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FUZZY DIFFERENTIAL EQUATIONS UNDER DISSIPATIVE AND COMPACTNESS TYPE CONDITIONS

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ABSTRACT. Fuzzy differential equation with right-hand side defined as a sum of two almost continuous functions is studied. The first function satisfies dissipative-type condition with respect to Lyapunov-like function. The second maps bounded sets into relatively compact sets. The existence of solution is proved with aid of Schauder's fixed point theorem.

1. Introduction

Starting from [6], the theory of fuzzy differential equations is rapidly developed due to many applications in the real world problems. Notice only the basic work in this direction [5, 8, 11, 12]. As it is shown in [5], the set of fuzzy numbers is not locally compact. It means that the classical Peano theorem is (probably) no longer valid and some extra conditions along with continuity of right-hand side are needed.

In [14] the existence of solutions of fuzzy differential equation with uniformly continuous right-hand side is proved under compactness-type condition. The existence and uniqueness of solution under dissipative-type conditions when the right-hand side is continuous is studied in [4, 10, 13]. In this paper we study fuzzy differential equation whose right-hand side is a sum of two almost continuous functions, one satisfies dissipative-type condition, and another maps bounded sets into relatively compact sets. To the authors knowledge there are not related results in the literature.

We study the fuzzy differential equation

$$\dot{x}(t) = f(t,x) + g(t,x);\ x(0) = x_0,\ t \in I, \tag{1.1}$$

where $f : I \times \mathbb{E} \to \mathbb{E}$ satisfies dissipative-type condition and $g : I \times \mathbb{E} \to \mathbb{E}$ satisfies compactness-type assumption. Here and further in the paper $I = [0,1]$. $\mathbb{E} = \{x : \mathbb{R}^n \to [0,1]; x \text{ satisfies (1)–(4)}\}$ is the space of fuzzy numbers:

(1) x is normal i.e. there exists $y_0 \in \mathbb{R}^n$ such that $x(y_0) = 1$,
(2) x is fuzzy convex i.e. $x(\lambda y + (1-\lambda)z) \geq \min\{x(y), x(z)\}$ whenever $y, z \in \mathbb{R}^n$ and $\lambda \in [0,1]$,
(3) x is upper semicontinuous i.e. for any $y_0 \in \mathbb{R}^n$ and $\varepsilon > 0$ there exists $\delta(y_0, \varepsilon) > 0$ such that $x(y) < x(y_0) + \varepsilon$ whenever $|y - y_0| < \delta,\ y \in \mathbb{R}^n$,

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(4) The closure of the set $\{y \in \mathbb{R}^n;\ x(y) > 0\}$ is compact.

The set $[x]^\alpha = \{y \in \mathbb{R}^n;\ x(y) \geq \alpha\}$ is called α-level set of x.

It follows from (1)–(4) that the α-level sets $[x]^\alpha$ are convex compact subsets of $\mathbb{R}^n$ for all $\alpha \in (0,1]$. The fuzzy zero is defined by

$$\hat{0}(y) = \begin{cases} 0 & \text{if } y \neq 0, \\ 1 & \text{if } y = 0. \end{cases}$$

The metric in $\mathbb{E}$ is defined by $D(x,y) = \sup_{\alpha \in (0,1]} D_H([x]^\alpha, [y]^\alpha)$, where

$$D_H(A,B) = \max\{\max_{a\in A} \min_{b \in B} |a-b|, \max_{b \in B} \min_{a \in A} |a-b|\}$$

is the Hausdorff distance between the convex compact subsets of $\mathbb{R}^n$.

The map $F : I \times \mathbb{E} \to \mathbb{E}$ is said to be continuous at (s,y) when for every $\varepsilon > 0$ there exists $\delta > 0$ such that $D(F(s,y), F(t,x)) < \varepsilon$ for every $t \in I$ and $x \in \mathbb{E}$ with $|t-s| + D(x,y) < \delta$. The map $F : I \times \mathbb{E} \to \mathbb{E}$ is said to be almost continuous if there exists a sequence $\{I_k\}_{k=1}^\infty$ of pairwise disjoint compact sets with $\operatorname{meas}(I_k) > 0$ and $\operatorname{meas}\big(\cup_{k=1}^\infty I_k\big) = \operatorname{meas}(I)$ such that $F : I_k \times \mathbb{E} \to \mathbb{E}$ is continuous for every k.

Since I_k is compact for every k, one has that $\cup_{k=1}^n I_k$ is also compact and hence $(0,1) \setminus \cup_{k=1}^n I_k = \cup_{i=1}^\infty (a_i, b_i)$ is open, because every open set in $\mathbb{R}$ is a union of countable sets of pairwise disjoint open intervals.

Throughout this paper both $f : I \times \mathbb{E} \to \mathbb{E}$ and $g : I \times \mathbb{E} \to \mathbb{E}$ are assumed to be almost continuous.

Remark 1.1. *Due to Lusin's theorem (see e.g.* [9] *for short proof)* $\Lambda : I \to \mathbb{E}$ *is strongly measurable if and only if it satisfies Lusin property, i.e. for all* $\varepsilon > 0$ *there exists* $I_\varepsilon \subset I$ *with* $\operatorname{meas}(I \backslash I_\varepsilon) \leq \varepsilon$ *such that* $\Lambda : I_\varepsilon \to \mathbb{E}$ *is continuous.*

A mapping $\Upsilon : I \to \mathbb{E}$ is said to be differentiable at $t \in I$ if for sufficiently small $h > 0$ the differences $\Upsilon(t+h) - \Upsilon(t)$, $\Upsilon(t) - \Upsilon(t-h)$ (in sense of Hukuhara) exist and there exists $\dot{\Upsilon}(t) \in \mathbb{E}$ such that the limits $\lim_{h \to 0^+} \frac{\Upsilon(t+h) - \Upsilon(t)}{h}$ and $\lim_{h\to 0^+} \frac{\Upsilon(t) - \Upsilon(t-h)}{h}$ exist, and are equal to $\dot{\Upsilon}(t)$. At the end points of I we consider only the one sided derivative.

The integral of fuzzy function $\Upsilon : I \to \mathbb{E}$ is defined levelwise, i.e. there exists $\Lambda : I \to \mathbb{E}$ such that $[\Lambda(t)]^\alpha = \int_0^t [\Upsilon(s)]^\alpha ds$, where the integral is in Auman sense. Every such function $\Lambda(\cdot)$ is absolutely continuous (AC).

The sequence of strongly measurable functions $\{y_n(\cdot)\}_{n=1}^\infty$ is said to be integrally bounded if there exists $\lambda(t) \in L_1(I, \mathbb{R}^+)$ (non negative valued integrable function) such that $D(y_n(t), \hat{0}) \leq \lambda(t)$ for every n and a.a. $t \in I$.

The Caratheodory function $v : I \times \mathbb{R}^+ \to \mathbb{R}^+$ is said to be Kamke function if it is integrally bounded on the bounded sets, $v(t,0) = 0$ and the unique solution of $\dot{r}(t) = v(t, r(t))$ with $r(0) = 0$ is $r(t) \equiv 0$.

2. Fuzzy differential equation under dissipative-type condition

In this section we consider the fuzzy differential equation

$$\dot{x}(t) = f(t,x), \quad x(0) = x_0, \tag{2.1}$$

where $f : I \times \mathbb{E} \to \mathbb{E}$ satisfies dissipative-type condition. We extend the results of [12] to the case of fuzzy differential equations with almost continuous right-hand side. We need the following hypothesis:

(F1) $D(f(t,x),\hat{0}) \le \lambda(t)(1+D(x,\hat{0}))$ for some $\lambda(t) \in L_1(I,\mathbb{R}^+)$.
(F2) There exists a Lyapunov-like function $W : \mathbb{E}\times\mathbb{E} \to \mathbb{R}^+$ for (2.1).

A continuous map $W : \mathbb{E}\times\mathbb{E} \to \mathbb{R}^+$ is said to be Lyapunov-like function for (2.1) if the following conditions hold (cf. [7]):

(1) $W(x,x) = 0$, $W(x,y) > 0$ for $x \ne y$ and $\lim_{m\to\infty} W(x_m,y_m) = 0$ implies $\lim_{m\to\infty} D(x_m,y_m) = 0$,
(2) There exists a constant $L > 0$ such that
$$|W(x_1,y_1) - W(x_2,y_2)| \le L\left(D(x_1,x_2) + D(y_1,y_2)\right),$$
(3) There exists a Kamke function $v : I\times\mathbb{R}^+ \to \mathbb{R}^+$ such that
$$\lim_{h\to 0^+} h^{-1}\left[W\left(x+hf(t,x), y+hf(t,y)\right) - W(x,y)\right] \le v(t,W(x,y))$$
for any $x,y \in \mathbb{E}$.

Lemma 2.1. *Let* (F1) *holds, then for $\varepsilon > 0$ and $\delta > 0$ there exists an AC function $x_\varepsilon(t)$ such that $D(\dot{x}_\varepsilon(t), f(t,x_\varepsilon(t))) \le \varepsilon$ for all $t \in I_\varepsilon \subset I$, where I_ε is a compact set with measure greater than $1-\delta$.*

Proof. Since $f : I\times\mathbb{E} \to \mathbb{E}$ is almost continuous there exists a sequence $\{I_k\}_{k=1}^{\infty}$ of pairwise disjoint compact sets such that meas$\left(\cup_{k=1}^{\infty} I_k\right) = $ meas(I) and $f : I_k\times\mathbb{E} \to \mathbb{E}$ is continuous for every k. For large n we have meas$(I_\delta) > 1-\delta$, where $I_\delta = \left(\cup_{k=1}^{n} I_k\right)$. Let the needed solution $x_\varepsilon(\cdot)$ be defined on $[0,\tau]$ where $\tau \le 1$ ($\tau = 0$ is possible). If $\tau = 1$ then we have done, otherwise two cases would be possible:

(i) $\tau \in (a_l,b_l)$ where $(0,1)\setminus I_\delta = \cup_{l=1}^{\infty}(a_l,b_l)$. In this case we extend $x_\varepsilon(\cdot)$ on $[\tau,b_l)$ by $x_\varepsilon(t) = x_\varepsilon(\tau)$ and denote $\tau_1 = b_l > \tau$,

(ii) $\tau \notin \cup_{i=1}^{\infty}[a_i,b_i)$ then we define
$$x_\varepsilon(t) = x_\varepsilon(\tau) + (t-\tau)f(\tau,x_\varepsilon(\tau)), \quad t \in [\tau,\tau_1]\cap I_\delta.$$
Since $f(\cdot,x_\varepsilon(\cdot))$ is continuous on I_δ, then $D(\dot{x}_\varepsilon(t) = f(t,x_\varepsilon(t)), f(\tau,x_\varepsilon(\tau))) \le \varepsilon$, $\forall t \in [\tau,\tau_1]\cap I_\delta$.

One can continue by induction. Suppose the largest interval on which $x_\varepsilon(\cdot)$ satisfies lemma conditions is $[0,\bar{\tau})$. Since $D(f(t,x_\varepsilon),\hat{0}) \le \lambda(t)(1+D(x_\varepsilon(t),\hat{0}))$, one has that
$$D(\dot{x}_\varepsilon(t),\hat{0}) \le \lambda(t)(1+D(x_\varepsilon(t),\hat{0})) + \varepsilon \quad \text{for } t \in [0,\bar{\tau}).$$
Consequently,
$$D(x_\varepsilon(t),\hat{0}) \le e^{\int_0^{\bar{\tau}}\lambda(s)ds} D(x_0,\hat{0}) + \varepsilon,$$
$$D(\dot{x}_\varepsilon(t),\hat{0}) \le \lambda(t)(1+N_\varepsilon) + \varepsilon,$$
where
$$N_\varepsilon = e^{\int_0^{\bar{\tau}}\lambda(s)ds}\big(D(x_0,\hat{0}) + 2\big).$$
Therefore, $D(\dot{x}_\varepsilon(t),\hat{0}) \in L_1(I,\mathbb{R}^+)$. Furthermore, since $x_\varepsilon(\cdot)$ is AC, then one can conclude that $x_\varepsilon(\cdot)$ is uniformly continuous on $[0,\bar{\tau})$. Thus $\lim_{t\uparrow\bar{\tau}} x_\varepsilon(t) = x(\bar{\tau})$ exists, which is a contradiction to the fact that $[0,\tau]$ is maximum interval of existence. If $\bar{\tau} = 1$ then the proof is complete.

If $\bar{\tau} < 1$ then we can continue this process by defining
$$x_\varepsilon(t) = x_\varepsilon(\bar{\tau}) + (t-\bar{\tau})f(\bar{\tau},x_\varepsilon(\bar{\tau})),\ t \in [\bar{\tau},\tilde{\tau}]$$

for $\bar{\tau} \notin \cup_{l=1}^{\infty}[a_l, b_l)$ or $\tilde{\tau} = b_l$ if $\bar{\tau} \in [a_l, b_l)$ for some l, therefore there exists a $\tilde{\tau}_1 > \tilde{\tau}$ such that $x_\varepsilon(\cdot)$ satisfies the conclusion of the lemma on $[0, \tilde{\tau}_1]$. Continuing in the same way the so defined $x_\varepsilon(\cdot)$ will satisfy the conclusion of the lemma on $[0, 1]$ □

Theorem 2.2. *Let* (F1) *and* (F2) *hold, then* (2.1) *admits unique solution.*

Proof. Denote $\chi_n(t) = \lambda(t)(1 + N_\varepsilon) + \frac{\varepsilon}{2^n}$, where N_ε is from Lemma 2.1. Let $I_{\delta_n} = \cup_{n=1}^{k_{\delta_n}} I_n$ be such that $\text{meas}(I_{\delta_n}) > 1 - \frac{\delta}{2^n}$ and $f : I_n \times \mathbb{E} \to \mathbb{E}$ is continuous. Consider the sequence of approximate solutions $\{x_n(\cdot)\}_{n=0}^{\infty}$ where $x_n(\cdot)$ is the AC function defined in Lemma 2.1 when ε is replaced by $\frac{\varepsilon}{2^n}$. Therefore $D(\dot{x}_n(t), f(t, x_n(t))) \le \eta_n(t)$, where

$$\eta_n(t) = \begin{cases} \varepsilon/2^n & \text{if } t \in I_{\delta_n}, \\ \chi_n(t) & \text{if } t \notin I_{\delta_n}. \end{cases}$$

We have to prove that $\{x_n(\cdot)\}_{n=0}^{\infty}$ is a Cauchy sequence. To this end we take $\{x_n(\cdot)\}$, $\{x_m(\cdot)\}$, where $n < m$. Without loss of generality we can assume that $\dot{x}_n(\cdot)$, $\dot{x}_m(\cdot)$ and $f(\cdot, x(\cdot))$ are continuous on J_n, where $J_n \subset I_{\delta_n}$ with $\text{meas}(J_n) > 1 - \frac{\delta}{2^n}$. If $t \in J_n$, then

$$\begin{aligned}
&D^+W(x_n(t), x_m(t)) \\
&= \lim_{h \to 0^+} \frac{W(x_n(t+h), x_m(t+h)) - W(x_n(t), x_m(t))}{h} \\
&\le \lim_{h \to 0^+} \frac{W(x_n(t) + h\dot{x}_n(t), x_m(t) + h\dot{x}_m(t)) - W(x_n(t), x_m(t)) + o(h)}{h} \\
&\le \lim_{h \to 0^+} \frac{W(x_n(t) + h\dot{x}_n(t), x_m(t) + h\dot{x}_m(t)) - W(x_n(t), x_m(t))}{h} \\
&\le \lim_{h \to 0^+} \frac{W(x_n(t) + hf(t, x_n(t)), x_m(t) + hf(t, x_m(t))) - W(x_n(t), x_m(t))}{h} \\
&\quad + \lim_{h \to 0^+} \frac{Lh\,[D(\dot{x}_n(t), f(t, x_n(t))) + D(\dot{x}_m(t), f(t, x_m(t)))]}{h} \\
&\le v(t, D(x_n(t), x_m(t))) + \frac{2L\varepsilon}{2^n}.
\end{aligned}$$

For almost all $t \notin J_n$, we have

$$\begin{aligned}
&D^+W(x_n(t), x_m(t)) \\
&= \lim_{h \to 0^+} \frac{W(x_n(t+h), x_m(t+h)) - W(x_n(t), x_m(t))}{h} \\
&\le \lim_{h \to 0^+} \frac{W(x_n(t) + h\dot{x}_n(t), x_m(t) + h\dot{x}_m(t)) - W(x_n(t), x_m(t)) + o(h)}{h} \\
&\le \lim_{h \to 0^+} \frac{W(x_n(t) + h\dot{x}_n(t), x_m(t) + h\dot{x}_m(t)) - W(x_n(t), x_m(t))}{h} \\
&\le \lim_{h \to 0^+} \frac{W(x_n(t) + hf(t, x_n(t)), x_m(t) + hf(t, x_m(t))) - W(x_n(t), x_m(t))}{h} \\
&\quad + \lim_{h \to 0^+} \frac{Lh\,[D(\dot{x}_n(t), f(t, x_n(t))) + D(\dot{x}_m(t), f(t, x_m(t)))]}{h} \\
&\le v(t, D(x_n(t), x_m(t))) + 2L\chi_n(t).
\end{aligned}$$

Consequently, $D^+W(x_n(t), x_m(t)) \le v(t, D(x_n(t), x_m(t))) + 2L\eta_n(t)$, because $n < m$.

Thus $W(x_n(t), x_m(t)) \leq r_n(t)$, where $r_n(t)$ is the maximal solution of $\dot{r}(t) = v(t, r(t)) + 2L\eta_n(t)$.

Clearly $\eta_n(\cdot)$ is integrally bounded (as a sequence of real valued functions), and $\lim_{n\to\infty} \eta_n(t) = 0$ for almost all $t \in I$. Since $v(\cdot,\cdot)$ is Kamke function, then $\lim_{n\to\infty} r_n(t) = 0$ uniformly on I. Therefore there exists a sequence of continuous real valued functions $S_n(t)$ with $\lim_{n\to\infty} D(x_n(t), x_m(t)) \leq S_n(t)$ for all $m \geq n$ and $\lim_{n\to\infty} S_n(t) = 0$ uniformly on I. Thus the sequence $\{x_n(\cdot)\}_{n=1}^{\infty}$ is a Cauchy sequence and hence $\lim_{n\to\infty} x_n(t) = x(t)$ uniformly on I. Consequently $f(t, x_n(t)) \to f(t, x(t))$ for a.a. $t \in I$. Furthermore, $D(f(t, x_n(t)), \hat{0}) \leq \chi_n(t) \leq \chi_1(t)$. Due to dominated convergence theorem we get

$$x(t) = x_0 + \int_0^t f(s, x(s))ds. \tag{2.2}$$

The proof is complete thanks to Lemma 2.3 given below. □

Lemma 2.3. *If $f : I \times \mathbb{E} \to \mathbb{E}$ is almost continuous and integrally bounded then every solution of* (2.1) *is a solution of* (2.2) *and vice versa.*

Proof. The space $\mathbb{E}$ can be embedded as a closed convex cone in a Banach space $\mathbb{X}$. The embedding map $j : \mathbb{E} \to \mathbb{X}$ is an isometry and isomorphism. From (cf[3]) we know that $j(\dot{x}(t)) = \frac{d}{dt} j(x(t))$. The fact that every solution of (2.2) is at the same time a solution of (2.1) is tautology because $\int_0^t \dot{x}(s)ds = \int_0^t f(s, x(s))ds$. Let $x : I \to \mathbb{E}$ be a solution of (2.2). Since $x : I \to \mathbb{E}$ is continuous, therefore $f : I \times \mathbb{E} \to \mathbb{E}$ satisfies Lusin property and hence

$$g(t) = \frac{d}{dt}\Big(\int_0^t g(s)ds\Big)$$

for a.a. $t \in I$. i.e. $\dot{x}(t) = g(t) = f(t, x(t))$.

Evidently, $x : I \to \mathbb{E}$ is AC, i.e. $x(\cdot)$ is a solution of (2.1). □

Remark 2.4. Let us consider the equation

$$\dot{x}_n = f(t, x_n(t)) + \varphi_n(t), \; x_n(0) = x_0. \tag{2.3}$$

If $\{\varphi_n(\cdot)\}_{n=1}^{\infty}$ is integrally bounded and $\lim_{n\to\infty} \varphi_n(t) = 0$, then $\lim_{n\to\infty} x_n(t) = x(t)$, where $\dot{x}(t) = f(t, x(t))$, $x(0) = x_0$. Therefore the solution of (2.3) depends continuously on the right-hand side.

3. Compact perturbations of dissipative fuzzy system

In this section we prove the existence of solution of the differential equation (1.1). We will use the additional hypotheses:

- (F3) $W(x + z, y + z) = W(x, y)$ for any fuzzy number z.
- (G1) $g(t, \cdot)$ maps the bounded subsets of $\mathbb{E}$ into relatively compact subsets of $\mathbb{E}$ for a.a. $t \in I$.
- (G2) $D(g(t, x), \hat{0}) \leq \nu(t)(1 + D(x, \hat{0}))$, where $\nu(\cdot) \in L_1(I, \mathbb{R}^+)$.

Condition (F3) is essential here. Notice that it holds automatically if $W(x, y) = \zeta(D(x, y))$, where ζ is some continuous function such that $W(x, y)$ is Lyapunov-like function for (2.1).

If $x(\cdot)$ is a solution of (1.1) then $D(\dot{x}(t),\hat{0}) \le \big(\lambda(t)+\nu(t)\big)(1+D(x(t),\hat{0}))$. Therefore,

$$D(x(t),\hat{0}) \le D(x_0,\hat{0}) + e^{\int_0^t(\lambda(s)+\nu(s))ds}\Big(D(x_0,\hat{0}) + \int_0^t [\lambda(s)+\nu(s)]ds\Big).$$

We can assume without loss of generality that $D(x(t),\hat{0}) \le N$ and $D(\dot{x}(t),\hat{0}) \le \gamma(t)$, where $\gamma(t) = (\lambda(t)+\nu(t))(1+N)$ is Lebesgue integrable. Let $A = \{y \in \mathbb{E} : D(y,x_0) \le N\}$. It follows from (G1) that $g(t,A) \subset K(t)$, where $K(t) \subset \mathbb{E}$ is a convex compact set for a.a. $t \in I$.

Theorem 3.1. *Let* (F1), (F2), (F3), (G1), (G2) *hold, then the differential equation* (1.1) *admits a solution.*

We need the following lemma for proving Theorem 3.1.

Lemma 3.2. *Let $\{\varphi_n(\cdot)\}_{n=1}^\infty$ be an integrally bounded (by an integrable function $c(\cdot)$) sequence of strongly measurable functions from I to $\mathbb{E}$ such that*

$$\overline{\text{co}}\big\{ \cup_{i=1}^\infty \{\varphi_i(t)\}\big\} = K(t)$$

is compact for a.a. $t \in I$ and

$$\dot{x}_n(t) = f(t,x_n(t)) + \varphi_n(t),\ x_n(0) = x_0. \tag{3.1}$$

Passing to subsequence, if necessarily, $x_n(\cdot)$ converges uniformly to $x(\cdot)$, such that

$$\dot{x}(t) \in f(t,x(t)) + K(t).$$

Proof. Clearly $D(\varphi_n(t),\hat{0}) \le c(t)$ implies that $z_n(t) = \int_0^t \varphi_n(s)ds$ is equicontinuous sequence. Furthermore,

$$\int_0^t \big[\cup_{n=1}^\infty \varphi_n(s)\big]ds \subset \int_0^t K(s)ds = R(t),$$

where $\overline{\cup_{t\in[0,1]}\{R(t)\}}$ is a compact subset of $\mathbb{E}$. Then the sequence $z_n(t) = \int_0^t \varphi_n(s)ds$ is $C(I,\mathbb{E})$ precompact. By Arzela Ascoli theorem, passing to subsequence we have $z_n(t) \to z(t)$ uniformly on I.

As we pointed out, $\mathbb{E}$ can be embedded as a closed convex cone in a Banach space $\mathbb{X}$ with a continuous embedding map $j : \mathbb{E} \to \mathbb{X}$. Thus $j(K) \subset \mathbb{X}$ is compact. Then due to Diestel criterion (see proposition 9.4 of [2]) the set $\{j(\varphi_n(\cdot))\}_{n=1}^\infty$ is weakly precompact in $L_1(I,\mathbb{X})$. Thus passing to subsequence in $L_1(I,\mathbb{X})$ we have $j(\varphi_n(t)) \rightharpoonup s(t)$. Since $s(t) \in j(K)$, then there exists $\varphi(t)$ such that $j(\varphi(t)) = s(t)$ and $z(t) = \int_0^t \varphi(s)ds$.

We denote for convenience $y(t) = j(x(t))$, $y_n(t) = j(x_n(t))$, $p(t) = j(z(t))$, $\psi(t) = j(\varphi(t))$, $y(t) - p(t) = u(t)$, $y_n(t) - p_n(t) = u_n(t)$ and $q(t,y) = j(f(t,x))$.

Consider the functions $y_n(t) - p_n(t) = u_n(t)$. We have

$$\begin{aligned}
&W(u(t)+h\dot{u}(t), u_n(t)+h\dot{u}_n(t))\\
&= W(u(t)+hq(t,y(t)), u_n(t)+hq(t,y_n(t))) + o(h)\\
&= W(u(t)+p_n(t)+hq(t,y(t)), y_n(t)+hq(t,y_n(t))) + o(h)\\
&= W(u(t)+p_n(t)+hq(t,y(t)-p(t)+p_n(t)), y_n(t)+hq(t,y_n(t)))\\
&\quad + h|q(t,y(t)) - q(t,y(t)-p(t)+p_n(t))| + o(h).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\lim_{h\to 0^+} \frac{W(u(t)+h(\dot u(t), u_n(t)+h\dot u_n(t))-W(u(t),u_n(t))}{h}\\
&=\lim_{h\to 0^+} \frac{W(u(t+h),u_n(t+h))-W(u(t),u_n(t))}{h}\\
&=\lim_{h\to 0^+} \frac{W(u(t)+h\dot u(t), u_n(t)+h\dot u_n(t))-W(u(t),u_n(t))}{h}\\
&\le v(t,|u(t)-u_n(t)|)+|q(t,y(t)-p(t)+p_n(t))-q(t,y(t))|.
\end{aligned}$$

Thus

$$\begin{aligned}
D^+W(y(t)-p(t),y_n(t)-p_n(t)) &\le v(t,|y(t)-p(t)-(y_n(t)-p_n(t))|)\\
&\quad +|q(t,y(t))-q(t,y(t)-p(t)+p_n(t))|.
\end{aligned}$$

The latter implies that

$$W(y(t)-p(t),y_n(t)-p_n(t))\le r_n(t),$$

where

$$\dot r_n(t)=v(t,r_n(t))+|q(t,y(t)-p(t)+p_n(t))-q(t,y(t))|,\ r_n(0)=0.$$

Since $v(\cdot,\cdot)$ is Kamke function and since

$$\lim_{n\to\infty}|q(t,y(t)-p(t)+p_n(t))-q(t,y(t))|=0 \text{ for a.a } t\in I,$$

one has that $\lim_{n\to\infty} r_n(t)=0$, which implies that $\lim_{n\to\infty} W(y(t)-p(t),y_n(t)-p_n(t))=0$. Thus $y_n(t)\to y(t)$ uniformly on I, where $\dot y(t)=q(t,y(t))+\psi(t)$, i.e $\dot x(t)=f(t,x(t))+\varphi(t)$. □

Proof of Theorem 3.1. Consider the set

$$Q=\{z(\cdot)\in C(I,K):\ D(\dot z(t),\hat 0)\le\gamma(t),\ z(0)=x_0\}.$$

It is easy to see that $Q\subset C(I,\mathbb{E})$ is closed, bounded and convex. Consider the map $\xi: z(\cdot)\to x_z(\cdot)$, where $x_z(\cdot)$ is the unique solution of

$$\dot x_z(t)=f(t,x_z(t))+g(t,z(t));\ x_z(0)=x_0,\ t\in I.$$

Due to Remark 2.4 the map $\xi: Q\to Q$ is continuous. Furthermore, $\overline{\xi(Q)}\subset Q$ is compact by Lemma 3.2. It follows from Schauder's theorem that there exist a fixed point $z(\cdot)\in Q$ such that $\xi(z)=z$. This function $z(\cdot)$ is a solution of (1.1). □

Notice that the linear growth conditions **(F1), (G2)** can be relaxed in order to prove only local existence, i.e. we can assume that $f: I\times\mathbb{E}\to\mathbb{E}$ is integrally bounded on the bounded sets. In that case, Theorem 2.2 is formulated as follows.

Theorem 3.3. *Let $f: I\times\mathbb{E}\to\mathbb{E}$ be integrally bounded on the bounded sets. Then under (F2) there exists $a>0$ such that the system* (2.1) *admits unique solution on* $[0,a]$.

Proof. Let $M>0$. There exists an integrable function $\zeta: I\to\mathbb{R}^+$ with

$$\sup_{|x-x_0|\le M}|f(t,x)|\le\zeta(t).$$

Let $a>0$ be such that $\int_0^a(\zeta(t)+\varepsilon)dt\le M$. On the interval $[0,a]$ every δ solution $x_\delta(t)$ satisfies $|x_\delta(t)|\le M$ and $|\dot x_\delta(t)|\le\zeta(t)+\varepsilon$. Therefore, one can continue as in the proof of Theorem 2.2. □

Theorem 3.1 can be obviously formulated as:

Theorem 3.4. *Let* $f : I \times \mathbb{E} \to \mathbb{E}$ *and* $g : I \times \mathbb{E} \to \mathbb{E}$ *be integrally bounded on the bounded set. Then under* (F2), (F3), (G1) *there exists* $a > 0$ *such that the system* (1.1) *admits a solution on* $[0, a]$.

Proof. As in the proof of Theorem 3.3 we can see that there exists $a > 0$ and $\varepsilon > 0$ such that every ε-solution of (1.1) is extendable on $[0, a]$ and $|x_\varepsilon(t) - x_0| \le M$. Let $g(t, x_0 + M\mathbb{B}) \subset A(t)$, where $A(t) \subset \mathbb{E}$ is a convex compact set. It follows from Theorem 3.3 that for every strongly measurable $\varphi(t) \in A(t)$, the fuzzy differential equation

$$\dot{x}(t) = f(t, x(t)) + \varphi(t), \quad x(0) = x_0$$

admits unique solution on $[0, a]$. One can then continue as in the proof of Theorem 3.1, proving of course the corresponding variant of Lemma 2.1. □

4. Conclusion

As it is pointed out in the introduction the space $\mathbb{E}$ is not locally compact. This implies that it would be very difficult (if it is possible at all) to prove analogue of the classical Peano theorem, when the right-hand side of (2.1) is only jointly continuous. On the other hand up to author's knowledge there is no example of such a system without solutions.

In authors opinion it is very interesting open question to give an example of fuzzy differential equation without local solution, when the right-hand side is jointly continuous.

In optimal control problems the controls are measurable functions and it is one of the main motivation to study differential equations with almost continuous right-hand sides.

In this paper we proved existence (and uniqueness) of the solution of (2.1) under as weak as it is possible dissipative-type condition w.r.t. Lyapunov-like function. We also show the existence of solution when the right-hand side is the sum of a function satisfying such condition along with almost continuous function mapping bounded sets into relatively compact ones. For example such function is $g(t, \cdot)$ which takes values in a locally compact set $\mathbb{E}_K \subset \mathbb{E}$. It seems that it is impossible to relax compactness-type assumptions on g without using stronger dissipative-type conditions on f. We refer the reader to the paper [1], where it is shown by example that if $v(\cdot, \cdot)$ is a Kamke function, then it is possible that the function $w(t, r) = v(t, r) + L(t)r$ is not a Kamke function.

Of course in our proof we essentially used (F3), which is in general not valid for arbitrary Lyapunov-like function. It is an open question does the solution exists, when the last condition is dispensed with?

Now we give a simple example of fuzzy system which satisfies our conditions.

Example 4.1. Consider the system of crisp first equation and fuzzy second:

$$\dot{x} = -\sqrt[3]{x} + f(t, x, y), \quad x(0) = 0$$
$$\dot{y}(t) = g(t, x, y), \quad y(0) = y_0.$$

Here x is crisp variable, $f : I \times \mathbb{R} \times \mathbb{E} \to \mathbb{R}$ is continuous and Lipschitzian on x and on y. Furthermore $g : I \times \mathbb{R} \times \mathbb{E} \to \mathbb{E}$ is continuous, Lipschitzian on x and takes values in a locally compact subset of $\mathbb{E}$. If some growth condition holds, than the system satisfies all the conditions of Theorem 3.1.

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