Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 53, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

INFINITELY MANY SOLUTIONS FOR ELLIPTIC BOUNDARY VALUE PROBLEMS WITH SIGN-CHANGING POTENTIAL

WEN ZHANG, XIANHUA TANG, JIAN ZHANG

Abstract. In this article, we study the elliptic boundary value problem

$$\begin{gathered} -\Delta u + a(x)u = g(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$ and the potential $a(x)$ is allowed to be sign-changing. We establish the existence of infinitely many nontrivial solutions by variant fountain theorem developed by Zou for sublinear nonlinearity.

1. Introduction

We study the semilinear elliptic boundary-value problem

$$\begin{gathered} -\Delta u + a(x)u = g(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $g \in C(\bar{\Omega} \times \mathbb{R}^N, \mathbb{R}^N)$ and $a \in L^{N/2}(\Omega)$. In this article, we are interested in the existence and multiplicity of solutions for problem (1.1) when $g(x,u)$ is sublinear.

The semilinear elliptic equation has found a great deal of interest in the previous years. With the aid of variational methods, the existence and multiplicity of nontrivial solutions for problem (1.1) or similar (1.1) have been extensively investigated in the literature over the past several decades. See [2, 3, 4, 5, 7, 8, 10, 6, 11, 12, 21, 22] and the references therein.

There are some works devoted to the superquadratic situation and asymptotically quadratic situation for problem (1.1), see for instance [6, 7, 8, 9]. In [6], Li and Willem [1] established the existence of a nontrivial solution for (1.1) under the following Ambrosetti-Rabinowitz type superquadratic condition

(G1) there exist $\mu > 2$ and $L > 0$ such that

$$0 < \mu G(x,u) \leq u g(x,u), \quad \text{for all } |u| \geq L,$$

where $G(x,u) = \int_0^u g(x,t)dt$. The role of (G_1) is to ensure the boundedness of the Palais-Smale (PS) sequences of the energy functional. This is very crucial

2000 *Mathematics Subject Classification.* 35J25, 35J60.
Key words and phrases. Semilinear elliptic equations; boundary value problems; sublinear; sign-changing potential.

Submitted September 8, 2013. Published February 21, 2014.

in applying the critical point theory. However, there are many functions which are superlinear at infinity, but do not satisfy the condition (G_1), for example the superlinear function

$$G(x,u)=|u|^2\Big(\ln(\frac{1}{3}|u|^4-|u|^2+1)\Big)^3.$$

Jiang and Tang [7] used the Li-Willem local linking theorem [6] to obtain a nontrivial solution under the following weak superquadratic condition and other basic conditions,

- (G2) $G(x,u)/|u|^2\to\infty$, as $|u|\to\infty$ uniformly in x,
- (G3) there are constants $\beta>\frac{2N(p-1)}{N+2}$ $(2<p<2^*)$, $a_1>0$ and $L>0$ such that

$$ug(x,u)-2G(x,u)\geq a_1|u|^\beta,\quad \text{for all } |u|\geq L.$$

This result generalized the one of Li and Willem. Very recently, Zhang and Liu [9] also considered the (G2) type superquadratic condition, but the authors weakened the condition (G3) to the following condition

- (G4) there exists constant $\varrho>\max\{2N/(N+2),N(p-2)/2\}$ and $d>0$ such that

$$\liminf_{|u|\to\infty}\frac{ug(x,u)-2G(x,u)}{|u|^\varrho}\geq d\quad \text{uniformly for } x\in\Omega.$$

They obtained the existence and multiplicity of solutions by variant fountain theorem developed by Zou [21] when $g(x,u)$ is odd. From this, we know that the result in [9] also generalized the one in [7] even [6]. For other superquadratic problem with pinching condition, we refer readers to [10]. For the asymptotically quadratic situation, He and Zou [8] obtained the existence of infinitely many nontrivial solutions under the following assumptions:

- (G5) $G(x,u)=\frac{1}{2}\alpha|u|^2+F(x,u)$, where $\alpha\notin\sigma(-\Delta+a)$, σ denotes the spectrum;
- (G6) there exist $\gamma_i\in(1,2)$, $b_i>0$, $i=1,2$ such that $b_1|u|^{\gamma_1}\leq F(x,u)$, $|F_u(x,u)|\leq b_2|u|^{\gamma_1-1}$ for all $(x,u)\in\Omega\times\mathbb{R}$.

However, for the subquadratic case, there is no work concerning on this case up to now. Motivated by the above fact, in this paper our aim is to study the existence of infinitely many solutions for problem (1.1) when $f(x,u)$ satisfies sublinear in u at infinity. Our tool is the variant fountain theorem established in Zou [21]. Compared to the above two cases, our result is different and extend the above results to some extent.

We will use the following assumptions:

- (F1) $G(x,u)\geq 0$, for all $(x,u)\in\Omega\times\mathbb{R}$, and there exist constants $\mu\in[1,2)$ and $r_1>0$ such that

$$g(x,u)u\leq\mu G(x,u),\quad \forall x\in\Omega,\ |u|\geq r_1;$$

- (F2) $\lim_{|u|\to 0}\frac{G(x,u)}{|u|^2}=\infty$ uniformly for $x\in\Omega$, and there exist constants $c_1,r_2>0$ such that

$$G(x,u)\leq c_1|u|,\quad \forall x\in\Omega,\ |u|\leq r_2;$$

- (F3) There exists a constant $d>0$ such that

$$\liminf_{|u|\to\infty}\frac{G(x,u)}{|u|}\geq d>0\quad \text{uniformly for } x\in\Omega;$$

The main result of this article is the following theorem.

Theorem 1.1. *Suppose that* (F1)–(F3)*, and that* $g(x,u)$ *is odd hold. Then* (1.1) *possesses infinitely many nontrivial solutions.*

As a motivation we recall that there are a large number of articles devoted to the study of the sublinear case. Among these problems are the second-order Hamiltonian system in Tang and Lin [14] and Sun et al.[15], the Schrödinger equation in Zhang and Wang [17], the Schrödinger-Maxwell equation in Sun [16], the fourth-order elliptic equations in Ye and Tang [18] and Zhang et al.[13]. lt is worth pointing out that these papers all considered the definite case that the quadratic of energy functional is positive definite. In the present article, we study the indefinite case, compared to the definite case, the indefinite case becomes more general.

2. Variational setting and proof of the main result

First we establish the variational setting for problem (1.1) to prove our main result. Since $a \in L^{N/2}(\Omega)$, we know that the following form defined on $H_0^1(\Omega)$ is bounded (see [20, Proposition VI.1.2]).

$$\mathcal{Q}(u,v) = \int_\Omega (\langle \nabla u, \nabla v\rangle + a(x)uv)dx, \quad \forall u,v \in H_0^1(\Omega), \tag{2.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathbb{R}^N$. Denote $A_0 = -\Delta + a$ the associated self-adjoint operator in $L^2 \equiv L^2(\Omega)$ with domain $D(A_0)$. From [20, Theorem VI.1.4], we know that $D(A_0)$ is dense as a subspace of $H_0^1(\Omega)$ and the spectrum of A_0 consists of only eigenvalues numbered $\lambda_1 \le \lambda_2 \le \dots \to \infty$ (counted with multiplicity) and the corresponding eigenfunctions $\{e_i\}_{i\in\mathbb{N}}(A_0 e_i = \lambda_i e_i)$, forming an orthogonal basis in L^2. Let $|A_0|$ be the absolute value of A_0 and $|A_0|^{1/2}$ be the square root of $|A_0|$ with domain $D(|A_0|^{1/2})$. Let $E = D(|A_0|^{1/2})$ and define the inner product on E as

$$(u,v)_0 := (|A_0|^{1/2}u, |A_0|^{1/2}v)_2 + (u,v)_2,$$

and the induced norm

$$\|u\|_0 := (u,u)_0^{1/2},$$

where $(\cdot,\cdot)_2$ denotes the usual inner product in L^2. Then E is a Hilbert space. The following Lemma is the Lemma 2.1 in [9], here we omit its proof.

Lemma 2.1. *The norm* $\|\cdot\|_0$ *in* $E = H_0^1(\Omega)$ *is equivalent to the usual Sobolev norm* $\|\cdot\|_{1,2}$ *in* $H_0^1(\Omega)$.

Set

$$n^- = \sharp\{i|\lambda_i < 0\}, \quad n^0 = \sharp\{i|\lambda_i = 0\}, \quad \bar{n} = n^- + n^0, \tag{2.2}$$

and let

$$L^2 = L^- \oplus L^0 \oplus L^+ \tag{2.3}$$

be the orthogonal decomposition in L^2 with

$$L^- = \operatorname{span}\{e_1,\dots,e_{n^-}\}, \quad L^0 = \operatorname{span}\{e_{n^-+1},\dots,e_{\bar{n}}\},$$
$$L^+ = (L^- \oplus L^0)^\perp = \overline{\operatorname{span}\{e_{\bar{n}+1},\dots\}}.$$

Now we introduce the following inner product on $E = H_0^1(\Omega)$,

$$(u,v) = (|A_0|^{1/2}u, |A_0|^{1/2}v)_2 + (u^0,v^0)_2$$

and the corresponding norm

$$\|u\| = (u, u)^{1/2},$$

where $u = u^- + u^0 + u^+$ and $v = v^- + v^0 + v^+$ with respect to the decomposition (2.3). Clearly, the norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent. Throughout the following sections, we take $(E, (\cdot,\cdot), \|\cdot\|)$ as our working space and denote by E^* its dual space with the associated operator norm $\|\cdot\|_{E^*}$. It is easy to check that E possesses the orthogonal decomposition

$$E = E^- \oplus E^0 \oplus E^+ \tag{2.4}$$

with

$$E^- = L^-, \quad E^0 = L^0, \quad E^+ = E \cap L^+ = \overline{\operatorname{span}\{e_{\bar{n}+1}, \dots\}}, \tag{2.5}$$

where the closure is taken with respect to the norm $\|\cdot\|$. Evidently, the above decomposition is also orthogonal in L^2. Similar to [9, Lemma 2.3], we have the following Lemma

Lemma 2.2. *The space E is compactly embedded in $L^p = L^p(\Omega)$ for $1 \le p < 2^*$ and continuously embedded in $L^{2^*} = L^{2^*}(\Omega)$, hence for every $1 \le p < 2^*$, there exists $\tau_p > 0$ such that*

$$|u|_p \le \tau_p \|u\|, \ \forall u \in E, \tag{2.6}$$

where $|\cdot|_p$ denotes the usual norm in L^p for all $1 \le p < 2^*$, ($2^* = \frac{2N}{N-2}$).

Let $A_0 = U|A_0|$ be the polar decomposition of A_0 (see [19]), where U is the partial isometry and commutes with A_0, $|A_0|$ and $|A_0|^{1/2}$. For any $u \in D(A_0)$ and $v \in E$, we have

$$\begin{aligned} \mathcal{Q}(u, v) &= \int_\Omega (\langle \nabla u, \nabla v\rangle + a(x)uv)dx \\ &= (A_0 u, v)_2 = (|A_0|Uu, v)_2 \\ &= (|A_0|^{1/2}Uu, |A_0|^{1/2}v)_2. \end{aligned} \tag{2.7}$$

Since $D(A_0)$ is dense in E, then (2.7) holds for all $u, v \in E$. Moreover, by definition,

$$\mathcal{Q}(u, v) = ((P^+ - P^-)u, u) = \|u^+\|^2 - \|u^-\|^2 \tag{2.8}$$

for all

$$u = u^- + u^0 + u^+ \in E = E^- \oplus E^0 \oplus E^+,$$

where $P^\pm : E \to E^\pm$ are the respective orthogonal projections.

Now, we define a functional Φ on E by

$$\begin{aligned} \Phi(u) &= \frac{1}{2}\int_\Omega (|\nabla u|^2 + a(x)u^2)dx - \Psi(u) \\ &= \frac{1}{2}\mathcal{Q}(u, u) - \Psi(u) \\ &= \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \Psi(u), \end{aligned} \tag{2.9}$$

where $\Psi(u) = \int_\Omega G(x, u)dx$ for all $u = u^- + u^0 + u^+ \in E = E^- \oplus E^0 \oplus E^+$. By (F1) and (F2), there exists a constant $c_2 > 0$ such that

$$G(x, u) \le c_2(1 + |u|^\mu), \quad \forall (x, u) \in \Omega \times \mathbb{R}. \tag{2.10}$$

From (2.10) and Lemma 2.1, we know Φ and Ψ are well defined. Furthermore, by virtue of [9, Proposition 2.4], we have the following Lemma.

Lemma 2.3. *Under assumptions* (F1) *and* (F2), $\Psi \in C^1(E, \mathbb{R})$ *and* $\Psi' : E \to E^*$ *is compact, and hence* $\Phi \in C^1(E, \mathbb{R})$. *Moreover,*

$$\begin{aligned}\langle \Psi'(u), v\rangle &= \int_\Omega g(x,u)v dx,\\ \langle \Phi'(u), v\rangle &= (u^+, v) - (u^-, v) - \langle \Psi'(u), v\rangle \\ &= (u^+, v) - (u^-, v) - \int_\Omega g(x,u)v dx.\end{aligned} \tag{2.11}$$

for all $u, v \in E = E^- \oplus E^0 \oplus E^+$ *with* $u = u^- + u^0 + u^+$ *and* $v = v^- + v^0 + v^+$, *respectively.*

Let E be a Banach space with the norm $\|\cdot\|$ and $E = \overline{\oplus_{j\in\mathbb{N}} X_j}$ with $\dim X_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \oplus_{j=1}^k X_j$ and $Z_k = \overline{\oplus_{j=k}^\infty X_j}$. Consider the C^1-functional $\Phi_\lambda : E \to \mathbb{R}$ defined by

$$\Phi_\lambda(u) := A(u) - \lambda B(u),\ \lambda \in [1, 2].$$

The following variant fountain theorem was established in [21].

Theorem 2.4 ([21, Theorem 2.2])**.** *Assume that the functional* Φ_λ *defined above satisfies*

- (T1) Φ_λ *maps bounded sets to bounded sets uniformly for* $\lambda \in [1, 2]$. *Moreover,* $\Phi_\lambda(-u) = \Phi_\lambda(u)$ *for all* $(\lambda, u) \in [1, 2] \times E$.
- (T2) $B(u) \geq 0$, *for all* $u \in E$*; and* $B(u) \to \infty$ *as* $\|u\| \to \infty$ *on any finite dimensional subspace of* E.
- (T3) *There exists* $\rho_k > r_k > 0$ *such that*
$$a_k(\lambda) := \inf_{u\in Z_k \|u\|=\rho_k} \Phi_\lambda(u) \geq 0 > \beta_k(\lambda) := \max_{u\in Y_k, \|u\|=r_k} \Phi_\lambda(u),$$
for all $\lambda \in [1, 2]$, *and*
$$\xi_k(\lambda) := \inf_{u\in Z_k, \|u\|\leq\rho_k} \Phi_\lambda(u) \to 0 \quad \text{as } k \to \infty \text{ uniformly for } \lambda \in [1, 2].$$

Then there exist $\lambda_n \to 1, u_{\lambda_n} \in Y_n$ *such that*

$$\Phi'_{\lambda_n}|_{Y_n}(u_{\lambda_n}) = 0, \quad \Phi_{\lambda_n}(u_{\lambda_n}) \to \eta_k \in [\xi_k(2), \beta_k(1)] \quad \text{as } n \to \infty.$$

Particularly, if $\{u_{\lambda_n}\}$ *has a convergent subsequence for every* k, *then* Φ_1 *has infinitely many nontrivial critical points* $\{u_k\} \in E \setminus \{0\}$ *satisfying* $\Phi_1(u_k) \to 0^-$ *as* $k \to \infty$.

To apply Theorem 2.4 in the proof of our main result, on the space E, we define the functionals A, B, Φ_λ as follows:

$$A(u) = \frac{1}{2}\|u^+\|^2, \quad B(u) = \frac{1}{2}\|u^-\|^2 + \int_\Omega G(x,u)dx, \tag{2.12}$$

$$\Phi_\lambda(u) = A(u) - \lambda B(u) = \frac{1}{2}\|u^+\|^2 - \lambda(\frac{1}{2}\|u^-\|^2 + \int_\Omega G(x,u)dx) \tag{2.13}$$

for all $u \in E$ and $\lambda \in [1, 2]$. From Lemma 2.3, we know that $\Phi_\lambda \in C^1(E, \mathbb{R})$ for all $\lambda \in [1, 2]$. We choose an orthonormal basis $\{e_j : j \in \mathbb{N}\}$ and let $X_j = \operatorname{span}\{e_j\}$ for all $j \in \mathbb{N}$. Note that $\Phi_1 = \Phi$, where Φ is the functional defined in (2.9). We also need the following lemmas:

Lemma 2.5. *Let* (F1) *and* (F3) *be satisfied. Then* $B(u) \geq 0$ *for all* $u \in E$. *Furthermore,* $B(u) \to \infty$ *as* $\|u\| \to \infty$ *on any finite dimensional subspace of* $\tilde{E} \subset E$.

Proof. From (F1) and (2.12), we know that $B(u) \geq 0$. We claim that for any finite dimensional subspace $\tilde{E} \subset E$, there exists $\varepsilon > 0$ such that

$$\operatorname{meas}(\{x \in \Omega : |u(x)| \geq \varepsilon\|u\|\}) \geq \varepsilon, \quad \forall u \in \tilde{E} \setminus \{0\}, \tag{2.14}$$

where meas$(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^N$. Arguing indirectly, we assume that there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \tilde{E} \setminus \{0\}$ such that

$$\operatorname{meas}(\{x \in \Omega : |u_n(x)| \geq \frac{\|u_n\|}{n}\}) < \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

Let $v_n = \frac{u_n}{\|u_n\|} \in \tilde{E}$. Then $\|v_n\| = 1$ for all $n \in \mathbb{N}$, and

$$\operatorname{meas}(\{x \in \Omega : |v_n(x)| \geq \frac{1}{n}\}) < \frac{1}{n}, \quad \forall n \in \mathbb{N}. \tag{2.15}$$

Passing to a subsequence if necessary, we may assume $v_n \to v_0$ in E, for some $v_0 \in \tilde{E}$. Since $\tilde{E}$ is of finite dimension. Evidently, $\|v_0\| = 1$. In view of Lemma 2.2 and the equivalent of any two norms on $\tilde{E}$, we have

$$\int_\Omega |v_n - v_0|dx \to 0 \quad \text{as } n \to \infty. \tag{2.16}$$

Since $v_0 \neq 0$, there exists a constant $\delta_0 > 0$ such that

$$\operatorname{meas}(\{x \in \Omega : |v_0(x)| \geq \delta_0\}) \geq \delta_0. \tag{2.17}$$

For each $n \in \mathbb{N}$, let

$$\Lambda_n = \{x \in \Omega : |v_n(x)| < \frac{1}{n}\}, \quad \Lambda_n^c = \Omega \setminus \Lambda_n = \{x \in \Omega : |v_n(x)| \geq \frac{1}{n}\}.$$

Set

$$\Lambda_0 = \{x \in \Omega : |v_0(x)| \geq \delta_0\},$$

where δ_0 is the constant in (2.17). Then for n large enough, by (2.15) and (2.17), we have

$$\operatorname{meas}(\Lambda_n \cap \Lambda_0) \geq \operatorname{meas}(\Lambda_0) - \operatorname{meas}(\Lambda_n^c) \geq \delta_0 - \frac{1}{n} \geq \frac{\delta_0}{2}.$$

Consequently, for n large enough, there holds

$$\begin{aligned}
\int_\Omega |v_n - v_0|dx &\geq \int_{\Lambda_n\cap\Lambda_0} |v_n - v_0|dx \\
&\geq \int_{\Lambda_n\cap\Lambda_0} (|v_0| - |v_n|)dx \\
&\geq (\delta_0 - \frac{1}{n}) \cdot \operatorname{meas}(\Lambda_n \cap \Lambda_0) \\
&\geq \frac{\delta_0^2}{4} > 0.
\end{aligned}$$

This is in contradiction to (2.16). Therefore (2.14) holds. For the ε given in (2.14), let

$$\Lambda_u = \{x \in \Omega : |u(x)| \geq \varepsilon\|u\|\}, \ \forall u \in \tilde{E} \setminus \{0\}.$$

Then by (2.14), we have

$$\operatorname{meas}(\Lambda_u) \geq \varepsilon, \quad \forall u \in \tilde{E} \setminus \{0\} \tag{2.18}$$

By (F3), we know there exists $r_3 > 0$ such that

$$G(x, u) \geq \frac{d}{2}|u|, \quad \forall (x, u) \in \Omega \times \mathbb{R} \text{ with } |u| \geq r_3. \tag{2.19}$$

Combing (2.12) and (2.19), we obtain

$$\begin{aligned} B(u) &= \frac{1}{2}\|u^-\|^2 + \int_\Omega G(x,u)dx \\ &\geq \int_\Omega \frac{d}{2}|u|dx \geq \int_{\Lambda_u} \frac{d}{2}|u|dx \\ &\geq \frac{d}{2}\varepsilon\|u\| \cdot \mathrm{meas}(\Lambda_u) \\ &\geq \frac{d}{2}\varepsilon^2\|u\|. \end{aligned}$$

This implies that $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace $\tilde{E} \subset E$. The proof is complete. □

Lemma 2.6. *Suppose that* (F1)–(F3) *hold. Then there exists $k_1 > 0$ and a sequence $\rho_k \to 0^+$ as $k \to \infty$ such that*

$$\alpha_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \Phi_\lambda(u) > 0, \quad \forall k > k_0, \tag{2.20}$$

$$\xi_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} \Phi_\lambda(u) \to 0 \quad \textit{as } k \to \infty \textit{ uniformly for } \lambda \in [1,2], \tag{2.21}$$

$$\beta_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \Phi_\lambda(u), \quad \forall k \in \mathbb{N}, \tag{2.22}$$

where $Y_k = \oplus_{j=1}^k X_j$ and $Z_k = \overline{\oplus_{j=k}^\infty X_j}$ for all $k \in \mathbb{N}$.

Proof. (a) Firstly, we show that (2.20) and (2.21) hold. Choosing appropriate k, so that $Z_k \subset E^+$ for $k > k_1 = \bar{n} + 1$. For any $u \in E$ with $\|u\| \leq \varepsilon$, for all $0 < \varepsilon < r_2$, we claim that there holds

$$|u| \leq \varepsilon < r_2, \tag{2.23}$$

where r_2 is the constant in (F2). If not, then there exists a positive constant ε_0 such that $|u| \geq \varepsilon_0$. Therefore, $\|u\| \geq c|u|_1 \geq c\varepsilon_0 \cdot \mathrm{meas}(\Omega)$ for some $c > 0$, which contradicts with $\|u\| \leq \varepsilon$. Thus, $|u| \leq \varepsilon < r_2$. Then for any $k > k_1 = \bar{n} + 1$ and $u \in Z_k \subset E^+$ with $\|u\| \leq \varepsilon$, for all $0 < \varepsilon < r_2$, by (F2) and the definitions of $\Phi_\lambda(u)$ and G, we have

$$\begin{aligned} \Phi_\lambda(u) &= \frac{1}{2}\|u^+\|^2 - \lambda(\frac{1}{2}\|u^-\|^2 + \int_\Omega G(x,u)dx) \\ &\geq \frac{1}{2}\|u^+\|^2 - 2\int_\Omega G(x,u)dx \\ &\geq \frac{1}{2}\|u^+\|^2 - 2\int_\Omega c_1|u|dx \\ &\geq \frac{1}{2}\|u^+\|^2 - 2c_1|u|_1 \\ &= \frac{1}{2}\|u\|^2 - 2c_1|u|_1. \end{aligned} \tag{2.24}$$

Let

$$l_k = \sup_{u \in Z_k \setminus \{0\}} \frac{|u|_1}{\|u\|^2}, \quad \forall k \in \mathbb{N}. \tag{2.25}$$

Then

$$l_k \to 0 \quad \text{as } k \to \infty, \tag{2.26}$$

by the Rellich embedding theorem (see [22]), consequently, (2.24) and (2.25) imply that

$$\Phi_\lambda(u) \geq \frac{1}{2}\|u\|^2 - 2c_1 l_k \|u\|, \tag{2.27}$$

for any $k > k_1 = \bar{n}+1$ and $u \in E^+$ with $\|u\| \leq \varepsilon$, for all $0 < \varepsilon < r_2$. For any $k \in \mathbb{N}$, let

$$\rho_k = 8c_1 l_k. \tag{2.28}$$

Then by (2.26), we have

$$\rho_k \to 0 \quad \text{as } k \to \infty, \tag{2.29}$$

Thus, for any $k > k_1 = \bar{n}+1$, by direct computation, we obtain

$$\alpha_k(\lambda) := \inf_{u\in Z_k, \|u\|=\rho_k} \Phi_\lambda(u) \geq \frac{\rho_k^2}{4} > 0.$$

By (2.27), for any $k \geq k_1$ and $u \in Z_k$ with $\|u\| \leq \rho_k$, we have

$$\Phi_\lambda(u) \geq -2c_1 k_k \rho_k \tag{2.30}$$

for all $\lambda \in [1,2]$ and $u \in Z_k$ with $\|u\| \leq \rho_k$. Therefore

$$-2c_1 k_k \rho_k \leq \inf_{u\in Z_k, \|u\|\leq\rho_k} \Phi_\lambda(u) \leq 0, \quad \forall k \geq k_1,$$

which together with $(2.26) and (2.29)$ implies

$$\xi_k(\lambda) := \inf_{u\in Z_k, \|u\|\leq\rho_k} \Phi_\lambda(u) \to 0 \quad \text{as } k \to \infty \text{ uniformly for } \lambda \in [1,2].$$

(b) Now, we show that (2.22) holds. For any $k \in \mathbb{N}$, there exists a constant $M_k > 0$ such that

$$|u|_2 \geq M_k \|u\|, \quad \forall u \in Y_k, \tag{2.31}$$

which dues to norms $|\cdot|_2$ and $\|\cdot\|$ are equivalent on finite dimensional subspace Y_k. By (F2) and the definition of G, for any $k \in \mathbb{N}$, there exists a constant $\delta_k > 0$ such that

$$G(x,u) \geq \frac{|u|^2}{M_k^2}, \quad \forall |u| \leq \delta_k. \tag{2.32}$$

For any $k \in \mathbb{N}$ and $u \in E$ with $\|u\| \leq \varepsilon$, for all $0 < \varepsilon \leq \delta_k$, similar to (2.23), we have

$$|u| \leq \varepsilon \leq \delta_k.$$

Thus, by (2.31) and (2.32), for any $k \in \mathbb{N}$ and $u \in Y_k$ with $\|u\| \leq \varepsilon$, for all $0 < \varepsilon \leq \delta_k$, we have

$$\begin{aligned}
\Phi_\lambda(u) &\leq \frac{1}{2}\|u^+\|^2 - \int_\Omega G(x,u)dx \\
&\leq \frac{1}{2}\|u\|^2 - \frac{|u|_2^2}{M_k^2} \\
&\leq \frac{1}{2}\|u\|^2 - \|u\|^2 = -\frac{1}{2}\|u\|^2, \quad \forall \lambda \in [1,2].
\end{aligned} \tag{2.33}$$

Now for any $k \in \mathbb{N}$, if we choose

$$0 < r_k < \min\{\rho_k, \varepsilon\}, \quad \forall 0 < \varepsilon \leq \delta_k,$$

then (2.33) implies

$$\beta_k(\lambda) := \max_{u\in Y_k, \|u\|=r_k} \Phi_\lambda(u) \leq \frac{-r_k^2}{2} < 0, \quad \forall k \in \mathbb{N}.$$

The proof is complete. $\square$

Proof of Theorem 1.1. By (2.6) and (2.13), we easily obtain that Φ_λ maps bounded sets uniformly for $\lambda \in [1,2]$. Obviously, $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times E$ since $G(x,u)$ is even in u. Consequently, condition (T1) of Theorem 2.4 holds. Lemma 2.5 shows that condition (T2) holds, while Lemma 2.6 implies that condition (T3) holds for all $k \geq k_1$, where k_1 is given there. Therefore, by Theorem 2.4, for each $k \geq k_1$, there exist $\lambda_n \to 1, u_{\lambda_n} \in Y_n$ such that

$$\Phi'_{\lambda_n}|_{Y_n}(u_{\lambda_n}) = 0, \quad \Phi_{\lambda_n}(u_{\lambda_n}) \to \eta_k \in [\xi_k(2), \beta_k(1)], \quad \text{as } n \to \infty. \tag{2.34}$$

For the sake of simplicity, in the remaining proof of Theorem 2.4, we let $u_n = u_{\lambda_n}$ for all $n \in \mathbb{N}$. Note that Y_n is a finite dimensional subspace, thus we only need to prove the following claims to complete the proof of Theorem 1.1.

Claim 1. $\{u_n\}$ is bounded in E. By the assumptions on $G(x,u)$, for the constant r_3 given in (2.19), there exists a constant $R_1 > 0$ such that

$$|G(x,u) - \frac{1}{2}g(x,u)u| \leq R_1 \quad \forall x \in \Omega,\ |u| \leq r_3. \tag{2.35}$$

By (2.19), (2.34), (2.35) and (F1), we have

$$\begin{aligned}
-\Phi_{\lambda_n}(u_n) &= \frac{1}{2}\Phi'_{\lambda_n}|_{Y_n}(u_n)u_n - \Phi_{\lambda_n}(u_n) \\
&= \lambda_n \int_\Omega [G(x,u_n) - \frac{1}{2}g(x,u_n)u_n]dx \\
&\geq \lambda_n \int_{\Omega_n} [G(x,u_n) - \frac{1}{2}g(x,u_n)u_n]dx - \lambda_n R_1 \cdot \operatorname{meas}(\Omega) \\
&\geq \frac{\lambda_n(2-\mu)}{2} \int_{\Omega_n} G(x,u_n)dx - \lambda_n R_1 \cdot \operatorname{meas}(\Omega) \\
&\geq \frac{d\lambda_n(2-\mu)}{4} \int_{\Omega_n} |u_n|dx - \lambda_n R_1 \cdot \operatorname{meas}(\Omega), \quad \forall n \in \mathbb{N},
\end{aligned}$$

where $\Omega_n := \{x \in \Omega : |u_n| \geq r_3\}$, d and r_3 are the constants in (2.19). It follows from (2.34) that there exists a constant $R_2 > 0$ such that

$$\int_{\Omega_n} |u_n|dx \leq R_2, \quad \forall n \in \mathbb{N}. \tag{2.36}$$

For any $n \in \mathbb{N}$, let $\chi_n : \Omega \to \mathbb{R}$ be the indicator of Ω_n; that is,

$$\chi_n(x) = \begin{cases} 1, & x \in \Omega_n, \\ 0, & x \notin \Omega_n, \end{cases} \forall n \in \mathbb{N}.$$

Then by the definition of Ω_n and (2.36), we know that

$$|(1-\chi_n)u_n|_\infty \leq r_3, \quad |\chi_n u_n|_1 \leq R_2, \quad \forall n \in \mathbb{N}.$$

Since any two norms on finite-dimensional space $E^0 \oplus E^-$ are equivalent, we obtain

$$\begin{aligned}
\|u_n^- + u_n^0\|^2 &= (u_n^- + u_n^0, u_n) \\
&= (u_n^- + u_n^0, (1-\chi_n)u_n) + (u_n^- + u_n^0, \chi_n u_n) \\
&\leq \|(1-\chi_n)u_n\| \cdot \|u_n^- + u_n^0\| + \|\chi_n u_n\| \cdot \|u_n^- + u_n^0\| \\
&\leq (c_3|(1-\chi_n)u_n|_1 + c_4|\chi_n u_n|_1)\|u_n^- + u_n^0\| \\
&\leq (c_3|(1-\chi_n)u_n|_\infty \cdot \operatorname{meas}(\Omega) + c_4|\chi_n u_n|_1)\|u_n^- + u_n^0\|
\end{aligned}$$

$$\leq (c_3 r_3 \cdot \operatorname{meas}(\Omega) + c_4 R_2)\|u_n^- + u_n^0\|, \quad \forall n \in \mathbb{N},$$

where $c_3, c_4 > 0$. Therefore,

$$\|u_n^- + u_n^0\| \leq c_3 r_3 \cdot \operatorname{meas}(\Omega) + c_4 R_2 := R_3, \quad \forall n \in \mathbb{N}. \tag{2.37}$$

Similar arguments as in the proof of (2.37) imply that

$$\|u_n^-\| \leq R_3, \quad \forall n \in \mathbb{N}. \tag{2.38}$$

Note that

$$\|u_n^+\|^2 = 2\Phi_{\lambda_n}(u_n) + \lambda_n \|u_n^-\|^2 + 2\lambda_n \int_\Omega G(x,u)dx, \quad \forall n \in \mathbb{N}.$$

Combining (2.10), (2.34), (2.37), (2.38) with the Sobolev embedding theorem, we obtain

$$\begin{aligned}
\|u_n\|^2 &= \|u_n^- + u_n^0\|^2 + \|u_n^+\|^2 \\
&= \|u_n^- + u_n^0\|^2 + 2\Phi_{\lambda_n}(u_n) + \lambda_n\|u_n^-\|^2 + 2\lambda_n \int_\Omega G(x,u)dx \\
&\leq R_4 + 4c_2 |u_n|_\mu^\mu \\
&\leq R_4 + 4c_2 c_5 \|u_n\|^\mu, \quad \forall n \in \mathbb{N},
\end{aligned} \tag{2.39}$$

for some $R_4, c_5 > 0$, c_2 is the constant in (2.10). Since $\mu < 2$, (2.39) implies that $\{u_n\}$ is bounded in E.

Claim 2. $\{u_n\}$ has a convergent subsequence in E. Since $\{u_n\}$ is bounded in E, E is reflexible and $\dim(E^0 \oplus E^-) < \infty$, without loss of generality, we assume

$$u_n^- \to u_0^-, \quad u_n^0 \to u_0^0, \quad u_n^+ \rightharpoonup u_0^+, \quad u_n \rightharpoonup u_0 \quad \text{as } n \to \infty \tag{2.40}$$

for some $u_0 = u_0^- + u_0^0 + u_0^+ \in E = E^- \oplus E^0 \oplus E^+$. By the Riesz Representation Theorem, $\Phi'_{\lambda_n}|_{Y_n} : Y_n \to Y_n^*$ and $\Psi' : E \to E^*$ can be viewed as $\Phi'_{\lambda_n}|_{Y_n} : Y_n \to Y_n$ and $\Psi' : E \to E$, respectively, where Y_n^* is the dual space of Y_n. Note that

$$0 = \Phi'_{\lambda_n}|_{Y_n}(u_n) = u_n^+ - \lambda_n P_n \Psi'(u_n), \quad \forall n \in \mathbb{N},$$

where $P_n : E \to Y_n$ is the orthogonal projection for all $n \in \mathbb{N}$; that is,

$$u_n^+ = \lambda_n P_n \Psi'(u_n), \quad \forall n \in \mathbb{N}. \tag{2.41}$$

In view of the compactness of Ψ' and (2.40), the right-hand of (2.41) converges strongly in E and hence $u_n^+ \to u_0^+$ in E. Together with (2.40), we get $u_n \to u_0$ in E.

Now, from the last assertion of Theorem 2.4, we know that $\Phi = \Phi_1$ has infinitely many nontrivial critical points. Therefore, (1.1) possesses infinitely many nontrivial solutions. The proof of Theorem 1.1 is complete. $\square$

Acknowledgments. This work is partially supported by the NNSF (No. 11171351) and the SRFDP (No. 20120162110021) of China and Hunan Provincial Innovation Foundation for Postgraduate.

References

[1] A. Ambrosetti, P. H. Rabinowitz; *Dual variational methods in critical point theory and applications*, J. Funct. Anal. 14 (1973) 349-381.

[2] T. Bartsch, M. Willem; *Infinitely many solutions of a symmetric Drichlet problem*, Nonlinear Anal. 20 (1993) 1205-1216.

[3] D. G. Costa; *Variational elliptic problems which are nonquadratic at infinity*, Nonlinear Anal. 69 (2008) 523-529.

[4] Y. H. Ding; *Infinitely many entire solutions of an elliptic system with symmetry*, Topol. Methods Nonlinear Anal. 9 (1997) 313-323.

[5] Y. H. Ding, S. J. Li; *Existence of entire solutions for some elliptic systems*, Bull. Aust. Math. Soc. 50 (1994) 501-519.

[6] S. Li, M. Willem; *Applications of local linking to critical point theory*, J. Math. Anal. Appl. 189 (1995) 6-32.

[7] Q. Jiang, C. L. Tang; *Existence of a nontrivial solution for a class of superquadratic elliptic problems*, Nonlinear Anal. 69 (2008) 523-529.

[8] X. He, W. Zou; *Multiplicity of solutions for a class of elliptic boundary value problems*, Nonlinear Anal. 71 (2009) 2606-2613.

[9] Q. Y. Zhang, C. G. Liu; *Multiple solutions for a class of semilinear elliptic equations with general potentials*, Nonlinear Anal. 75 (2012) 5473-5481.

[10] X. Zhang; *Existence and multiplicity of solutions for a class of elliptic boundary value problems*, J. Math. Anal. Appl. doi: 10.1016/j.jmaa.2013.08.001.

[11] X. H. Tang; *Infinitely many solutions for semilinear Schröinger equations with sign-changing potential and nonlinearity*, J. Math. Anal. Appl. 401 (2013) 407-415.

[12] J. Zhang, F. K. Zhao; *Multiple solutions for a semiclassical Schrödinger equation*, Nonlinear Anal. 75 (2012) 1834-1842.

[13] W. Zhang, X. H. Tang, J. Zhang; *Infinitely many solutions for fourth-order elliptic equations with general potentials*, J. Math. Anal. Appl. 407 (2013) 359-368.

[14] X. H. Tang, X. Y. Lin; *Infinitely many homoclinic orbits for Hamiltonian systems with indefinite sign subquadratic potentials*, Nonlinear Anal. 74 (2011) 6314-6325.

[15] J. Sun, H. Chen, J. J. Nieto; *Homoclinic solutions for a class of subquadratic second-order Hamiltonian systems*, J. Math. Anal. Appl. 373 (2011) 20-29.

[16] J. T. Sun; *Infinitely many solutions for a class of sublinear Schrödinger-Maxwell equations*, J. Math. Anal. Appl. 390 (2012) 514-522.

[17] Q. Y. Zhang, Q. Wang; *Multiple solutions for a class of sublinear Schrödinger equations*, J. Math. Anal. Appl. 389 (2012) 511-518.

[18] Y. W. Ye, C. L. Tang; *Infinitely many solutions for fourth-order elliptic equations*, J. Math. Anal. Appl. 394 (2012) 841-854.

[19] T. Kato; *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1980.

[20] D. Edmunds, W. Evans; *Spectral Theory and Differential Operators*, Clarendon Press, Oxford, 1987.

[21] W. M. Zou; *Variant fountain theorems and their applications*, Manuscripta Math. 104 (2001) 343-358.

[22] M. Willem; *Minimax Theorems*, Birkhäuser, Berlin, 1996.

Wen Zhang
School of Mathematics and Statistics, Central South University, Changsha, 410083 Hunan, China
E-mail address: zwmath2011@163.com

Xianhua Tang
School of Mathematics and Statistics, Central South University, Changsha, 410083 Hunan, China
E-mail address: tangxh@mail.csu.edu.cn

Jian Zhang
School of Mathematics and Statistics, Central South University, Changsha, 410083 Hunan, China
E-mail address: zhangjian433130@163.com