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EMDEN-FOWLER PROBLEM FOR DISCRETE OPERATORS WITH VARIABLE EXPONENT

MARIA MĂLIN

ABSTRACT. In this article, we prove the existence of homoclinic solutions for a $p(\cdot)$-Laplacian difference equation on the set of integers, involving a coercive weight function and a reaction term satisfying the Ambrosetti-Rabinowitz condition. The proof of the main result is obtained by using critical point theory combined with adequate variational techniques, which are mainly based on the mountain pass theorem.

1. INTRODUCTION AND MAIN RESULT

This article concerns the difference non-homogeneous equation

$$\begin{gathered} -\Delta\phi_{p(k-1)}(\Delta u(k-1)) + a(k)\phi_{p(k)}(u(k)) = f(k,u(k)) \quad \text{for all } k\in\mathbb{Z} \\ u(k)\to 0 \quad \text{as } |k|\to\infty, \end{gathered} \tag{1.1}$$

where $\phi_{p(k)}(t) = |t|^{p(k)-2}t$ for all $t\in\mathbb{R}$ and for each $k\in\mathbb{Z}$. We have denoted by Δ the difference operator, which is defined by $\Delta u(k-1) = u(k) - u(k-1)$ for each $k\in\mathbb{Z}$. Moreover,

$$\Delta\phi_{p(k-1)}(\Delta u(k-1)) = |\Delta u(k)|^{p(k)-2}\Delta u(k) - |\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1) \tag{1.2}$$

for each $k\in\mathbb{Z}$. The solutions of problem (1.1) are referred to as homoclinic solutions of the equations

$$-\Delta\phi_{p(k-1)}(\Delta u(k-1)) + a(k)\phi_{p(k)}(u(k)) = f(k,u(k)), \quad k\in\mathbb{Z}.$$

The goal of this article is to use the critical point theory to establish the existence of nontrivial homoclinic solutions for (1.1). Our idea is to transfer the problem of the existence of solutions for (1.1) into the problem of existence of critical points for some associated energy functional.

To explain the notion of homoclinic solution we go back to the definition of homoclinic orbit, which was introduced by Poincaré [14] for continuous Hamiltonian systems. In the theory of differential equations, a trajectory $x(t)$, which is asymptotic to a constant as $|t|\to\infty$, is called a doubly asymptotic or homoclinic orbit. Since we are seeking solutions $u(k)$ for (1.1) satisfying $\lim_{|k|\to\infty} u(k) = 0$, then

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according to the notion of a homoclinic orbit by Poincaré [14], we are interested in finding homoclinic solutions for problem (1.1).

The presence of the nonconstant potential $p(\cdot)$ is an important feature of this paper. The study of difference equations involving non-homogeneous difference operators of type (1.2) was initiated by Mihăilescu, Rădulescu and Tersian in [12], where some eigenvalue problems were investigated.

The study of discrete boundary value problems has captured special attention in the previous decade. In this context we point out the results obtained in the papers of Agarwal, Perera and O'Regan [1], Cabada, Iannizzotto and Tersian [3], Fang and Zhao [6], Ma and Guo [11], Kristály, Mihăilescu, Rădulescu and Tersian [10]. The studies regarding such type of problems can be placed at the interface of certain mathematical fields such as nonlinear partial differential equations and numerical analysis.

For the rest of this article, we will use the notation

$$p^+ := \sup_{k\in\mathbb{Z}} p(k), \quad p^- := \inf_{k\in\mathbb{Z}} p(k).$$

We assume that

$$p(\cdot) : \mathbb{Z} \to (1, +\infty),\ 1 < p^- \le p(\cdot) < p^+ < +\infty. \tag{1.3}$$

We also assume that $a(\cdot) : \mathbb{Z} \to \mathbb{R}$ is a positive and coercive weight function and $f = f(k,t) : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfy the following hypotheses:

- (A1) $a(k) \ge a_0 > 0$ for all $k \in \mathbb{Z}$; $a_0 < p^+$; $a(k) \to +\infty$ as $|k| \to \infty$;
- (F1) $\lim_{|t|\to 0} \frac{f(k,t)}{|t|^{p^+-1}} = 0$ uniformly for all $k \in \mathbb{Z}$;
- (F2) there exist $\alpha > p^+$ and $r > 0$ such that $0 < \alpha F(k,t) \le f(k,t)t$ for all $k \in \mathbb{Z}$, $t \ge r > 0$, where $F : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$F(k,t) = \int_0^t f(k,s)ds \quad \text{for all } k \in \mathbb{Z},\ t \in \mathbb{R};$$

The main result of this work is given by the following theorem.

Theorem 1.1. *Suppose that* $p(\cdot)$ *satisfies* (1.3)*,* $a(\cdot)$ *satisfies* (A1) *and* f *satisfies* (F1)-(F2)*. Then* (1.1) *admits at least a nontrivial homoclinic solution.*

This article is organized as follows. In Section 2 we define the functional spaces and prove some of their useful properties and in Section 3 we prove the existence of homoclinic solutions of problem (1.1), employing the mountain pass theorem of Ambrosetti & Rabinowitz [2].

2. Auxiliary results

For each $p(\cdot) : \mathbb{Z} \to (1, \infty)$, we introduce the space

$$l^{p(\cdot)} := \big\{u : \mathbb{Z} \to \mathbb{R}; \rho_{p(\cdot)}(u) := \sum_{k\in\mathbb{Z}} |u(k)|^{p(k)} < \infty\big\}.$$

On $l^{p(\cdot)}$ we introduce the Luxemburg norm

$$|u|_{p(\cdot)} := \inf\big\{\lambda > 0 : \sum_{k\in\mathbb{Z}} \big|\frac{u(k)}{\lambda}\big|^{p(k)} \le 1\big\}.$$

We also consider the space

$$l^\infty = \{u : \mathbb{Z} \to \mathbb{R}; |u|_\infty := \sup_{k\in\mathbb{Z}} |u(k)| < \infty\}.$$

We recall some useful properties of the space $l^{p(\cdot)}$. Firstly, by classical results of functional analysis we know that, for all $1 < p^- \leq p^+ < \infty$, $(l^{p(\cdot)}, |\cdot|_{p(\cdot)})$ is a reflexive Banach space whose dual is $(l^{q(\cdot)}, |\cdot|_{q(\cdot)})$ with $\frac{1}{p(k)} + \frac{1}{q(k)} = 1$. Moreover, for all $1 < p^- \leq p^+ < \infty$ there exists $c > 0$ such that

$$(\phi_{p(k)}(x) - \phi_{p(k)}(y))(x - y) \geq c|x - y|^{p(k)} \quad \text{for all } x, y \in \mathbb{R},\ k \in \mathbb{Z},\ \text{if } p(k) \geq 2; \tag{2.1}$$

or

$$(\phi_{p(k)}(x) - \phi_{p(k)}(y))(x - y) \geq c(|x| + |y|)^{p(k)-2}|x - y|^2 \quad \text{for all } x, y \in \mathbb{R},\ k \in \mathbb{Z}, \tag{2.2}$$

if $1 < p(k) < 2$.

Proposition 2.1 ([7, Proposition 2.3])**.** *If $u \in l^{p(\cdot)}$, $(u_n) \subset l^{p(\cdot)}$ and $p^+ < +\infty$, then the following properties hold:*

$$|u|_{p(\cdot)} > 1 \textit{ implies } |u|_{p(\cdot)}^{p^-} \leq \rho_{p(\cdot)}(u) \leq |u|_{p(\cdot)}^{p^+}; \tag{2.3}$$

$$|u|_{p(\cdot)} < 1 \textit{ implies } |u|_{p(\cdot)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq |u|_{p(\cdot)}^{p^-}; \tag{2.4}$$

$$|u_n|_{p(\cdot)} \to 0 \textit{ if and only if } \rho_{p(\cdot)}(u_n) \to 0 \textit{ as } n \to \infty; \tag{2.5}$$

On the other hand, it is useful to introduce other space which is still infinite dimensional but compactly embedded into $l^{p(\cdot)}$. The main reason is that such compact embedding is a key tool to prove Palais-Smale condition. We set

$$X = \{u : \mathbb{Z} \to \mathbb{R}; \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)} |u(k)|^{p(k)} < \infty\};$$

$$\|u\|_{p(\cdot)} := \inf\{\lambda > 0 : \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)} |\frac{u(k)}{\lambda}|^{p(k)} \leq 1\}.$$

Remark 2.2. We deduce that

$$|u|_{p(\cdot)} \leq \Big(\frac{a_0}{p^+}\Big)^{-1} \|u\|_{p(\cdot)} \quad \text{for all } u \in X. \tag{2.6}$$

Moreover, we have the following properties of the above norm, which are similarly to the properties of the norm $|\cdot|_{p(\cdot)}$.

Proposition 2.3. *Let $u \in X$, $(u_n) \subset X$ and $p^+ < +\infty$. Then the following properties hold*

$$\|u\|_{p(\cdot)} > 1 \textit{ implies } \|u\|_{p(\cdot)}^{p^-} \leq \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)} |u(k)|^{p(k)} \leq \|u\|_{p(\cdot)}^{p^+}; \tag{2.7}$$

$$\|u\|_{p(\cdot)} < 1 \textit{ implies } \|u\|_{p(\cdot)}^{p^+} \leq \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)} |u(k)|^{p(k)} \leq \|u\|_{p(\cdot)}^{p^-}; \tag{2.8}$$

$$\|u_n\|_{p(\cdot)} \to 0 \textit{ if and only if } \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)} |u_n(k)|^{p(k)} \to 0 \textit{ as } n \to \infty. \tag{2.9}$$

Proof. For (2.7), let $\|u\|_{p(\cdot)} > 1$. Then

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}|u(k)|^{p(k)} &= \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}\Big|\frac{u(k)}{\|u\|_{p(\cdot)}} \times \|u\|_{p(\cdot)}\Big|^{p(k)} \\ &\geq \|u\|_{p(\cdot)}^{p^-} \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}\Big|\frac{u(k)}{\|u\|_{p(\cdot)}}\Big|^{p(k)} \\ &= \|u\|_{p(\cdot)}^{p^-}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}|u(k)|^{p(k)} &= \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}\Big|\frac{u(k)}{\|u\|_{p(\cdot)}} \times \|u\|_{p(\cdot)}\Big|^{p(k)} \\ &\leq \|u\|_{p(\cdot)}^{p^+} \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}\Big|\frac{u(k)}{\|u\|_{p(\cdot)}}\Big|^{p(k)} \\ &= \|u\|_{p(\cdot)}^{p^+}.\end{aligned}$$

Thus,

$$\|u\|_{p(\cdot)}^{p^-} \leq \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}|u(k)|^{p(k)} \leq \|u\|_{p(\cdot)}^{p^+};$$

The proof of (2.8) is similar to that for (2.7).

For (2.9) we consider two cases. Case 1. $\|u_n\|_{p(\cdot)} \to 0$ as $n \to \infty$, then

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}|u_n(k)|^{p(k)} &= \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}\Big|\frac{u_n(k)}{\|u_n\|_{p(\cdot)}} \times \|u_n\|_{p(\cdot)}\Big|^{p(k)} \\ &\leq \|u_n\|_{p(\cdot)}^{\alpha} \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}\Big|\frac{u_n(k)}{\|u_n\|_{p(\cdot)}}\Big|^{p(k)} \\ &= \|u_n\|_{p(\cdot)}^{\alpha},\end{aligned}$$

where $\alpha = \begin{cases} p^+ & \text{if } \|u_n\|_{p(\cdot)} \geq 1 \\ p^- & \text{if } \|u_n\|_{p(\cdot)} < 1. \end{cases}$ So, $\sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}|u_n(k)|^{p(k)} \to 0$ as $n \to \infty$.

Case 2. $\sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}|u_n(k)|^{p(k)} \to 0$ as $n \to \infty$, then

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}|u_n(k)|^{p(k)} &= \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}\Big|\frac{u_n(k)}{\|u_n\|_{p(\cdot)}} \times \|u_n\|_{p(\cdot)}\Big|^{p(k)} \\ &\geq \|u_n\|_{p(\cdot)}^{\alpha} \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}\Big|\frac{u_n(k)}{\|u_n\|_{p(\cdot)}}\Big|^{p(k)} \\ &= \|u_n\|_{p(\cdot)}^{\alpha},\end{aligned}$$

where $\alpha = \begin{cases} p^- & \text{if } \|u_n\|_{p(\cdot)} \geq 1 \\ p^+ & \text{if } \|u_n\|_{p(\cdot)} < 1. \end{cases}$ So, $\|u_n\|_{p(\cdot)} \to 0$ as $n \to \infty$. □

Next we have a Discrete Hölder type inequality.

Proposition 2.4 ([7, Theorem 2.1]). *Let $u \in l^{p(\cdot)}$ and $v \in l^{q(\cdot)}$ be such that $\frac{1}{p(k)} + \frac{1}{q(k)} = 1$ for all $k \in \mathbb{Z}$. Then*
$$\sum_{k\in\mathbb{Z}} |uv| \le \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{p(\cdot)}|v|_{q(\cdot)}. \tag{2.10}$$

Moreover, we have the following relations
$$\frac{1}{p^-} + \frac{1}{q^+} = 1;\ \frac{1}{p^+} + \frac{1}{q^-} = 1. \tag{2.11}$$

Proposition 2.5. *$(X, \|\cdot\|_{p(\cdot)})$ is a reflexive Banach space and the embedding $X \hookrightarrow l^{p(\cdot)}$ is compact.*

Proof. Clearly, X is a Banach space. Next, we prove that it is uniformly convex. Indeed, we consider the σ-finite measure space $(\mathbb{Z}, \mu)$ where the measure is defined by
$$\mu(\emptyset) = 0 \quad \text{and} \quad \mu(S) = \sum_{k\in S} \frac{a(k)}{p(k)} \quad \text{for all } S \subseteq \mathbb{Z},\ S \ne \emptyset.$$
Then X is the usual $l^{p(\cdot)}$ space defined on $(\mathbb{Z}, \mu)$. The Clarkson theorem (see [5]) implies that X is uniformly convex, hence reflexive by the Milman-Pettis theorem (see [5]).

By Remark 2.2 we have that the embedding $X \hookrightarrow l^{p(\cdot)}$ is continuous.

Finally, we prove that $X \hookrightarrow l^{p(\cdot)}$ is compact. Let (u_n) be a bounded sequence in X; i.e., there exists $M > 0$ such that
$$\|u_n\|_{p(\cdot)}^{p^+} < \frac{M}{p^+} \quad \text{for all } n \in \mathbb{N},\ p^+ > 1. \tag{2.12}$$
By reflexivity, passing to a subsequence we have $u_n \rightharpoonup u$ in X for some $u \in X$. We may assume $u = 0$. In particular, $u_n(k) \to 0$ as $n \to \infty$ for all $k \in \mathbb{Z}$. By the coercivity of $a(\cdot)$ and the relation (1.3), for all $\varepsilon > 0$, we can find $h \in \mathbb{N}$ such that
$$\frac{a(k)}{p(k)} > \frac{1+M}{\varepsilon p^+} \quad \text{for all } |k| > h.$$
By continuity of the finite sum, there exists $\nu \in \mathbb{N}$ such that
$$\sum_{|k|\le h} |u_n(k)|^{p(k)} < \frac{\varepsilon}{1+M} \text{ for all } n > \nu.$$
Hence, for all $n > \nu$ we have
$$\begin{aligned}
\sum_{k\in\mathbb{Z}} |u_n(k)|^{p(k)} &= \sum_{|k|\le h} |u_n(k)|^{p(k)} + \sum_{|k|>h} |u_n(k)|^{p(k)} \\
&\le \frac{\varepsilon}{1+M} + \frac{\varepsilon p^+}{1+M} \sum_{|k|>h} \frac{a(k)}{p(k)} |u_n(k)|^{p(k)} \\
&\le \frac{\varepsilon}{1+M} + \frac{\varepsilon p^+}{1+M} \|u_n\|_{p(\cdot)}^{p^+} \quad \text{(see (2.7))} \\
&< \frac{\varepsilon}{1+M}\Big(1 + p^+ \frac{M}{p^+}\Big) \quad \text{(see (2.12))} \\
&= \varepsilon.
\end{aligned}$$
Thus, $u_n \to 0$ in $l^{p(\cdot)}$. □

In the proof of our theorem, we will need the following technical result.

Proposition 2.6. *If S is a compact subset of $l^{p(\cdot)}$, then for all $\varepsilon > 0$ there exists $h \in \mathbb{N}$ such that*

$$\Big(\sum_{|k|>h} |u(k)|^{p^-}\Big)^{1/p^-} < \varepsilon \quad \textit{for all } u \in l^{p(\cdot)}.$$

Proof. Arguing by contradiction, assume that there exists $\varepsilon > 0$ and a sequence $(u_n) \subseteq S$ such that

$$\Big(\sum_{|k|>n} |u_n(k)|^{p^-}\Big)^{1/p^-} > \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

Due to compactness of S, passing if necessary to a subsequence we may assume that $u_n \to u$ for some $u \in l^{p(\cdot)}$.

So, there exists $h \in \mathbb{N}$ such that

$$\Big(\sum_{|k|>h} |u(k)|^{p^-}\Big)^{1/p^-} < \frac{\varepsilon}{2}.$$

Moreover, there exists $n \geq h$ such that

$$\begin{aligned}
|u_n - u|_{p(\cdot)} &\leq \Big(\sum_{|k|>n} |u_n(k) - u(k)|^{p(k)}\Big)^{1/p^-} \\
&\leq \Big(\sum_{|k|>n} |u_n(k) - u(k)|^{p^-}\Big)^{1/p^-} \\
&< \frac{\varepsilon}{2} \quad \text{(see (2.3))}.
\end{aligned}$$

By the classical Minkowski inequality and the above inequalities, we have

$$\begin{aligned}
\varepsilon &< \Big(\sum_{|k|>n} |u_n(k)|^{p^-}\Big)^{1/p^-} \\
&= \Big(\sum_{|k|>n} |u_n(k) - u(k) + u(k)|^{p^-}\Big)^{1/p^-} \\
&\leq \Big(\sum_{|k|>n} |u_n(k) - u(k)|^{p^-}\Big)^{1/p^-} + \Big(\sum_{|k|>n} |u(k)|^{p^-}\Big)^{1/p^-} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

which is a contradiction. □

We denote by $(X^*, \|\cdot\|_{p^*(\cdot)}))$ the topological dual of $(X, \|\cdot\|_{p(\cdot)})$. We recall that a functional $J \in C^1(X, \mathbb{R})$, is said to satisfy the *Palais-Smale condition at the level* c, where c is a given real number, ($(PS)_c$ for short) if every sequence (u_n) in X satisfying $J(u_n) \to c$ and $J'(u_n) \to 0$ in X^* contains a convergent subsequence. Such condition is an essential hypothesis in the following mountain pass theorem, due to Ambrosetti and Rabinowitz [2].

Theorem 2.7. *Let X be a real Banach space and assume that $J \in C^1(X, \mathbb{R})$ satisfies the following geometric conditions:*

(H1) *there exist two numbers* $R > 0$ *and* $c_0 > 0$ *such that* $J(u) \geq c_0$ *for all* $u \in X$ *with* $\|u\| = R$*;*

(H2) $J(0) < c_0$ *and* $J(e) < 0$ *for some* $e \in X$ *with* $\|e\| > R$*.*

With an additional compactness condition of Palais-Smale type it then follows that the functional J *has a critical point* $u_0 \in X \setminus \{0, e\}$ *with critical value* $c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t))$*,* $c \geq c_0$*, where* $\Gamma := \{\gamma \in C([0,1], X); \gamma(0) = 0, \gamma(1) = e\}$*.*

3. Existence of positive homoclinic solutions

In this section, we investigate the existence of homoclinic solutions for problems of type (1.1).

Definition 3.1. We say that a function $u \in X$ is a *weak homoclinic solution* for the problem (1.1) if

$$\sum_{k \in \mathbb{Z}} \phi_{p(k-1)}(\Delta u(k-1)) \Delta v(k-1) + \sum_{k \in \mathbb{Z}} a(k) \phi_{p(k)}(u(k)) v(k) - \sum_{k \in \mathbb{Z}} f(k, u(k)) v(k) = 0,$$

for all $v \in X$ and $\lim_{|k| \to \infty} u(k) = 0$.

The basic idea in proving Theorem 1.1 is to consider the associate energetic functional of problem (1.1) and to show that it possesses a nontrivial critical point by using Theorem 2.7.

We denote $t^+ = \max\{+t, 0\}$, $t^- = \max\{-t, 0\}$ and set

$$f_+(k,t) = f(k, t^+), \quad F_+(k,t) = \int_0^t f_+(k,s) ds \quad \text{for all } k \in \mathbb{Z},\ t \in \mathbb{R}.$$

Note that $f_+ : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is continuous as $f(k, 0) = 0$ for all $k \in \mathbb{Z}$ (by (F1)). Next, we define the energetic functional associated to problem (1.1) as $J : X \to \mathbb{R}$,

$$J(u) := \sum_{k \in \mathbb{Z}} \frac{1}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} + \sum_{k \in \mathbb{Z}} \frac{1}{p(k)} a(k) |u(k)|^{p(k)} - \sum_{k \in \mathbb{Z}} F_+(k, u(k)).$$

Standard arguments assure that J is well-defined on the space X and is of class $C^1(X, \mathbb{R})$, with the derivative given by

$$\begin{aligned} \langle J'(u), v \rangle = &\sum_{k \in \mathbb{Z}} \phi_{p(k-1)}(\Delta u(k-1)) \Delta v(k-1) + \sum_{k \in \mathbb{Z}} a(k) \phi_{p(k)}(u(k)) v(k) \\ &- \sum_{k \in \mathbb{Z}} f_+(k, u(k)) v(k), \quad \text{for all } u, v \in X. \end{aligned} \tag{3.1}$$

Thus, we have established that the critical points of J correspond to the weak solutions of problem (1.1).

Remark 3.2. Note that weak solutions are usual solutions of (1.1). Indeed, fix $h \in \mathbb{Z}$ and define $e_h \in X$ by setting $e_h(k) = \delta_{h,k}$ (where $\delta_{h,k} = 1$ if $h = k$ and $\delta_{h,k} = 0$ if $h \neq k$) for all $k \in \mathbb{Z}$. Taking $v = e_h$ in (3.1) we obtain

$$-\Delta \phi_{p(h-1)}(\Delta u(h-1)) + a(h) \phi_{p(h)}(u(h)) = f(h, u(h)).$$

Moreover, clearly $u(h) \to 0$ as $|h| \to +\infty$. So, u is in fact a solution of problem (1.1).

To prove that J has a nontrivial critical point our idea is to show that actually J possesses a mountain pass geometry and J satisfies $(PS)_c$ condition.

Firstly we show that J satisfies the geometric conditions (H1) and (H2) of Theorem 2.7. With that end in view, we start by proving the following two results.

Lemma 3.3. *There exist two numbers $R > 0$ and $c_0 > 0$ such that $J(u) \geq c_0$ for all $u \in X$ with $\|u\|_{p(\cdot)} = R$.*

Proof. Fix $0 < \varepsilon < \frac{a_0}{2}$. By (F1), there exists $\delta \in (0, 1)$ such that

$$F_+(k,t) \leq \frac{\varepsilon}{p^+}|t|^{p^+} \leq \frac{a_0}{2p^+}|t|^{p^+} \leq \frac{a_0}{2p^+}|t|^{p(k)}, \tag{3.2}$$

for all $k \in \mathbb{Z}$ and all $|t| \leq \delta$. Define

$$R := \Big(\frac{a_0}{p^+}\Big)^{1/p^-} \delta^{\frac{p^+}{p^-}}.$$

By condition (A1) we deduce that $R \in (0, 1)$. Then for all $u \in X$ with $\|u\|_{p(\cdot)} = R$ relation (2.8) implies

$$\begin{aligned} R^{p^-} &= \frac{a_0}{p^+}\delta^{p^+} = \|u\|_{p(\cdot)}^{p^-} \\ &\geq \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}|u(k)|^{p(k)} \\ &\geq \frac{a_0}{p^+}|u(k)|^{p(k)}, \end{aligned}$$

for all $k \in \mathbb{Z}$. It follows that

$$1 > \delta^{p^+} \geq |u(k)|^{p(k)}, \quad \text{for all } k \in \mathbb{Z}.$$

Therefore, $|u(k)| < 1$ for every $k \in \mathbb{Z}$ and thus

$$|u(k)|^{p(k)} \geq |u(k)|^{p^+}, \quad \text{for all } k \in \mathbb{Z}.$$

The above inequalities show that $\delta \geq |u(k)|$, for all $k \in \mathbb{Z}$. Next, by (3.2) we deduce

$$\sum_{k\in\mathbb{Z}} F_+(k, u(k)) \leq \frac{a_0}{2p^+}\sum_{k\in\mathbb{Z}} |u(k)|^{p(k)} \leq \frac{1}{2}\sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}|u(k)|^{p(k)}, \tag{3.3}$$

with $\|u\|_{p(\cdot)} = R$. Define $c_0 := \frac{R^{p^+}}{2}$ and for each u with $\|u\|_{p(\cdot)} = R$, by (2.8) and (3.3) we deduce

$$\begin{aligned} J(u) &= \sum_{k\in\mathbb{Z}} \frac{1}{p(k-1)}|\Delta u(k-1)|^{p(k-1)} + \sum_{k\in\mathbb{Z}} \frac{1}{p(k)}a(k)|u(k)|^{p(k)} - \sum_{k\in\mathbb{Z}} F_+(k, u(k)) \\ &\geq \sum_{k\in\mathbb{Z}} \frac{1}{p(k)}a(k)|u(k)|^{p(k)} - \sum_{k\in\mathbb{Z}} F_+(k, u(k)) \\ &\geq \sum_{k\in\mathbb{Z}} \frac{1}{p(k)}a(k)|u(k)|^{p(k)} - \frac{1}{2}\sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}|u(k)|^{p(k)} \\ &= \frac{1}{2}\sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}|u(k)|^{p(k)} \\ &\geq \frac{1}{2}\|u\|_{p(\cdot)}^{p^+} = \frac{R^{p^+}}{2} = c_0. \end{aligned}$$

Hence, we conclude that Lemma 3.3 holds and in fact J satisfies condition (H1) of Theorem 2.7. □

Lemma 3.4. *$J(0) < c_0$ and $J(e) < 0$ for some $e \in X$ with $\|e\|_{p(\cdot)} > R$ (where c_0 and R is given in Lemma 3.3).*

Proof. Clearly $J(0) = 0 < c_0$. By standard integration, (F2) implies that there exist two constants $c_1 > 0$ and $c_2 > 0$ such that

$$F_+(k,t) \geq c_1|t|^\alpha - c_2, \quad \text{for all } k \in \mathbb{Z} \text{ and all } t \in \mathbb{R},\ \alpha > p^+. \tag{3.4}$$

Defining $v \in X$ by $v(k) = \begin{cases} a > 0 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0. \end{cases}$ Using (3.4), for each $\eta > 0$, we obtain

$$\begin{aligned} &J(\eta v) \\ &= \sum_{k\in\mathbb{Z}} \frac{1}{p(k-1)}|\Delta(\eta v)(k-1)|^{p(k-1)} + \sum_{k\in\mathbb{Z}} \frac{1}{p(k)} a(k)|\eta v(k)|^{p(k)} - \sum_{k\in\mathbb{Z}} F_+(k,\eta v(k)) \\ &\leq (2 + a(0))\frac{\eta^{p(0)}a^{p(0)}}{p(0)} - c_1\eta^\alpha a^\alpha + c_2, \end{aligned}$$

which approaches $-\infty$ as $\eta \to +\infty$ (since by relation (1.3) we have $\alpha > p^+ > p(0)$). So, we can choose $\eta > 0$ big enough and set $e = \eta v$, such that $\|e\|_{p(\cdot)} > c_0$ and $J(e) < 0$.

The proof of Lemma 3.4 is complete and in fact J satisfies condition (H2) of Theorem 2.7. □

Lemma 3.5. *If* (1.3), (A1), (F1), (F2) *are satisfied, then J satisfies $(PS)_c$.*

Proof. Let (u_n) be a sequence in X such that

$$J(u_n) \to c \text{ and } J'(u_n) \to 0 \quad \text{in } X^* \text{ as } \ n \to \infty. \tag{3.5}$$

The existence of such a sequence is given in the proof of Theorem 1.1.

Firstly, we prove that (u_n) is bounded in X. Assume that $\|u_n\|_{p(\cdot)} > 1$ for each n and by condition (F2) and relations (1.3) and (2.7) we deduce that

$$\begin{aligned} &\alpha J(u_n) - \langle J'(u_n), u_n\rangle \\ &= \sum_{k\in\mathbb{Z}} \big(\frac{\alpha}{p(k-1)} - 1\big)|\Delta u_n(k-1)|^{p(k-1)} + \sum_{k\in\mathbb{Z}} \big(\frac{\alpha}{p(k)} - 1\big)a(k)|u_n(k)|^{p(k)} \\ &\quad + \sum_{k\in\mathbb{Z}} \big[f_+(k,u_n(k))u_n(k) - \alpha F_+(k,u_n(k))\big] \\ &\geq \sum_{k\in\mathbb{Z}} \big[\frac{\alpha}{p(k)} a(k)|u_n(k)|^{p(k)} - \frac{1}{p(k)} a(k)|u_n(k)|^{p(k)}p(k)\big] \quad \text{(see (F2))} \\ &\geq \alpha\|u_n\|_{p(\cdot)}^{p^-} - p^+\|u_n\|_{p(\cdot)}^{p^-} \quad \text{(see (2.7) and (1.3))} \\ &= (\alpha - p^+)\|u_n\|_{p(\cdot)}^{p^-}, \end{aligned}$$

for all n. The above estimates and condition (3.5) imply that (u_n) is bounded in X. By Proposition 2.5 combined with this information, passing if necessary to a subsequence, we obtain that there exists $u \in X$ such that $u_n \rightharpoonup u$ in X and $u_n \to u$ in $l^{p(\cdot)}$.

Next, we prove that $u_n \to u$ in X. We assume $p(k) \geq 2$ and choose $\tilde{c} > 0$ such that

$$|u_n|_{p(\cdot)} < \tilde{c} \quad \text{for all } n \in \mathbb{N}. \tag{3.6}$$

Fix $\varepsilon > 0$. By (F1), there exists $\delta \in (0,1)$ such that

$$|f_+(k,t)| \leq |t|^{p^+-1} \quad \text{for all } k \in \mathbb{Z} \text{ and } |t| < \delta. \tag{3.7}$$

Choosing h even bigger if necessary, we have $|u_n(k)| < \delta$ for all $n \in \mathbb{N}$, $|k| > h$ and $|u(k)| < \delta$ for all $|k| > h$. By Proposition 2.6, there exists $h \in \mathbb{N}$ such that

$$\begin{gathered}\Big(\sum_{|k|>h} |u_n(k)|^{p^-}\Big)^{1/p^-} < \frac{\varepsilon}{2^{p^-+2}3\tilde{c}^{p^+-1}} \quad \text{for all } n \in \mathbb{N}, \\ \Big(\sum_{|k|>h} |u(k)|^{p^-}\Big)^{1/p^-} < \frac{\varepsilon}{2^{p^-+2}3\tilde{c}^{p^+-1}}.\end{gathered} \tag{3.8}$$

Due to the continuity of the finite sum, for $n \in \mathbb{N}$ big enough, we have

$$\sum_{|k|\le h} \big|\phi_{p(k-1)}(\Delta u_n(k-1)) - \phi_{p(k-1)}(\Delta u(k-1))\big||\Delta u_n(k-1) - \Delta u(k-1)| < \frac{\varepsilon}{6} \tag{3.9}$$

and

$$\sum_{|k|\le h} |f_+(k, u_n(k)) - f_+(k, u(k))||u_n(k) - u(k)| < \frac{\varepsilon}{6}. \tag{3.10}$$

Since $J'(u_n) \to 0$ in X^*, for $n \in \mathbb{N}$ big enough we have

$$|\langle J'(u_n), u_n - u\rangle| < \frac{\varepsilon}{6}. \tag{3.11}$$

Furthermore, $u_n \rightharpoonup u$ in X yields for $n \in \mathbb{N}$ big enough,

$$|\langle J'(u), u_n - u\rangle| < \frac{\varepsilon}{6}. \tag{3.12}$$

Assume that $\|u_n\|_{p(\cdot)} > 1$. For $n \in \mathbb{N}$ big enough we have

$$\begin{aligned}
&c\|u_n - u\|_{p(\cdot)}^{p^-} \\
&\le c\sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}|u_n(k) - u(k)|^{p(k)} \quad \text{(see (2.7))} \\
&\le \frac{1}{p^-} c\sum_{k\in\mathbb{Z}} a(k)|u_n(k) - u(k)|^{p(k)} \quad \text{(see (1.3))} \\
&\le \frac{1}{p^-}\sum_{k\in\mathbb{Z}} a(k)\left(\phi_{p(k)}(u_n(k)) - \phi_{p(k)}(u(k))\right)(u_n(k) - u(k)) \quad \text{(see (2.1))} \\
&= \frac{1}{p^-}\langle J'(u_n), u_n - u\rangle - \frac{1}{p^-}\langle J'(u), u_n - u\rangle - \frac{1}{p^-}\sum_{k\in\mathbb{Z}}\big(\phi_{p(k-1)}(\Delta u_n(k-1)) \\
&\quad - \phi_{p(k-1)}(\Delta u(k-1))\big)(\Delta u_n(k-1) - \Delta u(k-1)) \\
&\quad + \frac{1}{p^-}\sum_{k\in\mathbb{Z}} (f_+(k, u_n(k)) - f_+(k, u(k)))(u_n(k) - u(k)) \quad \text{(see (3.1))} \\
&\le \frac{1}{p^-}\frac{2\varepsilon}{3} - \frac{1}{p^-}\sum_{|k|>h}\big(\phi_{p(k-1)}(\Delta u_n(k-1)) - \phi_{p(k-1)}(\Delta u(k-1))\big) \\
&\quad \times (\Delta u_n(k-1) - \Delta u(k-1)) + \frac{1}{p^-}\sum_{|k|>h}(f_+(k, u_n(k)) - f_+(k, u(k))) \\
&\quad \times (u_n(k) - u(k)) \quad \text{(see (3.9)–(3.12))} \\
&\le \frac{1}{p^-}\frac{2\varepsilon}{3} + \frac{1}{p^-}\big(\frac{1}{p^-} + \frac{1}{q^-}\big)\Big[\sum_{|k|>h}\big|\phi_{p(k-1)}(\Delta u_n(k-1))
\end{aligned}$$

$$
\begin{aligned}
&- \phi_{p(k-1)}(\Delta u(k-1))\big|^{q(k)}\Big]^{1/q^-} \Big[\sum_{|k|>h} |\Delta u_n(k-1) - \Delta u(k-1)|^{p(k)}\Big]^{1/p^-} \\
&+ \frac{1}{p^-}\big(\frac{1}{p^-} + \frac{1}{q^-}\big)\Big[\sum_{|k|>h} |f_+(k, u_n(k)) - f_+(k, u(k))|^{q(k)}\Big]^{1/q^-} \\
&\times \Big[\sum_{|k|>h} |u_n(k) - u(k)|^{p(k)}\Big]^{1/p^-} \quad \text{(see (2.3) and (2.10))} \\
\le\ & \frac{1}{p^-}\frac{2\varepsilon}{3} + \frac{1}{p^-}\big(\frac{1}{p^-} + \frac{1}{q^-}\big)\Big[\sum_{|k|>h} \big|\phi_{p(k-1)}(\Delta u_n(k-1)) \\
&- \phi_{p(k-1)}(\Delta u(k-1))\big|^{q^-}\Big]^{1/q^-} \Big[\sum_{|k|>h} |\Delta u_n(k-1) - \Delta u(k-1)|^{p^-}\Big]^{1/p^-} \\
&+ \frac{1}{p^-}\big(\frac{1}{p^-} + \frac{1}{q^-}\big)\Big[\sum_{|k|>h} |f_+(k, u_n(k)) - f_+(k, u(k))|^{q^-}\Big]^{1/q^-} \\
&\times \Big[\sum_{|k|>h} |u_n(k) - u(k)|^{p^-}\Big]^{1/p^-} \\
\le\ & \frac{1}{p^-}\frac{2\varepsilon}{3} + \frac{1}{p^-}\big(\frac{1}{p^-} + \frac{1}{q^-}\big)\Big[\Big(\sum_{|k|>h} \big|\phi_{p(k-1)}(\Delta u_n(k-1))\big|^{q^-}\Big)^{1/q^-} \\
&+ \Big(\sum_{|k|>h} \big|\phi_{p(k-1)}(\Delta u(k-1))\big|^{q^-}\Big)^{1/q^-}\Big] \\
&\times \Big[\Big(\sum_{|k|>h} |\Delta u_n(k-1)|^{p^-}\Big)^{1/p^-} + \Big(\sum_{|k|>h} |\Delta u(k-1)|^{p^-}\Big)^{1/p^-}\Big] \\
&+ \frac{1}{p^-}\big(\frac{1}{p^-} + \frac{1}{q^-}\big)\Big[\Big(\sum_{|k|>h} |f_+(k, u_n(k))|^{q^-}\Big)^{1/q^-} + \Big(\sum_{|k|>h} |f_+(k, u(k))|^{q^-}\Big)^{1/q^-}\Big] \\
&\times \Big[\Big(\sum_{|k|>h} |u_n(k)|^{p^-}\Big)^{1/p^-} + \Big(\sum_{|k|>h} |u(k)|^{p^-}\Big)^{1/p^-}\Big] \quad \text{(by Minkowski's inequality)} \\
\le\ & \frac{1}{p^-}\frac{2\varepsilon}{3} + \frac{1}{p^-}\big(\frac{1}{p^-} + \frac{p^+ - 1}{p^+}\big)\Big[\Big(\sum_{|k|>h} \big|\phi_{p(k-1)}(\Delta u_n(k-1))\big|^{\frac{p^+}{p^+-1}}\Big)^{\frac{p^+-1}{p^+}} \\
&+ \Big(\sum_{|k|>h} \big|\phi_{p(k-1)}(\Delta u(k-1))\big|^{\frac{p^+}{p^+-1}}\Big)^{\frac{p^+-1}{p^+}}\Big] \\
&\times \Big[\Big(\sum_{|k|>h} |\Delta u_n(k-1)|^{p^-}\Big)^{1/p^-} + \Big(\sum_{|k|>h} |\Delta u(k-1)|^{p^-}\Big)^{1/p^-}\Big] \\
&+ \frac{1}{p^-}\big(\frac{1}{p^-} + \frac{p^+ - 1}{p^+}\big)\Big[\Big(\sum_{|k|>h} \big(|u_n(k)|^{p^+-1}\big)^{\frac{p^+}{p^+-1}}\Big)^{\frac{p^+-1}{p^+}} \\
&+ \Big(\sum_{|k|>h} \big(|u(k)|^{p^+-1}\big)^{\frac{p^+}{p^+-1}}\Big)^{\frac{p^+-1}{p^+}}\Big]\Big[\Big(\sum_{|k|>h} |u_n(k)|^{p^-}\Big)^{1/p^-}
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(\sum_{|k|>h}|u(k)|^{p^-}\Big)^{1/p^-}\Big] \quad \text{(see (2.3), (2.11), (3.7))}\\
&\le \frac{1}{p^-}\frac{2\varepsilon}{3}+\frac{1}{p^-}\big(\frac{1}{p^-}+\frac{p^+-1}{p^+}\big)\Big[2^{p^--1}|u_n|_{p(\cdot)}^{p^+-1}+2^{p^--1}|u|_{p(\cdot)}^{p^+-1}\Big]\\
&\quad\times\Big[\Big(\sum_{|k|>h}|\Delta u_n(k-1)|^{p^-}\Big)^{1/p^-}+\Big(\sum_{|k|>h}|\Delta u(k-1)|^{p^-}\Big)^{1/p^-}\Big]\\
&\quad+\frac{1}{p^-}\Big(\frac{1}{p^-}+\frac{p^+-1}{p^+}\Big)\Big[|u_n|_{p(\cdot)}^{p^+-1}+|u|_{p(\cdot)}^{p^+-1}\Big]\\
&\quad\times\Big[\Big(\sum_{|k|>h}|u_n(k)|^{p^-}\Big)^{1/p^-}+\Big(\sum_{|k|>h}|u(k)|^{p^-}\Big)^{1/p^-}\Big]\\
&\quad\text{(by Minkowski inequality and (2.3))}\\
&\le \frac{1}{p^-}\frac{2\varepsilon}{3}+\frac{1}{p^-}\big(\frac{1}{p^-}+\frac{p^+-1}{p^+}\big)\Big(2^{p^-}\tilde{c}^{p^+-1}\Big)\frac{4\varepsilon}{2^{p^-+2}3\tilde{c}^{p^+-1}}\\
&\quad+\frac{1}{p^-}\big(\frac{1}{p^-}+\frac{p^+-1}{p^+}\big)2\tilde{c}^{p^+-1}\frac{2\varepsilon}{2^{p^-+2}3\tilde{c}^{p^+-1}} \quad \text{(see (3.6), (3.8))}\\
&<\Big(\frac{2}{3p^-}+\frac{1}{p^-}\big(\frac{1}{p^-}+\frac{p^+-1}{p^+}\big)\big(\frac{1}{3}+\frac{1}{2^{p^-}\cdot 3}\big)\Big)\varepsilon\,.
\end{aligned}
$$

Hence, $u_n \to u$ in X. If $1 < p(k) < 2$ we argue in an analogous way using (2.2) instead of (2.1). Thus, J satisfies $(PS)_c$. □

Proof of Theorem 1.1. By Lemma 3.3 and Lemma 3.4 and the mountain pass theorem, namely Theorem 2.7, we deduce the existence of a sequence $(u_n) \subset X$ such that

$$J(u_n) \to c > 0 \text{ and } J'(u_n) \to 0 \quad \text{in } X^*, \text{ as } n \to \infty.$$

We also we know by Lemma 3.5 that J satisfies $(PS)_c$ condition. Then there exists a subsequence, still denoted by (u_n), and $u_0 \in X$ such that (u_n) converges to u_0 in X. So, we have $J(u_0) = c > 0$ and $J'(u_0) = 0$ which implies that u_0 is a critical point of J and consequently a solution of (1.1)(see Remark 3.2). Moreover, since $J(u_0) > 0$ we have $u_0 \neq 0$. Hence, we conclude that u_0 is a nontrivial solution of (1.1). □

Theorem 3.6. *Assume that the hypotheses of Theorem 1.1 are satisfied and in addition, the following condition holds*

$$f(k,t) \ge 0 \text{ for all } t < 0 \text{ and all } k \in \mathbb{Z}\,. \tag{3.13}$$

Then (1.1) *has a positive homoclinic solution.*

Proof. By Theorem 1.1 we know that (1.1) has a nontrivial homoclinic solution. Now, arguing by contradiction, suppose that $u(k_0) < 0$ for some $k_0 \in \mathbb{Z}$ and let k_1 be such that $u(k_1) = \min\{u(k), k \in \mathbb{Z}\} < 0$. In consequence $\Delta\phi_{p(k_1-1)}(\Delta u(k_1-1)) \ge 0$, which by equation (1.1) implies that

$$f(k_1, u(k_1)) = -\Delta\phi_{p(k_1-1)}(\Delta u(k_1-1)) + a(k_1)\phi_{p(k_1)}(u(k_1)) < 0,$$

a contradiction with (3.13). So, $u(k) \ge 0$ for all $k \in \mathbb{Z}$.

Next, we prove that $u(k) > 0$ for all $k \in \mathbb{Z}$. Indeed, arguing by contradiction, assume that $u(k_2) = 0$ for some $k_2 \in \mathbb{Z}$. By (1.1) we have

$$\phi_{p(k_2)}(\Delta u(k_2)) = \phi_{p(k_2-1)}(\Delta u(k_2 - 1))$$

(recall that $f(k, 0) = 0$). Note that if $u(k_2 + 1) = 0$ or $u(k_2 - 1) = 0$, the solution is identically zero by a recursion, which is a contradiction with $u \neq 0$. So, $u(k) > 0$ for all $k \in \mathbb{Z}$. □

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Maria Mălin
Department of Mathematics, University of Craiova, 200585, Romania
E-mail address: amy.malin@yahoo.com