P(t), \mu(t)) + (I_n + \mu(t)A_i)^T P^{\Delta}(I_n + \mu(t)A_i) \prec 0, \quad i \in I, \text{ for all } t \in \mathbb{T}.$$

Since the theory in this paper deals with constant $P \succ 0$, this condition is trivially satisfied. In addition, Theorem 4.9 also generalizes a major result of [8], since sets of simultaneously diagonal matrices are trivially simultaneously triangularizable.

One can also view the statement of Theorem 4.9 and its corollary in terms of the Lie algebra generated by the subsystem matrices $\{A_i\}_{i\in I}$, an approach that many authors [1, 14, 15] utilize. Recall that the *Lie algebra generated by a set of matrices* $\{A_i\}_{i\in I}$ is the smallest finite-dimensional vector space closed under the Lie bracket ($[A,B] := AB - BA$) which contains $\{A_i\}_{i\in I}$. For more information regarding Lie algebras and their properties, the reader is referred to [23].

As a result of Lie's Theorem [23], which states that every solvable Lie algebra has a basis for which each matrix in the algebra has upper triangular form (i.e., the matrices are simultaneously triangularizable), Theorem 4.9 is a generalization to hybrid domains of the central result in [15]. This result is stated in its newfound generality below.

Corollary 4.11. *Let $\mathbb{T}$ be given. If the matrices $\{A_i\}_{i\in I}$ are Hilger and their generated Lie algebra $\{A_i : i \in I\}_{\mathrm{LA}}$ is solvable, then the switched system $x^\Delta \in \{A_i x\}_{i\in I}$ is globally uniformly exponentially stable under arbitrary switching.*

When viewing switched systems under arbitrary switching in terms of their generated Lie algebras, an alternate proof of Corollary 4.10 arises. Pairwise commutativity of a set of matrices implies that their Lie algebra generated is nilpotent, and thus solvable [23]. This allows one to appeal to Corollary 4.11 to imply the existence of a CQLF for a switched system comprised of pairwise commuting subsystem matrices.

While interesting properties of switched systems can be gleaned by studying their generated Lie algebras, the benefit of viewing switched systems in terms of simultaneous triangularizablity is that it can be quickly determined in $N \cdot n$ many steps whether a given set of matrices are mutually deflatable and hence simultaneously triangularizable. For this reason, we will continue throughout this paper to state our hypotheses in terms of simultaneous triangularizability. It is also important to keep in mind that the existence of a CQLF is not equivalent to the GUES of a switched system and as such there exist systems that are stable under arbitrary switching which do not have CQLFs.

To illustrate the proof of Theorem 4.9, we construct a CQLF for a given switched system.

Example 4.12. Let

$$A_1 = \begin{bmatrix} -1 & -3 & 1 \\ 0 & -3 & 0 \\ -1 & -1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & -1 & 1 \\ 0 & -1 & 0 \\ -1 & 1 & -2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1 & -1 & 2 \\ 0 & -2 & 0 \\ -2 & 2 & -1 \end{bmatrix},$$

and $\mathbb{T}$ be any time scale with a compact set of graininesses and $\mu_{\max} = \frac{1}{4}$. This generalizes a result from [22] since none of the three matrices commute with each other. These three matrices are simultaneously unitarily upper triangularizable by the unitary matrix

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix},$$

and are all Hilger, which can be seen by evaluating $g(\lambda, \mu_{\max})$ for each eigenvalue of A_i and noticing that $\operatorname{spec}(A_i)$ is bounded away from $\frac{-1}{\mu(t)}$ for all $t \in \mathbb{T}$. By Theorem 4.9, the switched system $x^\Delta \in \{A_i x\}_{i=1}^3, x(t_0) = x_0$, is stable under arbitrary switching.

To construct the CQLF detailed in the proof of Theorem 4.9, the unitary transform is used to triangularize each of the subsystem matrices:

$$T_1 = UA_1U^* = \begin{bmatrix} -1+i & 0 & -\frac{-3-i}{\sqrt{2}} \\ 0 & -1-i & \sqrt{4+3i} \\ 0 & 0 & -3 \end{bmatrix},$$
$$T_2 = UA_2U^* = \begin{bmatrix} -2+i & 0 & -(-1)^{1/3} \\ 0 & -2-i & (-1)^{3/4} \\ 0 & 0 & -1 \end{bmatrix},$$
$$T_3 = UA_3U^* = \begin{bmatrix} -1+2i & -1 & -\frac{1+2i}{\sqrt{2}} \\ 0 & -1-2i & -\frac{1-2i}{\sqrt{2}} \\ 0 & 0 & -2 \end{bmatrix}.$$

As argued in the proof, each leading principal minor of $-\mathcal{L}_a^{\mathbb{T}}(T_i, D_P, \mu_{\max})$ must be positive. Let D be a diagonal matrix with the entries $\{p_k\}_{k=1}^n$. Evaluating the $1,1$ entry yields the three expressions:

$$[-\mathcal{L}_a^{\mathbb{T}}(T_1, D_P, \mu_{\max})]_{1,1} = \frac{3}{2}p_1,$$
$$[-\mathcal{L}_a^{\mathbb{T}}(T_2, D_P, \mu_{\max})]_{1,1} = \frac{11}{4}p_1,$$
$$[-\mathcal{L}_a^{\mathbb{T}}(T_2, D_P, \mu_{\max})]_{1,1} = \frac{3}{4}p_1.$$

Since each A_i is Hilger, these expressions are positive for any choice of $p_1 > 0$; set $p_1 = 1$.

Next, in order for the second leading principal minors of $-\mathcal{L}_a^{\mathbb{T}}(T_i, D_P, \mu_{\max})$ to be positive for $i = 1, 2, 3$, the following three conditions must be satisfied:

$$\frac{9}{4}p_2 > 0, \quad \frac{121}{16}p_2 > 0, \quad \frac{9}{16}p_2 > 0.$$

As before, any $p_2 > 0$ can be chosen; for simplicity, let $p_2 = 1$.

Finally, $\det -\mathcal{L}_a^{\mathbb{T}}(T_i, D_P, \mu)$ must be positive, leading to the inequalities

$$\det(-\mathcal{L}_a^{\mathbb{T}}(T_1, D_P, \mu_{\max})) = -15 + \frac{135}{16}p_3 > 0,$$
$$\det(-\mathcal{L}_a^{\mathbb{T}}(T_2, D_P, \mu_{\max})) = -\frac{11}{2} + \frac{847}{64}p_3 > 0,$$
$$\det(-\mathcal{L}_a^{\mathbb{T}}(T_3, D_P, \mu_{\max})) = -\frac{15}{4} + \frac{27}{16}p_3 > 0.$$

A choice of $p_3 = 7/3$ satisfies the three inequalities. Thus,

$$\begin{aligned} P &= U^* D_P U \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}^* \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{7}{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -i/\sqrt{2} \\ 1/\sqrt{2} & 0 & i/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{7}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

produces the common quadratic Lyapunov function

$$V_P(x) = x^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{7}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} x.$$

To verify that this is indeed a *bona fide* common quadratic Lyapunov function, the spectrum of the algebraic Lyapunov operator outputs are evaluated below:

$$\begin{aligned} \operatorname{spec}(\mathcal{L}_a^{\mathbb{T}}(A_1, P, \mu_{\max})) &\approx \{-7.3233, -1.5, -0.4267\}, \\ \operatorname{spec}(\mathcal{L}_a^{\mathbb{T}}(A_2, P, \mu_{\max})) &\approx \{-4.0603, -2.75, -2.2730\}, \\ \operatorname{spec}(\mathcal{L}_a^{\mathbb{T}}(A_3, P, \mu_{\max})) &\approx \{-6.4613, -.75, -0.0387\}. \end{aligned}$$

Since the outputs of the operator are negative definite, $V_P(x(t)) = x^T Px$ is indeed a common quadratic Lyapunov function for the switched system $x^\Delta \in \{A_i x\}_{i=1}^3$, and the switched system is stable under arbitrary switching.

5. Using constrained switching to stabilize switched systems

If the requirement that switched systems must produce stable trajectories under arbitrary switching is relaxed, we can be less conservative about the placement of the subsystem eigenvalues. To account for this, we consider switched systems like (3.2); that is,

$$x^\Delta = A_{i(t)}x, \quad x(t_0) = x_0,$$

where $i(t) : \mathbb{T} \to I$ is a single switching signal being investigated regardless of whether or not it is necessarily known *a priori*. We wish to find a single $P \succ 0$ such that $\mathcal{L}_a^{\mathbb{T}}(A_{i(t)}, P, \mu(t))$ is negative definite at each $t \in \mathbb{T}$. This amounts to the identification of a single QLF for the time-varying "aggregate" system $x^\Delta = A_{i(t)}x$, and *not* the construction of a QLF that is common to each of the subsystems. In doing so, we first require a definition.

Definition 5.1. Let $\{A_i\}_{i\in I}$ be a collection of real invertible matrices with eigenvalues in the open left-half complex plane. Then a time scale $\mathbb{T}$ is said to be *admissible with respect to* $\{A_i\}_{i\in I}$ if for every $t \in \mathbb{T}$, there exists an $i \in I$ such that $\operatorname{spec}(A_i) \subset \mathcal{H}(t)$. The time scale is *completely admissible* if it is admissible and for every $i \in I$, there exists a $t \in \mathbb{T}$ such that $\operatorname{spec}(A_i) \in \mathcal{H}(t)$.

Theorem 5.2 ([7])**.** *Let $\{A_i\}_{i\in I} \subset \mathbb{R}^{n\times n}$ be a compact collection of simultaneously triangularizable matrices and $\mathbb{T}$ a time scale with whose graininesses form a compact set. If $\mathbb{T}$ is (completely) admissible, then there exists a (complete) stable switching pattern for the switched system* (3.2)*.*

Proof. Let A_i each be triangularizable by the unitary matrix U. Set

$$S_{\mu(t)} := \{i \in I \mid \operatorname{spec}(A_i) \subset \mathcal{H}(t)\}, \tag{5.1}$$

and notice that $S_{\mu(t)}$ is nonempty for all $t \in \mathbb{T}$ since $\mathbb{T}$ is admissible (furthermore, if $\mathbb{T}$ is completely admissible, then $\bigcup_{t\in\mathbb{T}} S_{\mu(t)} = I$). Let $i(t) : \mathbb{T} \to I$ be defined such that $i(t) \in S_{\mu(t)}$ for each $t \in \mathbb{T}$, which can be defined to be a complete switching signal if the time scale is completely admissible. Once such a switching signal has been chosen, we can construct a QLF for the time-varying system (3.2).

We show that there exists a diagonal D with eigenvalues $\{p_k\}_{k=1}^n$ such that $\mathcal{L}_a^{\mathbb{T}}(T_{i(t)}, D, \mu(t))$ is negative definite for all $t \in \mathbb{T}$. Consider $\mu(t) = \mu_1$ and let

$i(t) \in S_{\mu_1}$. The first positive eigenvalue of D can be arbitrarily chosen, so let $p_1 = 1$. Now for each $1 < j \leq n$, choose $p_{j,\mu_1} > \max_{k \in \overline{S_{\mu_1}}} J_j(A_k, \mu_1)$, where $J_j(A_k, \mu_1)$ is the set defined in (4.1) and $\overline{S_{\mu_1}}$ is the closure of S_{μ_1}. It is necessary to take the closure since S_{μ_1} may be open, although it must be bounded due to the compactness of I.

Similarly, for each value of μ_r in the compact set $\{\mu(t)\}_{t \in \mathbb{T}}$ and for each $1 < j \leq n$, choose

$$p_{j,\mu_r} > \max_{k \in \overline{S_{\mu_r}}} \{J_j(A_k, \mu_r)\}.$$

Thus we can choose the eigenvalues of D to be

$$p_1 = 1,\ p_2 > \max_r \{p_{2,\mu_r}\}, \dots,\ p_n > \max_r \{p_{n,\mu_r}\},$$

all of which are obtainable values due to the compactness of $\{\mu(t)\}_{t \in \mathbb{T}}$. Evaluating at each $t \in \mathbb{T}$, the time-invariant matrix $\mathcal{L}_a^{\mathbb{T}}(T_{i(t)}, D, \mu(t))$ is negative definite according to Sylvester's Theorem, due to the chosen switching signal $i(t)$ and the eigenvalue construction of D. Therefore, $P = U^* D U$ is a QLF for the time-varying system $x^{\Delta} = A_{i(t)} x$. $\square$

The proof of Theorem 5.2 leads to the following corollary.

Corollary 5.3. *Let $\mathbb{T}$ be given and $\{A_i\}_{i \in I} \subset \mathbb{R}^{n \times n}$ be a compact collection of simultaneously triangularizable matrices. If the time-varying matrix $A_{i(t)}$ is Hilger, then there exists a QLF for the switched system* (3.2).

Example 5.4. We consider a switched system comprised of the subsystem matrices

$$\begin{bmatrix} 0.1124 & -2.3597 \\ 0.2887 & -1.6124 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -6.2887 & 0.6124 \\ 0.8067 & -7.7113 \end{bmatrix}$$

whose dynamics evolve over a time scale $\mathbb{T}$ comprised of only two graininesses which occur equally often in groups of five, $\mu_1 = 1$ and $\mu_2 = \frac{1}{5}$. The region of exponential stability and the associated Hilger circles for this type of time scale are shown in Figure 2, along with the spectrum of the two matrices; notice that A_2 is not exponentially stable over this time scale.

These matrices are both triangularizable by the unitary matrix

$$U = \begin{bmatrix} \frac{3}{\sqrt{11}} & \sqrt{\frac{2}{11}} \\ -\sqrt{\frac{2}{11}} & \frac{3}{\sqrt{11}} \end{bmatrix},$$

which gives

$$U^* A_1 U = \begin{bmatrix} -1 & -2.6484 \\ 0 & -0.5 \end{bmatrix}, \quad U^* A_2 U = \begin{bmatrix} -6 & -0.1943 \\ 0 & -8 \end{bmatrix}.$$

Since the spectrum of at least one of the matrices is contained in $\mathcal{H}_{\min}$ and each eigenvalue of A_1 and A_2 is contained in at least one Hilger circle, the sets defined by (5.1) are nonempty for all $t \in \mathbb{T}$; specifically, $S_{\mu_1} = \{1\}$ and $S_{\mu_2} = \{1, 2\}$. Therefore, any switching pattern which satisfies $i(t) \in S_{\mu(t)}$ at each $t \in \mathbb{T}$ will produce stable behavior. This can be interpreted in the following manner: for any $t \in \mathbb{T}$ with $\mu(t) = \mu_1$, the activated subsystem must be A_1, while for any $t \in \mathbb{T}$ where $\mu(t) = \mu_2$ either A_1 or A_2 can be activated.

To prove the stability of any such a switching pattern, we construct the QLF outlined in Theorem 5.2. The notation is taken from the proof of Theorem 4.3 with

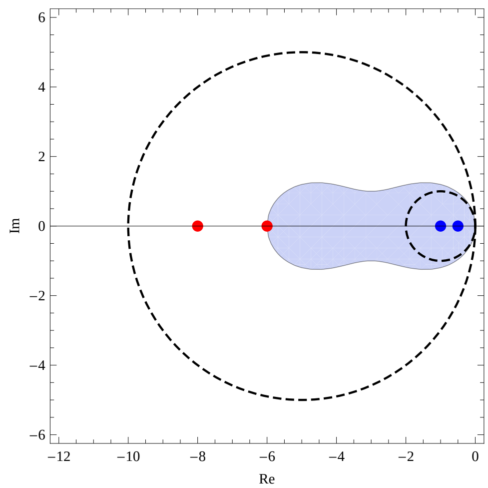

FIGURE 2. The region in the complex plane of exponential stability and the associated Hilger circles for $\mu_1 = 1$ and $\mu_2 = \frac{1}{5}$. The eigenvalues of A_1 are on the right and the eigenvalues of A_2 are on the left

the addition of superscripts to denote A_1 and A_2. Since the matrices are 2×2, the principal minors for the output of (3.6), denoted $M_{i,j}$, are scalars. We consider μ_1 and let $i(t) \in S_{\mu_1}$, that is $i(t) = 1$. The first positive eigenvalue of D is arbitrarily chosen to be $p_1 = 1$. We now compute p_{2,μ_1} and p_{2,μ_2}. Based on the proof of Theorem 5.2, p_{2,μ_1} must satisfy

$$\begin{aligned} p_{2,\mu_1} &> \max_{k \in \overline{S_{\mu_1}}} J_2(A_k, \mu_1) \\ &= J_2(A_1, 1) \\ &= \frac{p_1 |t^1_{1,2}|^2 M^1_{2,2} - (-1)^{2+1} \mathcal{L}^1_{2,1} M^1_{2,1}}{-M^1_{2,2} g(t^1_{2,2}, \mu_1)} \approx 9.3520, \end{aligned}$$

so let $p_{2,\mu_1} = 10$. Similarly, p_{2,μ_2} must satisfy

$$p_{2,\mu_2} > \max_{k \in \overline{S_{\mu_2}}} J_2(A_k, \mu_2) = \max\big\{J\big(A_1, 2, \frac{1}{5}\big), J\big(A_2, 2, \frac{1}{5}\big)\big\}$$

where

$$J\big(A_1, 2, \frac{1}{5}\big) = \frac{\frac{1}{5} p_1 |t^1_{1,2}|^2 M^1_{2,2} - (-1)^{2+1} \mathcal{L}^1_{2,1} M^1_{2,1}}{-M^1_{2,2} g(t^1_{2,2}, \frac{1}{5})} \approx 4.1017,$$

and

$$J\big(A_2, 2, \frac{1}{5}\big) = \frac{\frac{1}{5} p_1 |t^2_{1,2}|^2 M^2_{2,2} - (-1)^{2+1} \mathcal{L}^2_{2,1} M^2_{2,1}}{-M^2_{2,2} g(t^2_{2,2}, \frac{1}{5})} \approx 0.0025.$$

So p_{2,μ_2} must be chosen to be larger than 4.1017, say $p_{2,\mu_2} = 5$. Finally, in choosing the second eigenvalue of the quadratic Lyapunov function, p_2 must satisfy $p_2 > \max\{p_{2,\mu_1}, p_{2,\mu_2}\}$; let $p_2 = 11$. This yields the quadratic Lyapunov function

$$V_P(x) = x^T P x = x^T \begin{bmatrix} \frac{31}{11} & -\frac{30\sqrt{2}}{11} \\ -\frac{30\sqrt{2}}{11} & \frac{101}{11} \end{bmatrix} x,$$

where $P = U^*DU$. To verify that $V_P(x)$ is indeed a quadratic Lyapunov function for the time-varying system $x^\Delta = A_{i(t)}x$ (where $i(t) \in S_{\mu(t)}$ for all $t \in \mathbb{T}$), we examine the spectrum of the output of the time scale algebraic Lyapunov operator:

$$\begin{gathered}\operatorname{spec}(\mathcal{L}_a^{\mathbb{T}}(A_1, P, \mu_1)) \approx \{-1.2361, -1\}\\ \operatorname{spec}(\mathcal{L}_a^{\mathbb{T}}(A_1, P, \mu_2)) \approx \{-9.6212, -1.2261\}\\ \operatorname{spec}(\mathcal{L}_a^{\mathbb{T}}(A_2, P, \mu_1)) \approx \{-35.1925, -4.8000\}.\end{gathered}$$

So for all $t \in \mathbb{T}$, $\mathcal{L}_a^{\mathbb{T}}(A_{i(t)}, P, \mu(t)))$ is negative definite, and the switched system is GUES under the constrained switching pattern. It is worth noting that $V_P(x)$ is *not* a CQLF for the system since $\operatorname{spec}(\mathcal{L}_a^{\mathbb{T}}(A_2, P, \mu_1)) \approx \{528.0370, 23.9983\}$.

In Theorem 5.2 and Corollary 5.3, it is assumed that a time scale with a compact set of graininesses was given *a priori* and then a QLF was derived for certain switching signals implying the GUES of trajectories under those switching signals. Even if the subsystem matrices themselves were not all exponentially stable, certain switching signals (which still activate the systems which are not exponentially stable) yield GUES behavior, as demonstrated in Example 5.4. This is possible because the hypotheses of Theorem 5.2 can be met with matrices whose eigenvalues are in the open left-half complex plane but not in the time scale region of exponential stability $\mathcal{S}(\mathbb{T})$; however, to satisfy the hypotheses there must be at least one subsystem whose eigenvalues are in the smallest Hilger circle $\mathcal{H}_{\min} \subset \mathcal{S}(\mathbb{T})$.

In [3] the authors proved stability of switched systems over continuous domains comprised of unstable matrices (by appealing to the average dwell time of an unstable matrix and multiple Lyapunov functions). The situation described in Theorem 5.2 still has a single QLF which guarantees that solutions decrease monotonically with respect to the norm defined by the QLF; there is no analogous phenomenon when $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, since in those situations the set of exponential stability coincides with the active Hilger circle for all $t \in \mathbb{T}$. While results based on dwell time and multiple Lyapunov functions allow some subsystem matrices with eigenvalues in the right-half complex plane, Theorem 5.2 and Corollary 5.3 still require that subsystem matrices have their spectrum in the open left-half plane (which, however, is not equivalent to exponential stability on general time scales).

We now discuss stability results which arise when a particular switching *order* is desired from a given set of matrices. In this situation the only cog that is manipulated is the time scale domain, and priority is given to the order in which switching will occur without exact knowledge of specifically *when* the switching will take place (since switching can only occur at points in the timescale). However, depending on the graininesses that comprise the time scale, the exact times of which switching will occur can be given within a reasonable error defined by the graininesses. The next result shows that this is sufficient freedom to produce switched systems with stable trajectories which follow the desired switching order. A few required definitions are introduced first.

Definition 5.5. A *switching order* is an infinite sequence of N letters

$$\mathcal{O} = \{s_0, s_1, \dots, s_n, \dots\} \in N^\omega$$

coupled with a successor shift operation $\tilde{\sigma} : N^\omega \to N^\omega$, where

$$\tilde{\sigma}(\{s_0, s_1, \dots, s_n, \dots\}) := \{s_1, s_2, \dots, s_{n-1}, \dots\}.$$

A switching signal $i(t)$ is said to be *associated with a switching order* $\mathcal{O}$ if

$$i(\sigma^n(t)) = \tilde{\sigma}^n(\mathcal{O})$$

for all $n \in \mathbb{N}_0$.

Because we will be constructing time scales in the following proof, we must adjust our Hilger circle notation.

Definition 5.6. The *Hilger circle associated with graininess* μ, denoted $\mathcal{H}_\mu$, is the open region of the complex plane given by

$$\mathcal{H}_\mu := \big\{z \in \mathbb{C} : |z + \frac{1}{\mu}| < \frac{1}{\mu}\big\}.$$

The following theorem depends heavily on the QLF that was constructed in the proof to Theorem 5.2.

Theorem 5.7 ([7])**.** *Let $\{A_i\}_{i\in 1}^N$ be a collection of simultaneously triangularizable matrices with eigenvalues in the open left-half complex plane, and $\mathcal{O} \in N^\omega$ be a specified switching order. Then there exist time scales and at least one switching signal $i(t) : \mathbb{T} \to \{1, \dots, N\}$ associated with $\mathcal{O}$ such that $x^\Delta = A_{i(t)}x$ has a QLF.*

Proof. To begin, we define a base equivalence class of time scales; this is done by selecting grainինesses μ_j such that for each A_i, there exists at least one $j = 1, \dots, M$ such that $\operatorname{spec}(A_i) \subset \mathcal{H}_{\mu_j}$. That is, the graininesses which are chosen must give rise to stabilizing time scales. This collection of graininesses can be refined to include smaller graininesses if desired (possibly to have more control of *when* the switching occurs, as opposed to just what order it occurs); the smallest graininess will define the potential "error" (with respect to continuous time) possible in the timing of switching instances. Once refined, a stabilizing time scale can be constructed as follows.

Let s_0 be the first element of $\mathcal{O}$ and choose the next point in the time scale, denoted $\sigma(0)$, such that $\operatorname{spec}(A_{s_0}) \subset \mathcal{H}_{\mu(0)}$. Continuing in this manner for each point in the time scale, we define the stabilizing time scale domain to be given by the closed set

$$\mathbb{T} := \{\sigma^n(0) : \operatorname{spec}(A_{\tilde{\sigma}(i_0)}) \subset \mathcal{H}_{\mu(\sigma^n(0))}\,,\, n \in \mathbb{N}\}.$$

Constructing the time scale in this manner guarantees that the set

$$S_{\mathcal{H}(t)} := \{i \in \mathbb{N}_0 : \operatorname{spec}(A_i) \subset \mathcal{H}(t)\}$$

is nonempty for each $t \in \mathbb{T}$. Following the construction in the proof of Theorem 5.2, a QLF can be obtained for the time-varying system generated by the switching order $\mathcal{O}$. $\square$

It should be emphasized that it's possible for this construction to generate a time scale for which the eigenvalues of one or more of the subsystem matrices are not contained in the region of exponential stability. However, the time scale has been generated in such a way that the associated switching order will produce solutions which decrease monotonically with respect to the norm defined by the QLF.

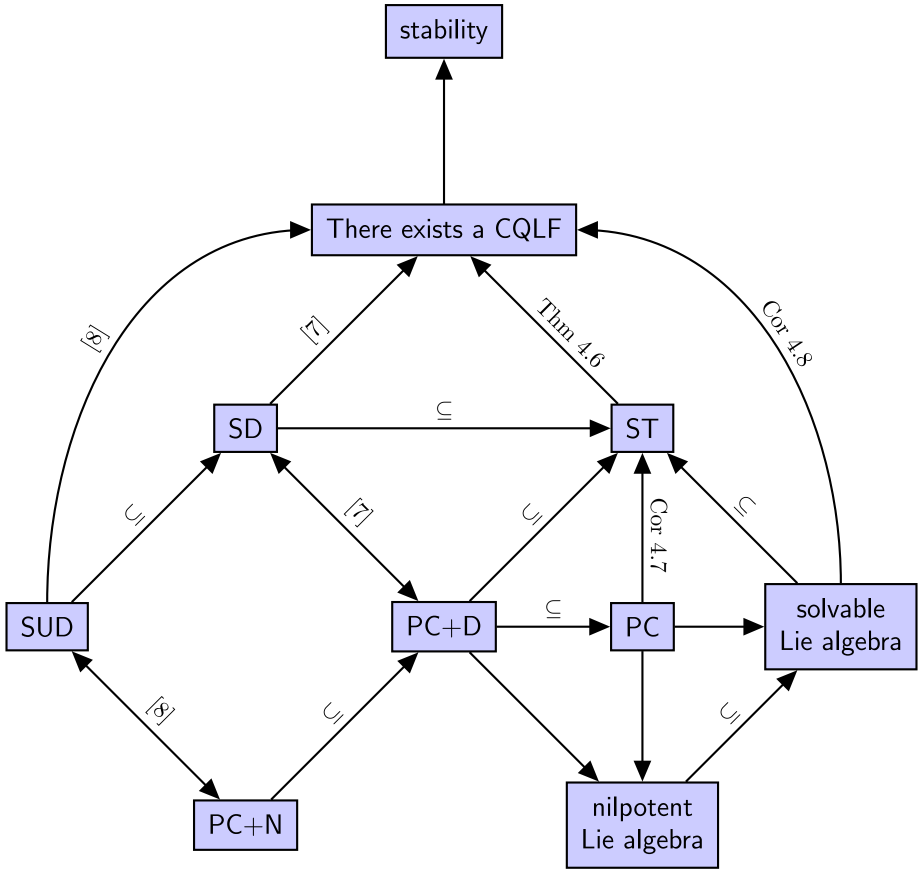

Figure 3. A schematic of how the various classes of switched systems interrelate with one another with respect to the existence of CQLFs, based on the results of [8], this work, and generalizations of [8] made possible by this work. Here,
$SD = \{$simultaneously diagonalizable$\}$,
$SUD = \{$simultaneously unitarily diagonalizable$\}$,
$ST = \{$simultaneously triangularizable$\}$,
$PC = \{$pairwise commutative$\}$,
$D = \{$diagonal$\}$,
$N = \{$normal$\}$.
Switched systems are assumed to be compact, Hilger stable, and uniformly regressive

6. Conclusions

We have extended a major result for switched systems on uniform domains [15] to hybrid domains and extended the theory in [8, 22] over time scales to include switched systems comprised of subsystem matrices which are not normal nor pairwise commutative. In doing so, the proofs for the results in [8, 15, 22] have been explained in a new light, highlighting the importance of the simultaneous triangularizability of a given set of matrices. The relationship of the results presented in this paper to the results presented by the authors in [8] is illustrated in Figure 3. In addition, new results concerning the construction of stabilizing switching patterns over hybrid domains were established for a larger class of matrices than those included in [8], which first introduced the concept for switched systems over time scale domains.

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