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NONLINEAR SCHRÖDINGER ELLIPTIC SYSTEMS INVOLVING EXPONENTIAL CRITICAL GROWTH IN $\mathbb{R}^2$

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ABSTRACT. This article concerns the existence and multiplicity of solutions for elliptic systems with weights, and nonlinearities having exponential critical growth. Our approach is based on the Trudinger-Moser inequality and on a minimax theorem.

1. Introduction

In this article, we consider the system

$$\begin{aligned} -\Delta u + V(|x|)u &= Q(|x|)f(u,v) \quad \text{in } \mathbb{R}^2, \\ -\Delta v + V(|x|)v &= Q(|x|)g(u,v) \quad \text{in } \mathbb{R}^2, \end{aligned} \tag{1.1}$$

where the nonlinear terms f and g are allowed to have exponential critical growth. By means of the Trudinger-Moser inequality and the radial potentials V and Q may be unbounded or decaying to zero. We shall consider the variational situation in which

$$(f(u,v), g(u,v)) = \nabla F(u,v)$$

for some function $F : \mathbb{R}^2 \to \mathbb{R}$ of class C^1, where ∇F stands for the gradient of F in the variables $w = (u,v) \in \mathbb{R}^2$. Aiming an analogy with the scalar case, we rewrite (1.1) in the matrix form

$$-\Delta w + V(|x|)w = Q(|x|)\nabla F(w) \quad \text{in } \mathbb{R}^2,$$

where we denote $\Delta = (\Delta, \Delta)$ and $Q(|x|)\nabla F(w) = (Q(|x|)f(w), Q(|x|)g(w))$. We make the following assumptions on the potentials $V(|x|)$ and $Q(|x|)$:

(V1) $V \in C(0,\infty)$, $V(r) > 0$ and there exists $a > -2$ such that

$$\liminf_{r\to+\infty} \frac{V(r)}{r^a} > 0.$$

(Q1) $Q \in C(0,\infty)$, $Q(r) > 0$ and there exist $b < (a-2)/2$ and $b_0 > -2$ such that

$$\limsup_{r\to 0} \frac{Q(r)}{r^{b_0}} < \infty \quad \text{and} \quad \limsup_{r\to+\infty} \frac{Q(r)}{r^b} < \infty.$$

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This type of potentials appeared in [2, 13, 14], in which the authors studied the existence and multiplicity of solutions for the scalar problem

$$-\Delta u + V(|x|)u = Q(|x|)f(u) \quad \text{in } \mathbb{R}^N$$
$$|u(x)| \to 0 \quad \text{as } |x| \to \infty,$$

where in [13, 14] the nonlinearity considered was $f(u) = |u|^{p-2}u$, with $2 < p < 2^* = 2N/(N-2)$ for $N \geq 3$ and $2 < p < \infty$ for $N = 2$. In [2] the authors considered the critical case in the sense of Trudinger-Moser inequality [9, 15].

Let us introduce the precise assumptions under which our problem is studied.

(F0) f and g have α_0-exponential critical growth, i.e., there exists $\alpha_0 > 0$ such that

$$\lim_{|w|\to+\infty} \frac{|f(w)|}{e^{\alpha|w|^2}} = \lim_{|w|\to+\infty} \frac{|g(w)|}{e^{\alpha|w|^2}} = \begin{cases} 0, & \forall \alpha > \alpha_0, \\ +\infty, & \forall \alpha < \alpha_0; \end{cases}$$

(F1) $f(w) = o(|w|)$ and $g(w) = o(|w|)$ as $|w| \to 0$;

(F2) there exists $\theta > 2$ such that

$$0 < \theta F(w) \leq w \cdot \nabla F(w), \quad \forall w \in \mathbb{R}^2 \backslash \{0\};$$

(F3) there exist constants $R_0, M_0 > 0$ such that

$$0 < F(w) \leq M_0 |\nabla F(w)|, \quad \forall |w| \geq R_0;$$

(F4) there exist $\nu > 2$ and $\mu > 0$ such that

$$F(w) \geq \frac{\mu}{\nu}|w|^\nu, \quad \forall w \in \mathbb{R}^2.$$

To establish our main results, we need to recall some notation about function spaces. In all the integrals we omit the symbol dx and we use $C, C_0, C_1, C_2, \dots$ to denote (possibly different) positive constants. Let $C_0^\infty(\mathbb{R}^2)$ be the set of smooth functions with compact support and

$$C_{0,rad}^\infty(\mathbb{R}^2) = \{u \in C_0^\infty(\mathbb{R}^2) : u \text{ is radial }\}.$$

Denote by $D_{\mathrm{rad}}^{1,2}(\mathbb{R}^2)$ the closure of $C_{0,rad}^\infty(\mathbb{R}^2)$ under the norm

$$\|\nabla u\|_2 = \Big(\int_{\mathbb{R}^2} |\nabla u|^2\Big)^{1/2}.$$

If $1 \leq p < \infty$ we define

$$L^p(\mathbb{R}^2; Q) \doteq \{u : \mathbb{R}^2 \to \mathbb{R} : u \text{ is mensurable}, \int_{\mathbb{R}^2} Q(|x|)|u|^p < \infty\}.$$

Similarly we define $L^2(\mathbb{R}^2; V)$. Then we set

$$H_{\mathrm{rad}}^1(\mathbb{R}^2; V) \doteq D_{\mathrm{rad}}^{1,2}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2; V),$$

which is a Hilbert space (see [14]) with the norm

$$\|u\|_{H_{\mathrm{rad}}^1(\mathbb{R}^2;V)} \doteq \Big(\int_{\mathbb{R}^2} |\nabla u|^2 + V(|x|)|u|^2\Big)^{1/2}.$$

We will denote $H_{\mathrm{rad}}^1(\mathbb{R}^2; V)$ by E and its norm by $\|\cdot\|_E$. In $E \times E$ we consider the scalar product

$$\langle w_1, w_2\rangle \doteq \int_{\mathbb{R}^2} [\nabla u_1 \nabla u_2 + V(|x|)u_1 u_2] + \int_{\mathbb{R}^2} [\nabla v_1 \nabla v_2 + V(|x|)v_1 v_2],$$

where $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$, to which corresponds the norm

$$\|w\| = \langle w, w\rangle^{1/2}.$$

Motivated by [2, 13, 14] and using a minimax procedure, we obtain existence and multiplicity results for system (1.1). As in the scalar problem treated in [2], there are at least two main difficulties in our problem; the possible lack of the compactness of the Sobolev embedding since the domain $\mathbb{R}^2$ is unbounded and the critical growth of the nonlinearities.

Denoting by $S_\nu > 0$ the best constant of the Sobolev embedding

$$E \hookrightarrow L^\nu(\mathbb{R}^2; Q)$$

(see Lemma 2.3 below), we have the following existence result for system (1.1).

Theorem 1.1. *Assume* (V1) *and* (Q1). *If* (F0)–(F4) *are satisfied, then* (1.1) *has a nontrivial weak solution* w_0 *in* $E \times E$ *provided*

$$\mu > \Big[\frac{2\alpha_0(\nu-2)}{\alpha'\nu}\Big]^{(\nu-2)/2} S_\nu^{\nu/2},$$

where $\alpha' \doteq \min\{4\pi, 4\pi(1 + b_0/2)\}$.

Our multiplicity result concerns the problem

$$-\Delta w + V(|x|)w = \lambda Q(|x|)\nabla F(w) \quad \text{in } \mathbb{R}^2, \tag{1.2}$$

where λ is a positive parameter. It can be stated as follows.

Theorem 1.2. *Assume* (V1) *and* (Q1). *If* F *is odd and* (F0)–(F4) *are satisfied, then for any given* $k \in \mathbb{N}$ *there exists* $\Lambda_k > 0$ *such that the system* (1.2) *has at least* $2k$ *pairs of nontrivial weak solutions in* $E \times E$ *provided* $\lambda > \Lambda_k$.

To close up this section, we remark that the main tool to prove Theorem 1.2, is the symmetric Mountain-Pass Theorem due to Ambrosetti-Rabinowitz [3]. It will be used in a more common version in comparison to the one used to prove the analogous theorem in the scalar case [2, Theorem 1.5], which leads us to a more direct conclusion of the result.

This article is organized as follows. Section 2 contains some technical results. In Section 3, we set up the framework in which we study the variational problem associated with (1.1) and we prove our existence result, Theorem 1.1. Finally, in Section 4 we prove Theorem 1.2.

2. Preliminaries

We start by recalling a version of the radial lemma due to Strauss in [12] (see [2, 13]). In the following, B_r denotes the open ball in $\mathbb{R}^2$ centered at the 0 with radius r and $B_R \setminus B_r$ denotes the annulus with interior radius r and exterior radius R.

Lemma 2.1. *Assume* (V1) *with* $a \geq -2$. *Then, there exists* $C > 0$ *such that for all* $u \in E$,

$$|u(x)| \leq C\|u\||x|^{-\frac{a+2}{4}}, \quad |x| \gg 1.$$

Next, we recall some basic embeddings (see Su et al. [13]). Let $\mathcal{A} \subset \mathbb{R}^2$ and define

$$H^1_{\rm rad}(\mathcal{A}; V) = \{u|_{\mathcal{A}} : u \in H^1_{\rm rad}(\mathbb{R}^2; V)\}.$$

Lemma 2.2. *Let $1 \leq p < \infty$. For any $0 < r < R < \infty$, with $R \gg 1$,*

(i) *the embeddings $H^1_{\rm rad}(B_R \setminus B_r; V) \hookrightarrow L^p(B_R \setminus B_r; Q)$ are compact;*
(ii) *the embedding $H^1_{\rm rad}(B_R; V) \hookrightarrow H^1(B_R)$ is continuous.*

In particular, as a consequence of (ii) we have that $H^1_{\rm rad}(B_R; V)$ is compactly immersed in $L^q(B_R)$ for all $1 \leq q < \infty$. If we assume that (V1) and (Q1) hold, by using Lemmas 2.1 and 2.2, a Hardy inequality with remainder terms (see [16]) and the same ideas from [13] we have:

Lemma 2.3. *Assume* (V1) *and* (Q1). *If $a \geq -2$ and $b < a$, then the embeddings $E \hookrightarrow L^p(\mathbb{R}^2; Q)$ are compact for all $2 \leq p < \infty$.*

Inspired by [1, 5, 9, 10, 15], to study system (1.1), the following version of the Trudinger-Moser inequality in the scalar case, obtained in [2, Theorem 1.1], plays an important rule.

Proposition 2.4. *Assume* (V1) *and* (Q1). *Then, for any $u \in E$ and $\alpha > 0$, we have that $Q(|x|)(e^{\alpha u^2} - 1) \in L^1(\mathbb{R}^2)$. Furthermore, if $\alpha < \alpha'$, then there exists a constant $C > 0$ such that*

$$\sup_{u \in E,\, \|u\|_E \leq 1} \int_{\mathbb{R}^2} Q(|x|)(e^{\alpha u^2} - 1) \leq C.$$

In line with Lions [8] and in order to prove our multiplicity result; Theorem 1.2, we establish an improvement of the Trudinger-Moser inequality on the space $E \times E$, considering our variational setting. Using Proposition 2.4 and following the same steps as in the proof of [7, Lemma 2.6] we have:

Corollary 2.5. *Assume* (V1) *and* (Q1). *Let (w_n) be in $E \times E$ with $\|w_n\| = 1$ and suppose that $w_n \rightharpoonup w$ weakly in $E \times E$ with $\|w\| < 1$. Then, for each $0 < \beta < \frac{\alpha'}{2}\big(1 - \|w\|^2\big)^{-1}$, up to a subsequence, it holds*

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{R}^2} Q(|x|)(e^{\beta |w_n|^2} - 1) < +\infty.$$

3. Variational setting

The natural functional associated with (1.1) is

$$I(w) = \frac{1}{2}\|w\|^2 - \int_{\mathbb{R}^2} QF(w),$$

$w \in E \times E$. Under our assumptions we have that I is well defined and it is C^1 on $E \times E$. Indeed, by (F1), for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|\nabla F(w)| \leq \varepsilon |w|$ always that $|w| < \delta$. On the other hand, for $\alpha > \alpha_0$, there exist constants $C_0, C_1 > 0$ such that $f(w) \leq C_0(\exp(\alpha |w|^2) - 1)$ and $g(w) \leq C_1(\exp(\alpha |w|^2) - 1)$ for all $|w| \geq \delta$. Thus, for all $w \in \mathbb{R}^2$ we have

$$\begin{aligned} |\nabla F(w)| &\leq \varepsilon |w| + |f(w)| + |g(w)| \\ &\leq \varepsilon |w| + C(\exp(\alpha |w|^2) - 1). \end{aligned} \tag{3.1}$$

Hence, using (F2), (3.1) and the Hölder's inequality, we have

$$\int_{\mathbb{R}^2} Q|F(w)|$$

$$
\begin{aligned}
&\le \varepsilon \int_{\mathbb{R}^2} Q|w|^2 + C\int_{\mathbb{R}^2} Q|w|(e^{\alpha|w|^2}-1)\\
&\le \varepsilon\Big(\int_{\mathbb{R}^2} Q|u|^2 + \int_{\mathbb{R}^2} Q|v|^2\Big) + C\Big(\int_{\mathbb{R}^2} Q|w|^r\Big)^{1/r}\Big(\int_{\mathbb{R}^2} Q(e^{s\alpha|w|^2}-1)\Big)^{1/s},
\end{aligned}
$$

with $r, s \ge 1$ such that $1/r + 1/s = 1$. Considering Lemma 2.3, for $r \ge 4$, we have

$$
\Big(\int_{\mathbb{R}^2} Q|w|^r\Big)^{1/r} = \|u^2+v^2\|^{1/2}_{L^{r/2}(\mathbb{R}^2;Q)} \le C\|u^2+v^2\|_E^{1/2} \le C\|w\| < \infty.
$$

On the other hand, by the Young's inequality and Proposition 2.4,

$$
\int_{\mathbb{R}^2} Q(e^{s\alpha|w|^2}-1) \le \frac{1}{2}\int_{\mathbb{R}^2} Q(e^{2s\alpha u^2}-1) + \frac{1}{2}\int_{\mathbb{R}^2} Q(e^{2s\alpha v^2}-1) < \infty. \tag{3.2}
$$

Hence, $QF(w) \in L^1(\mathbb{R}^2)$, which implies that I is well defined, for $\alpha > \alpha_0$. Using standard arguments, we can see that $I \in C^1(E\times E, \mathbb{R})$ with

$$
I'(w)z = \langle w, z\rangle - \int_{\mathbb{R}^2} Qz\cdot\nabla F(w)
$$

for all $z \in E\times E$. Consequently, critical points of the functional I are precisely the weak solutions of system (1.1).

In the next lemma we check that the functional I satisfies the geometric conditions of the Mountain-Pass Theorem.

Lemma 3.1. *Assume* (V1) *and* (Q1). *If* (F0)–(F2) *hold, then:*

(i) *there exist $\tau, \rho > 0$ such that $I(w) \ge \tau$ whenever $\|w\| = \rho$;*
(ii) *there exists $e_* \in E\times E$, with $\|e_*\| > \rho$, such that $I(e_*) < 0$.*

Proof. Just as we have obtained (3.1), we deduce that

$$
|\nabla F(w)| \le \varepsilon|w| + C|w|^{q-1}(e^{\alpha|w|^2}-1) \tag{3.3}
$$

for all $w \in \mathbb{R}^2$ and $q \ge 1$. Thus, using (F2), the Hölder's inequality and Lemma 2.3, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^2} Q|F(w)|\\
&\le \varepsilon\int_{\mathbb{R}^2} Q|w|^2 + C\int_{\mathbb{R}^2} Q|w|^q(e^{\alpha|w|^2}-1)\\
&\le \varepsilon\Big(\int_{\mathbb{R}^2} Q|u|^2 + \int_{\mathbb{R}^2} Q|v|^2\Big) + C\Big(\int_{\mathbb{R}^2} Q|w|^{qr}\Big)^{1/r}\Big(\int_{\mathbb{R}^2} Q(e^{s\alpha|w|^2}-1)\Big)^{1/s}\\
&\le C\varepsilon\|w\|^2 + C_0\|w\|^q\Big(\int_{\mathbb{R}^2} Q(e^{s\alpha|w|^2}-1)\Big)^{1/s},
\end{aligned}
$$

provided $r \ge 2$ and $s > 1$ such that $1/r+1/s = 1$. Now for $\|w\| \le M < [\alpha'/(2\alpha)]^{1/2}$, which implies that $2\alpha\|u\|_E^2 \le 2\alpha M^2 < \alpha'$ and $2\alpha\|v\|_E^2 \le 2\alpha M^2 < \alpha'$, and s sufficiently close to 1, it follows from (3.2) that

$$
\int_{\mathbb{R}^2} Q|F(w)| \le C\varepsilon\|w\|^2 + C_1\|w\|^q.
$$

Hence,

$$
I(w) \ge \Big(\frac{1}{2} - C\varepsilon\Big)\|w\|^2 - C_1\|w\|^q,
$$

which implies $i)$, if $q > 2$. In order to verify $ii)$, let $w \in E \times E$ with compact support G. Thus, using (F4) we obtain

$$I(tw) \le \frac{t^2}{2}\|w\|^2 - Ct^\nu \int_G Q|w|^\nu,$$

for all $t > 0$, which yields $I(tw) \to -\infty$ as $t \to +\infty$, provided $\nu > 2$. Setting $e_* = t_* w$ with $t_* > 0$ large enough, the proof is complete. □

To prove that a Palais-Smale sequence converges to a weak solution of system (1.1) we need to establish the following lemmas.

Lemma 3.2. *Assume* (F2)*. Let (w_n) be a sequence in $E \times E$ such that*

$$I(w_n) \to c \quad \text{and} \quad I'(w_n) \to 0.$$

Then

$$\|w_n\| \le C, \quad \int_{\mathbb{R}^2} QF(w_n) \le C, \quad \int_{\mathbb{R}^2} Qw_n \cdot \nabla F(w_n) \le C.$$

Proof. Let (w_n) be a sequence in $E \times E$ such that $I(w_n) \to c$ and $I'(w_n) \to 0$. Thus, for any $z \in E \times E$,

$$I(w_n) = \frac{1}{2}\|w_n\|^2 - \int_{\mathbb{R}^2} QF(w_n) = c + o_n(1) \tag{3.4}$$

and

$$I'(w_n)z = \langle w_n, z\rangle - \int_{\mathbb{R}^2} Qz \cdot \nabla F(w_n) = o_n(1). \tag{3.5}$$

Taking $z = w_n$ in (3.5) and using (F2) we have

$$\begin{aligned} c + \|w_n\| + o_n(1) &\ge I(w_n) - \frac{1}{\theta} I'(w_n)w_n \\ &= \big(\frac{1}{2} - \frac{1}{\theta}\big)\|w_n\|^2 + \int_{\mathbb{R}^2} Q\big[\frac{1}{\theta} w_n \cdot \nabla F(w_n) - F(w_n)\big] \\ &\ge \big(\frac{1}{2} - \frac{1}{\theta}\big)\|w_n\|^2. \end{aligned}$$

Consequently, $\|w_n\| \le C$. By (3.4) and (3.5) we obtain

$$\int_{\mathbb{R}^2} QF(w_n) \le C, \quad \int_{\mathbb{R}^2} Qw_n \cdot \nabla F(w_n) \le C.$$ □

We will also use the following convergence result.

Lemma 3.3. *Assume* (F2) *and* (F3)*. If $(w_n) \subset E \times E$ is a Palais-Smale sequence for I and w_0 is its weak limit then, up to a subsequence,*

$$\nabla F(w_n) \to \nabla F(w_0) \quad \text{in } L^1_{\rm loc}(\mathbb{R}^2, \mathbb{R}^2)$$

and

$$QF(w_n) \to QF(w_0) \quad \text{in } L^1(\mathbb{R}^2).$$

Proof. Suppose that (w_n) is a Palais-Smale sequence. According to Lemma 3.2, $w_n = (u_n, v_n) \rightharpoonup w_0 = (u_0, v_0)$ weakly in $E \times E$, that is, $u_n \rightharpoonup u_0$ and $v_n \rightharpoonup v_0$ weakly in E. Thus, recalling that $H^1_{\rm rad}(B_R; V) \hookrightarrow L^q(B_R)$ compactly for all $1 \le q < \infty$ and $R > 0$ (see the consequence of $ii)$ from Lemma 2.2), up to a subsequence, we can assume that $u_n \to u_0$ and $v_n \to v_0$ in $L^1(B_R)$. Hence, $w_n \to w_0$ in $L^1(B_R, \mathbb{R}^2)$

and $w_n(x) \to w_0(x)$ a.e. in $\mathbb{R}^2$. Since $\nabla F(w_n) \in L^1(B_R, \mathbb{R}^2)$, the first convergence follows from [6, Lemma 2.1]. Hence,

$$f(w_n) \to f(w_0) \quad \text{and} \quad g(w_n) \to g(w_0) \quad \text{in} \quad L^1_{\rm loc}(\mathbb{R}^2).$$

Thus, there exist $h_1, h_2 \in L^1(B_R)$ such that $Q|f(w_n)| \le h_1$ and $Q|g(w_n)| \le h_2$ a.e. in B_R. From (F3) we conclude that

$$|F(w_n)| \le \sup_{[-R_0,R_0]} |F(w_n)| + M_0|\nabla F(w_n)|$$

a.e. in B_R. Thus, by Lebesgue Dominated Convergence Theorem

$$QF(w_n) \to QF(w_0) \quad \text{in } L^1(B_R).$$

On the other hand, from (F2) and (3.3) with $q = 2$ we have

$$\int_{B_R^c} QF(w_n) \le \varepsilon \int_{B_R^c} Q|w_n|^2 + C \int_{B_R^c} Q|w_n|(e^{\alpha|w_n|^2} - 1), \tag{3.6}$$

for $\alpha > \alpha_0$. From Lemma 2.3, the Hölder's inequality, $\|w_n\| \le C$ and developing the exponential into a power series, we obtain

$$\varepsilon \int_{B_R^c} Q|w_n|^2 \le C\varepsilon \quad \text{and} \quad \int_{B_R^c} Q|w_n|(e^{\alpha|w_n|^2} - 1) \le \frac{C}{R^\xi},$$

for some $\xi > 0$. Hence, given $\delta > 0$, there exists $R > 0$ sufficiently large such that

$$\int_{B_R^c} Q|w_n|^2 < \delta \quad \text{and} \quad \int_{B_R^c} Q|w_n|(e^{\alpha|w_n|^2} - 1) < \delta.$$

Thus, from (3.6)

$$\int_{B_R^c} QF(w_n) \le C\delta \quad \text{and} \quad \int_{B_R^c} QF(w_0) \le C\delta.$$

Finally, since

$$\begin{aligned} &\Big| \int_{\mathbb{R}^2} QF(w_n) - \int_{\mathbb{R}^2} QF(w_0) \Big| \\ &\le \Big| \int_{B_R} QF(w_n) - \int_{B_R} QF(w_0) \Big| + \int_{B_R^c} QF(w_n) + \int_{B_R^c} QF(w_0), \end{aligned}$$

we obtain

$$\lim_{n\to\infty} \Big| \int_{\mathbb{R}^2} QF(w_n) - \int_{\mathbb{R}^2} QF(w_0) \Big| \le C\delta.$$

Since $\delta > 0$ is arbitrary, the result follows and the lemma is proved. □

In view of Lemma 3.1 the minimax level satisfies

$$c = \inf_{g\in\Gamma} \max_{0\le t\le 1} I(g(t)) \ge \tau > 0,$$

where

$$\Gamma = \{g \in C([0,1], E \times E) : g(0) = 0 \text{ and } I(g(1)) < 0\}.$$

Hence, by the Mountain-Pass Theorem without the Palais-Smale condition (see [3]) there exists a $(PS)_c$ sequence $(w_n) = ((u_n, v_n))$ in $E \times E$, that is,

$$I(w_n) \to c \quad \text{and} \quad I'(w_n) \to 0. \tag{3.7}$$

Lemma 3.4. *If*
$$\mu > \Big[\frac{2\alpha_0(\nu-2)}{\alpha'\nu}\Big]^{(\nu-2)/2} S_\nu^{\nu/2},$$
then $c < \alpha'/(4\alpha_0)$.

Proof. Since the embeddings $E \hookrightarrow L^p(\mathbb{R}^2; Q)$ are compacts for all $2 \le p < \infty$, there exists a function $\bar{u} \in E$ such that
$$S_\nu = \|\bar{u}\|_E^2 \quad \text{and} \quad \|\bar{u}\|_{L^\nu(\mathbb{R}^2;Q)} = 1.$$
Thus, considering $\overline{w} = (\bar{u}, \bar{u})$, by the definition of c and (F4), one has
$$\begin{aligned} c \le \max_{t\ge 0}\Big[S_\nu t^2 - \int_{\mathbb{R}^2} QF(t\overline{w})\Big] &\le \max_{t\ge 0}\Big[S_\nu t^2 - \frac{2^{\nu/2}\mu}{\nu} t^\nu\Big] \\ &= \frac{\nu-2}{2\nu}\frac{S_\nu^{\nu/(\nu-2)}}{\mu^{2/(\nu-2)}} < \frac{\alpha'}{4\alpha_0}. \end{aligned}$$
□

Now we are ready to prove our existence result.

Proof of Theorem 1.1. It follows from Lemmas 3.2 and 3.3 that the Palais-Smale sequence (w_n) is bounded and it converges weakly to a weak solution of (1.1) denoted by w_0. To prove that w_0 is nontrivial we argue by contradiction. If $w_0 \equiv 0$, Lemma 3.3 implies that
$$\lim_{n\to\infty} \int_{\mathbb{R}^2} QF(w_n) = 0.$$
Thus, by (3.4)
$$\lim_{n\to\infty} \|w_n\|^2 = 2c > 0. \tag{3.8}$$
From this and Lemma 3.4, given $\varepsilon > 0$, we have that $\|w_n\|^2 < \alpha'/(2\alpha_0) + \varepsilon$ for $n \in \mathbb{N}$ large. Thus, it is possible to choice $s > 1$ sufficiently close to 1 and $\alpha > \alpha_0$ close to α_0 such that $s\alpha\|w_n\|^2 \le \beta' < \alpha'/2$, which implies that
$$2s\alpha\|u_n\|_E^2 \le 2\beta' < \alpha' \quad \text{and} \quad 2s\alpha\|v_n\|_E^2 \le 2\beta' < \alpha'.$$
Thus, using (3.2), (3.1) in combination with the Hölder's inequality and Lemma 2.3, up to a subsequence, we conclude that
$$\lim_{n\to\infty} \int_{\mathbb{R}^2} Qw_n \cdot \nabla F(w_n) = 0.$$
Hence, by (3.5), we obtain that
$$\lim_{n\to\infty} \|w_n\|^2 = 0,$$
which is a contradiction with (3.8). Therefore, w_0 is a nontrivial weak solution of (1.1). □

4. Proof of Theorem 1.2

To prove our multiplicity result we shall use the following version of the Symmetric Mountain-Pass Theorem (see [3, 4, 11]).

Theorem 4.1. *Let $X = X_1 \oplus X_2$, where X is a real Banach space and X_1 is finite-dimensional. Suppose that J is a $C^1(X, \mathbb{R})$ functional satisfying the following conditions:*

(J1) *$J(0) = 0$ and J is even;*
(J2) *there exist $\tau, \rho > 0$ such that $J(u) \ge \tau$ if $\|u\| = \rho$, $\quad u \in X_2$;*

(J3) *there exists a finite-dimensional subspace* $W \subset X$ *with* $\dim X_1 < \dim W$ *and there exists* $\mathcal{S} > 0$ *such that* $\max_{u\in W} J(u) \le \mathcal{S}$*;*
(J4) J *satisfies the* $(PS)_c$ *condition for all* $c \in (0, \mathcal{S})$*.*

Then J *possesses at least* $\dim W - \dim X_1$ *pairs of nontrivial critical points.*

Given $k \in \mathbb{N}$, we apply this abstract result with $X = E \times E$, $X_1 = \{0\}$, $J = I_\lambda$ and $W = \widetilde{W} \times \widetilde{W}$ with $\widetilde{W} \doteq [\psi_1, \dots, \psi_k]$, where $\{\psi_i\}_{i=1}^k \subset C_0^\infty(\mathbb{R}^2)$ is a collection of smooth function with disjoint supports. We see that the energy functional associated with (1.2),

$$I_\lambda(w) \doteq \frac{1}{2}\|w\|^2 - \lambda \int_{\mathbb{R}^2} QF(w), \quad w \in E \times E,$$

is well defined and $I_\lambda \in C^1(E \times E, \mathbb{R})$ with derivative given by, for $w, z \in E \times E$,

$$I'_\lambda(w)z = \langle w, z\rangle - \lambda \int_{\mathbb{R}^2} Qz \cdot \nabla F(w).$$

Hence, a weak solution $w \in E \times E$ of (1.2) is exactly a critical point of I_λ. Furthermore, since $I_\lambda(0) = 0$ and F is odd, I_λ satisfies (J1) and with similar computations to prove (i) in Lemma 3.1 we conclude that I_λ also verifies (J2). In order to verify (J3) and (J4) we consider the following lemma.

Lemma 4.2. *Assume* (V1) *and* (Q1)*. If* F *satisfies (F0)-(F4), we have*

(i) *there exists* $\mathcal{S} > 0$ *such that* $\max_{w\in W} I_\lambda(w) \le \mathcal{S}$*;*
(ii) *the functional* I_λ *satisfies the* $(PS)_c$ *condition for all* $c \in (0, \mathcal{S})$*, that is, any sequence* (w_n) *in* $E \times E$ *such that*

$$I_\lambda(w_n) \to c \quad \text{and} \quad I'_\lambda(w_n) \to 0 \tag{4.1}$$

admits a convergent subsequence in $E \times E$*.*

Proof. By (F4),

$$\begin{aligned}
\max_{w\in W} I_\lambda(w) &= \max_{w\in W}\Big[\frac{1}{2}\|w\|^2 - \lambda\int_{\mathbb{R}^2} QF(w)\Big]\\
&\le \max_{w\in W}\Big[\frac{1}{2}\|u\|^2_{\widetilde{W}} + \frac{1}{2}\|v\|^2_{\widetilde{W}} - \frac{\mu\lambda}{\nu}\|u\|^\nu_{L^\nu(\mathbb{R}^2;Q)} - \frac{\mu\lambda}{\nu}\|v\|^\nu_{L^\nu(\mathbb{R}^2;Q)}\Big]\\
&\le \max_{u\in\widetilde{W}}\Big[\frac{1}{2}\|u\|^2_{\widetilde{W}} - \frac{\mu\lambda}{\nu}\|u\|^\nu_{L^\nu(\mathbb{R}^2;Q)}\Big] + \max_{v\in\widetilde{W}}\Big[\frac{1}{2}\|v\|^2_{\widetilde{W}} - \frac{\mu\lambda}{\nu}\|v\|^\nu_{L^\nu(\mathbb{R}^2;Q)}\Big].
\end{aligned}$$

Now, once $\dim \widetilde{W} < \infty$, the equivalence of the norms in this space gives a constant $C > 0$ such that

$$\max_{u\in\widetilde{W}}\Big[\frac{1}{2}\|u\|^2_{\widetilde{W}} - \frac{\mu\lambda}{C\nu}\|u\|^\nu_{\widetilde{W}}\Big] + \max_{v\in\widetilde{W}}\Big[\frac{1}{2}\|v\|^2_{\widetilde{W}} - \frac{\mu\lambda}{C\nu}\|v\|^\nu_{\widetilde{W}}\Big] = M_k(\lambda),$$

where

$$M_k(\lambda) \doteq \frac{\nu-2}{\nu}\big(\frac{C}{\mu}\big)^{2/(\nu-2)}\lambda^{2/(2-\nu)}.$$

Since $2/(2-\nu) < 0$ we have that $\lim_{\lambda\to+\infty} M_k(\lambda) = 0$, which implies that there exists $\Lambda_k > 0$ such that $M_k(\lambda) < \alpha'/(4\alpha_0) \doteq \mathcal{S}$ for any $\lambda > \Lambda_k$. Therefore, *i)* is proved. For *ii)*, by Lemma 3.2, (w_n) is bounded in $E \times E$ and so, up to a subsequence, $w_n \rightharpoonup w$ weakly in $E \times E$. We claim that

$$\int_{\mathbb{R}^2} Qw \cdot \nabla F(w_n) \to \int_{\mathbb{R}^2} Qw \cdot \nabla F(w) \quad \text{as} \quad n \to \infty. \tag{4.2}$$

Indeed, since $C_{0,rad}^{\infty}(\mathbb{R}^2)$ is dense in E, for all $\delta > 0$, there exists $v \in C_{0,rad}^{\infty}(\mathbb{R}^2, \mathbb{R}^2)$ such that $\|w - v\| < \delta$. Observing that

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^2} Qw \cdot [\nabla F(w_n) - \nabla F(w)] \Big| \\
&\leq \Big| \int_{\mathbb{R}^2} Q(w - v) \cdot \nabla F(w_n) \Big| + \|v\|_\infty \int_{supp(v)} Q|\nabla F(w_n) - \nabla F(w)| \\
&\quad + \Big| \int_{\mathbb{R}^2} Q(w - v) \cdot \nabla F(w) \Big|
\end{aligned}$$

and using Cauchy-Schwarz and the fact that $|I'_\lambda(w_n)(w - v)| \leq \varepsilon_n \|w - v\|$ with $\varepsilon_n \to 0$, we obtain

$$\Big| \int_{\mathbb{R}^2} Q(w - v) \cdot \nabla F(w_n) \Big| \leq \varepsilon_n \|w - v\| + \|w_n\| \|w - v\| \leq C\|w - v\| < C\delta,$$

where we have used that (w_n) is bounded in $E \times E$. Similarly, since the second limit in (4.1) implies that $I'_\lambda(w)(w - v) = 0$, we have

$$\Big| \int_{\mathbb{R}^2} Q(w - v) \cdot \nabla F(w_n) \Big| < C\delta.$$

From Lemma 3.3,

$$\lim_{n \to \infty} \int_{\mathrm{supp}(v)} Q|\nabla F(w_n) - \nabla F(w)| = 0.$$

Thus,

$$\lim_{n \to \infty} \Big| \int_{\mathbb{R}^2} Qw \cdot [\nabla F(w_n) - \nabla F(w)] \Big| < 2C\delta.$$

Since $\delta > 0$ is arbitrary, the claim follows. Hence, passing to the limit when $n \to \infty$ in

$$o_n(1) = I'_\lambda(w_n)w = \langle w_n, w \rangle - \lambda \int_{\mathbb{R}^2} Qw \cdot \nabla F(w_n)$$

and using that $w_n \rightharpoonup w$ weakly in $E \times E$, (4.2) and (F2) we obtain

$$\|w\|^2 = \lambda \int_{\mathbb{R}^2} Qw \cdot \nabla F(w) \geq 2\lambda \int_{\mathbb{R}^2} QF(w).$$

Hence

$$I_\lambda(w) \geq 0. \tag{4.3}$$

We have two cases to consider:
Case 1: $w = 0$. This case is similar to the checking that the solution w_0 obtained in the Theorem 1.1 is nontrivial. *Case 2:* $w \neq 0$. In this case, we define

$$z_n = \frac{w_n}{\|w_n\|} \quad \text{and} \quad z = \frac{w}{\lim \|w_n\|}.$$

It follows that $z_n \rightharpoonup z$ weakly in $E \times E$, $\|z_n\| = 1$ and $\|z\| \leq 1$. If $\|z\| = 1$, we conclude the proof. If $\|z\| < 1$, it follows from Lemma 3.3 and (4.1) that

$$\frac{1}{2} \lim_{n \to \infty} \|w_n\|^2 = c + \lambda \int_{\mathbb{R}^2} QF(w). \tag{4.4}$$

Setting

$$A \doteq \Big(c + \lambda \int_{\mathbb{R}^2} QF(w) \Big)(1 - \|z\|^2),$$

by (4.4) and the definition of z, we obtain $A = c - I_\lambda(w)$. Hence, coming back to (4.4) and using (4.3), we conclude that

$$\frac{1}{2}\lim_{n\to\infty}\|w_n\|^2 = \frac{A}{1-\|z\|^2} = \frac{c - I_\lambda(w)}{1-\|z\|^2} \le \frac{c}{1-\|z\|^2} < \frac{\alpha'}{4\alpha_0(1-\|z\|^2)}.$$

Consequently, for $n \in \mathbb{N}$ large, there are $r > 1$ sufficiently close to 1, $\alpha > \alpha_0$ close to α_0 and $\beta > 0$ such that

$$r\alpha\|w_n\|^2 \le \beta < \frac{\alpha'}{2}(1-\|z\|^2)^{-1}.$$

Therefore, from Corollary 2.5,

$$\int_{\mathbb{R}^2} Q(e^{\alpha|w_n|^2} - 1)^r < +\infty. \tag{4.5}$$

Next, we claim that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} Q(w_n - w)\cdot\nabla F(w_n) = 0.$$

Indeed, let $r, s > 1$ be such that $1/r + 1/s = 1$. Invoking (3.1) and the Hölder's inequality we conclude that

$$\begin{aligned}\Big|\int_{\mathbb{R}^2} Q(w_n - w)\cdot\nabla F(w_n)\Big| &\le \varepsilon\Big(\int_{\mathbb{R}^2} Q|w_n|^2\Big)^{1/2}\Big(\int_{\mathbb{R}^2} Q|w_n - w|^2\Big)^{1/2}\\ &\quad + C\Big(\int_{\mathbb{R}^2} Q(e^{\alpha|w_n|^2} - 1)^r\Big)^{1/r}\Big(\int_{\mathbb{R}^2} Q|w_n - w|^s\Big)^{1/s}.\end{aligned}$$

Then, from Lemma 2.3 and (4.5), the claim follows. This convergence together with the fact that $I'_\lambda(w_n)(w_n - w) = o_n(1)$ imply that

$$\lim_{n\to\infty}\|w_n\|^2 = \|w\|^2$$

and so $w_n \to w$ strongly in $E \times E$. The proof is complete. □

Proof of Theorem 1.2. Since I_λ satisfies (J1)–(J4), the result follows directly from Theorem 4.1. □

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