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ASYMPTOTIC BEHAVIOR OF SINGULAR SOLUTIONS TO SEMILINEAR FRACTIONAL ELLIPTIC EQUATIONS

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Abstract. In this article we study the asymptotic behavior of positive singular solutions to the equation

$$(-\Delta)^\alpha u + u^p = 0 \quad \text{in } \Omega \setminus \{0\},$$

subject to the conditions $u = 0$ in Ω^c and $\lim_{x\to 0} u(x) = \infty$, where $p \geq 1$, Ω is an open bounded regular domain in $\mathbb{R}^N$ $(N \geq 2)$ containing the origin, and $(-\Delta)^\alpha$ with $\alpha \in (0,1)$ denotes the fractional Laplacian. We show that the asymptotic behavior of positive singular solutions is controlled by a radially symmetric solution with Ω being a ball.

1. Introduction

In this article, we study the positive singular solutions to semilinear elliptic equations involving the fractional Laplacian

$$\begin{aligned} (-\Delta)^\alpha u + u^p = 0 \quad &\text{in } \Omega \setminus \{0\}, \\ u = 0 \quad &\text{in } \Omega^c, \\ \lim_{x\to 0} u(x) = \infty&, \end{aligned} \tag{1.1}$$

where Ω is an open bounded C^2 domain in $\mathbb{R}^N$ $(N \geq 2)$ with $0 \in \Omega$ and $p \geq 1$. The operator $(-\Delta)^\alpha$ with $\alpha \in (0,1)$ is the fractional Laplacian defined as

$$(-\Delta)^\alpha u(x) = P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2\alpha}} dy. \tag{1.2}$$

Here $P.V.$ denotes the principal value of the integral, and for notational simplicity we omit it in what follows.

In recent years, nonlinear elliptic equations involving general integro-differential operators, especially, fractional Laplacian, have been studied by many authors. Various regularity issues for fractional elliptic equations have been studied, see for instance [3, 5, 23, 26]. The existence of solutions to semilinear equations involving fractional Laplacian has been investigated by [13, 25] and others by using variational methods.

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When $\alpha = 1$ and $p \in (1, \frac{N}{N-2})$, it was made in [28] the description of the all possible singular behavior of positive solutions to equation (1.1) with $u = 0$ on $\partial\Omega$. In particular, there are only two types of singular behavior occur:
(i) either $u(x) \sim c_N k|x|^{2-N}$ when $x \to 0$ and k can take any positive value (u is said to have a *weak singularity* at 0);
(ii) or $u(x) \sim c_{N,p}|x|^{-\frac{2}{p-1}}$ when $x \to 0$ (u is said to have a *strong singularity* at 0).

For $\alpha \in (0, 1)$, it was shown in [10] the existence of singular solutions for (1.1) by using Perron's method. It was studied in [11] the existence of weak singular solutions and asymptotic behavior of solutions, and the that regularity of weak singular solutions was improved into classical solutions in [12]. In particular, for $p \in (1 + \frac{2\alpha}{N}, \frac{N}{N-2\alpha})$, there exist a solution u_p of problem (1.1) and some positive constant c_1 such that

$$\lim_{x\to 0} u_p(x)|x|^{\frac{2\alpha}{p-1}} = c_1.$$

For $p \in (0, \frac{N}{N-2\alpha})$, there exists a solution u_t of problem (1.1) for each $t > 0$ such that

$$\lim_{x\to 0} u_t(x)|x|^{N-2\alpha} = t.$$

Moreover, if $p \in (0, 1 + \frac{2\alpha}{N})$, the solutions $\{u_t\}$ blow up everywhere in Ω as $t \to \infty$. If $p \in (1 + \frac{2\alpha}{N}, \frac{N}{N-2\alpha})$, the limit of $\{u_t\}$ as $t \to \infty$ is a strongly singular solution of (1.1), which coincides with u_p for $p \in (\max\{\frac{2\alpha}{N-2\alpha}, 1 + \frac{2\alpha}{N}\}, \frac{N}{N-2\alpha})$. However, it does not make all the classification of the singularities of (1.1) in [10, 12]. Our purpose in this paper is to describe the asymptotic behavior of positive singular solutions of (1.1), more precisely, any positive solution of (1.1) with the general domain Ω is controlled by a radially symmetric solution of (1.1) with Ω being a ball. Moreover, we show that any positive solution of (1.1) is radially symmetric when Ω is a ball.

Before stating the main theorem we make precise the notion of solution that we use in this article. We say that a continuous function $u : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ is a classical solution of equation (1.1) if the fractional Laplacian of u is defined at any point of $\Omega \setminus \{0\}$, according to the definition given in (1.2), and if u satisfies the equation and the external condition in a pointwise sense. In this article, we deal with the singular solution only in the classical sense, since the viscosity solution of (1.1) (see Definition 2.4) could be improved into classical sense by regularity results [12, Lemma 3.1], [7, Theorem 2.1] and even the solution in the weak sense (see [12, Definition 1.1]) has been improved into classical sense by [12, Theorem 3.1].

Now we are ready to state our main result. It will be convenient for our description to define

$$r_0 = \max\{r > 0 : B_r(0) \subset \Omega\}. \tag{1.3}$$

Theorem 1.1. *Assume that $\alpha \in (0, 1)$, $p \geq 1$ and u is a positive classical solution of (1.1). Then there exists a radially symmetric solution u_s of (1.1) with $\Omega = B_{r_0}(0)$ such that*

$$u_s \leq u \leq u_s + c_2 \quad \text{in} \quad \Omega \setminus \{0\},$$

where $c_2 = \sup_{x\in\mathbb{R}^N\setminus B_{r_0}(0)} u(x)$.

We remark if the domain Ω is a ball centered at0, then any positive solution u of (1.1) is radially symmetric. In fact, when $\Omega = B_R(0)$ with $R > 0$, by the definition of r_0, it is obvious that $r_0 = R$. Using Theorem 1.1 and the fact of $c_2 = 0$, there exists a radially symmetric solution u_s of (1.1) such that $u = u_s$ in $\mathcal{B}_R := B_R(0) \setminus \{0\}$; therefore, u is radially symmetric. In precise statement, we have following corollary.

Corollary 1.2. *Assume that $\alpha \in (0,1)$, $p \geq 1$ and u is a positive classical solution of (1.1) with $\Omega = B_{r_0}(0)$. Then u is radially symmetric.*

To prove Theorem 1.1, we first show that problem (1.1) with $\Omega = B_{r_0}(0)$ admits a positive singular solution u_s by Perron's method. This will be done by constructing super and sub-solutions by using the solution u. Next, we prove that u_s is radially symmetric by the classical moving planes method which is developed in [2, 9, 14, 21] to obtain the symmetry results for the fractional semilinear problem. In this paper, we extend this method to obtain the symmetry property of isolated singular solutions to

$$(-\Delta)^\alpha u = h(u) + \eta \ \text{ in } \mathcal{B}_1, \quad u = 0 \text{ in } B_1^c(0), \tag{1.4}$$

where $\eta : \mathcal{B}_1 \to \mathbb{R}$ is radially symmetric and decreasing function. We note that the singularity at the origin gives rise to difficulties in the procedure of moving planes.

The paper is organized as follows. In section 2, we obtain the existence of solutions by Perron's Method. Section 3 is devoted to obtain symmetry property for general nonlinearity.

2. Existence of solutions

In this section, we show that problem (1.1) with $\Omega = B_{r_0}(0)$ admits a positive singular solution u_s under the hypotheses of Theorem 1.1. This result will be shown by the barrier method. To this purpose, we need some auxiliary lemmas.

Lemma 2.1. [7, Theorem 2.5] *Let $p > 0$ and $\mathcal{O}$ be an open bounded C^2 domain in $\mathbb{R}^N$. Suppose that $g : \mathcal{O}^c \to \mathbb{R}$ is in $L^1(\mathcal{O}^c, \frac{dx}{1+|x|^{N+2\alpha}})$ and it is of class C^2 in $\{z \in \mathcal{O}^c : \operatorname{dist}(z, \partial\mathcal{O}) \leq \delta\}$ for some $\delta > 0$ and $f : \bar{\mathcal{O}} \to \mathbb{R}$ is continuous, $f \in C^\beta_{\rm loc}(\mathcal{O})$ with $\beta \in (0,1)$. Then there exists a classical solution u of*

$$\begin{gathered} (-\Delta)^\alpha u(x) + |u|^{p-1}u(x) = f(x), \quad x \in \mathcal{O}, \\ u(x) = g(x), \quad x \in \mathcal{O}^c, \end{gathered} \tag{2.1}$$

which is continuous in $\bar{\mathcal{O}}$.

Next, we introduce the comparison principle.

Lemma 2.2. [7, Theorem 2.3] *Let u and v be classical super-solution and sub-solution of*

$$(-\Delta)^\alpha u + h(u) = \eta \quad \text{in } \mathcal{O},$$

respectively, where $\mathcal{O}$ is an open, bounded domain, the function $\eta : \mathcal{O} \to \mathbb{R}$ is continuous and $h : \mathbb{R} \to \mathbb{R}$ is increasing. Suppose further that u and v are continuous in $\bar{\mathcal{O}}$ and $v(x) \leq u(x)$ for all $x \in \mathcal{O}^c$. Then $u(x) \geq v(x)$, $x \in \mathcal{O}$.

Once we have a sub-solution and a super-solution of (1.1), we may find a solution of (1.1) by the Perron's method. More precisely, we have the following result.

Lemma 2.3. *Let $p > 0$ and $0 \in \Omega$ be an open bounded C^2 domain. Suppose that there are super-solution $\overline{U}$ and sub-solution $\underline{U}$ of* (1.1) *such that $\overline{U}$ and $\underline{U}$ are locally C^2 in $\Omega \setminus \{0\}$, and*

$$\overline{U} \geq \underline{U} \text{ in } \Omega \setminus \{0\}, \quad \lim_{x\to 0} \underline{U}(x) = +\infty, \quad \overline{U} \geq 0 \geq \underline{U} \quad \text{in } \Omega^c.$$

Then there exists at least one positive solution u of (1.1) *such that*

$$\underline{U} \leq u \leq \overline{U} \quad \text{in } \Omega \setminus \{0\}.$$

Before proving the above lemma, we introduce the definition of viscosity solution to (1.1).

Definition 2.4. We say that a function $u \in L^1(\mathbb{R}^N, \frac{dy}{1+|y|^{N+2\alpha}})$, continuous in $\mathbb{R}^N \setminus \{0\}$, is a viscosity super solution (sub solution) of (1.1) if

$$u \geq 0 \quad (\text{resp. } u \leq 0) \quad \text{in } \Omega^c$$

and for every point $x_0 \in \Omega \setminus \{0\}$ and some neighborhood V of x_0 with $\bar{V} \subset \Omega \setminus \{0\}$ and for any $\phi \in C^2(\bar{V})$ such that $u(x_0) = \phi(x_0)$ and

$$u(x) \geq \phi(x) \quad (\text{resp. } u(x) \leq \phi(x)) \quad \text{for all } x \in V,$$

defining

$$\tilde{u} = \begin{cases} \phi & \text{in } V, \\ u & \text{in } V^c, \end{cases}$$

we have

$$(-\Delta)^\alpha \tilde{u}(x_0) + u^p(x_0) \geq 0 \quad (\text{resp. } (-\Delta)^\alpha \tilde{u}(x_0) + u^p(x_0) \leq 0).$$

We say that u is a viscosity solution of (1.1) if it is a viscosity super solution and also a viscosity sub solution of (1.1).

Proof of Lemma 2.3. Let $\Omega_n = \{x \in \Omega : |x| > 1/n\}$. Then, Ω_n is of class C^2 for $n \geq N_0$, where N_0 is chosen large enough such that

$$\bar{B}_{1/N_0}(0) \subset \Omega \quad \text{and} \quad \underline{U} > 0 \quad \text{in } \bar{B}_{1/N_0}(0).$$

Since $\underline{U}, \overline{U}$ are locally C^2 in $\Omega \setminus \{0\}$, applying Lemma 2.1 with $\mathcal{O} = \Omega_n$ and $\delta = \frac{1}{4n}$, we find a solution u_n of the problem

$$\begin{gathered} (-\Delta)^\alpha u(x) + |u|^{p-1} u(x) = 0, \quad x \in \Omega_n, \\ u(x) = \underline{U}(x), \quad x \in \bar{B}_{\frac{1}{n}}(0) \setminus \{0\}, \\ u(x) = 0, \quad x \in \Omega^c \end{gathered} \tag{2.2}$$

and a solution v_n of the problem

$$\begin{gathered} (-\Delta)^\alpha u(x) + |u|^{p-1} u(x) = 0, \quad x \in \Omega_n, \\ u(x) = \overline{U}(x), \quad x \in \bar{B}_{\frac{1}{n}}(0) \setminus \{0\}, \\ u(x) = 0, \quad x \in \Omega^c. \end{gathered} \tag{2.3}$$

Now we show that for any $n \geq N_0$,

$$\underline{U} \leq u_n \leq v_n \leq \overline{U} \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{2.4}$$

In fact, since u_n is the solution of (2.2) in Ω_n, $\underline{U}$ is a sub-solution of (2.2) in Ω_n and $u_n = \underline{U}$ in $B_{\frac{1}{n}}(0) \setminus \{0\}$, $u_n = 0 \geq \underline{U}$ in Ω^c, then we apply Lemma 2.2 to obtain that $\underline{U} \le u_n$ in $\mathbb{R}^N \setminus \{0\}$. Similarly, we have $v_n \le \overline{U}$ in $\mathbb{R}^N \setminus \{0\}$. Since u_n and v_n are solutions of

$$(-\Delta)^\alpha u(x) + |u|^{p-1}u(x) = 0, \quad x \in \Omega_n \tag{2.5}$$

and $u_n = \underline{U} \le \overline{U} = v_n$ in $B_{\frac{1}{n}}$, $u_n = v_n = 0$ in Ω^c, by Lemma 2.2, we have $u_n \le v_n$ in $\mathbb{R}^N \setminus \{0\}$.

Next, we prove that for all $n \ge N_0$,

$$u_n \le u_{n+1} \le v_{n+1} \le v_n \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{2.6}$$

Since $u_{n+1} \ge \underline{U}$ in $\mathbb{R}^N \setminus \{0\}$ and $u_n = \underline{U}$ in $\bar{B}_{\frac{1}{n}}(0) \setminus \{0\}$, we obtain $u_{n+1} \ge u_n$ in $\bar{B}_{\frac{1}{n}}(0) \setminus \{0\}$. For u_{n+1} and u_n being solutions of (2.5), by Lemma 2.2, we have $u_n \le u_{n+1}$ in $\mathbb{R}^N \setminus \{0\}$. Similarly, $v_n \ge v_{n+1}$ in $\mathbb{R}^N \setminus \{0\}$. The inequality $u_{n+1} \le v_{n+1}$ follows from (2.4).

Therefore, we can define a function u by

$$u(x) = \lim_{n\to+\infty} u_n(x), \quad x \in \mathbb{R}^N \setminus \{0\},$$

which satisfies

$$\underline{U}(x) \le u_{N_0} \le u(x) \le v_{N_0}(x) \le \overline{U}(x), \quad x \in \mathbb{R}^N \setminus \{0\}. \tag{2.7}$$

Since $\underline{U}$ and $\overline{U}$ belong to $L^1(\mathcal{O}^c, \frac{dx}{1+|x|^{N+2\alpha}})$, then $u_n \to u$ in $L^1(\mathcal{O}^c, \frac{dx}{1+|x|^{N+2\alpha}})$ as $n \to \infty$. By interior estimates as given in [12, Lemma 3.1] or [23, Proposition 2.3], for any compact set K of $\Omega\setminus\{0\}$, there exists $N_K \ge N_0$ such that $\{u_n\}$ is uniformly bounded in $C^\theta(K)$ for $n \ge N_k$ and some $\theta \in (0, 2\alpha)$. By Ascoli-Arzelà Theorem we find that u is continuous in K and $u_n \to u$ uniformly in K. Together with that $u_{N_0} = v_{N_0} = 0$ in Ω^c and u_{N_0}, v_{N_0} are continuous up to the boundary of Ω, we have $u = 0$ in Ω^c, u is continuous in $\Omega \setminus \{0\}$ and up to $\partial\Omega$. By stability theorem [7, Theorem 2.4], u is a solution of (1.1) in the viscosity sense. Applying [6, Theorem 2.6], we find that u is $C^\theta_{\rm loc}(\mathcal{O})$, and using [7, Theorem 2.1] we conclude that u is a classical solution. □

Our main result in this section is as follows.

Proposition 2.5. *Let $0 \in \Omega$ be an open bounded C^2 domain in $\mathbb{R}^N$ and $\alpha \in (0,1)$. Assume that $p > 0$ and u is a positive classical solution of (1.1). Then there exists a positive singular solution u_s of (1.1) with $\Omega = B_{r_0}(0)$ such that*

$$u_s \le u \le u_s + c_2 \quad \text{in } \Omega \setminus \{0\},$$

where $c_2 = \sup_{x\in\mathbb{R}^N\setminus B_{r_0}(0)} u(x)$.

Proof. We will construct a sub-solution and a super-solution of (1.1). Let u be a solution of (1.1) and $u_0(x) = u(x) - c_2$ for $x \in \mathbb{R}^N \setminus \{0\}$. We observe that

$$(-\Delta)^\alpha u_0 = (-\Delta)^\alpha u \quad \text{in} B_{r_0}(0) \setminus \{0\}$$

and

$$|u_0|^{p-1}u_0 \le u^p \quad \text{in } B_{r_0}(0) \setminus \{0\}.$$

Hence, u_0 is a sub-solution of (1.1) with $\Omega = B_{r_0}(0)$. Since u is a super-solution of (1.1) with $\Omega = B_{r_0}(0)$ and $u \ge u_0$ in $\mathbb{R}^N$, by Lemma 2.3 there exists a solution u_s of (1.1) with $\Omega = B_{r_0}(0)$ such that

$$u - c_2 \le u_s \le u \quad \text{in } \Omega \setminus \{0\}. \tag{2.8}$$

The proof is complete. □

3. Symmetry of solutions

In this section, we will prove that the singular solution u_s of (1.1) with $\Omega = B_{r_0}(0)$ is radially symmetric. To this end, we investigate the radially symmetric property of positive singular solutions to more general semilinear elliptic equations

$$\begin{gathered} (-\Delta)^\alpha u = h(u) + \eta \quad \text{in } B_1(0)\setminus\{0\}, \\ u = 0 \qquad \text{in } B_1^c(0), \\ \lim_{x\to 0} u(x) = \infty. \end{gathered} \tag{3.1}$$

We assume that h and η satisfy

(H0) The function $h : [0,\infty) \to \mathbb{R}$ is Lipschitz continuous in $[0, R]$ for any $R > 0$.

(H1) The function $\eta : B_1(0)\setminus\{0\} \to \mathbb{R}$ is radially symmetric and decreasing in $|x|$.

The main results of this section reads as follows.

Proposition 3.1. *Suppose that* (H0) *and* (H1) *hold. If u is a positive classical solution of* (3.1)*, then u must be radially symmetric and strictly decreasing in $r = |x|$ for $r \in (0, 1)$.*

Proposition 3.1 will be proved by the moving plane method. However, since the solution u is singular at the origin, we need the following variant of the maximum principle for small domain.

Lemma 3.2. *Let Ω be an open and bounded subset of $\mathbb{R}^N$. Suppose that $\varphi : \Omega \to \mathbb{R}$ is in $L^\infty(\Omega)$ and $w \in L^\infty(\mathbb{R}^N)$ is a classical solution of*

$$\begin{gathered} -(-\Delta)^\alpha w(x) \le \varphi(x) w(x), \quad x \in \Omega, \\ w(x) \ge 0, \quad x \in \mathbb{R}^N \setminus \Omega. \end{gathered} \tag{3.2}$$

Then there is $\delta > 0$ such that whenever $|\Omega^-| \le \delta$, w has to be non-negative in Ω, where $\Omega^- = \{x \in \Omega : w(x) < 0\}$.

For a proof of the above lemma, see [14, Corollary 2.1] and [18, 24].

Now we use the moving plane method to show the radial symmetry and monotonicity of positive solutions to equation (3.1). For simplicity, we denote $\mathcal{B}_1 = B_1(0)\setminus\{0\}$,

$$\Sigma_\lambda = \{x = (x_1, x') \in \mathcal{B}_1 : x_1 > \lambda\}, \tag{3.3}$$

$$u_\lambda(x) = u(x_\lambda) \quad \text{and} \quad w_\lambda(x) = u_\lambda(x) - u(x), \tag{3.4}$$

where $\lambda \in (0, 1)$ and $x_\lambda = (2\lambda - x_1, x')$ for $x = (x_1, x') \in \mathbb{R}^N$.

By the moving plane method, we will prove that $w_\lambda > 0$ in Σ_λ for all $\lambda \in (0, 1)$. This is proved in a indirect way. Suppose, on the contrary, that $\Sigma_\lambda^- = \{x \in \Sigma_\lambda : w_\lambda(x) < 0\} \ne \emptyset$ for $\lambda \in (0, 1)$. Let us define

$$w_\lambda^+(x) = \begin{cases} w_\lambda(x), & x \in \Sigma_\lambda^-, \\ 0, & x \in \mathbb{R}^N \setminus \Sigma_\lambda^-, \end{cases} \tag{3.5}$$

$$w_\lambda^-(x) = \begin{cases} 0, & x \in \Sigma_\lambda^-, \\ w_\lambda(x), & x \in \mathbb{R}^N \setminus \Sigma_\lambda^-. \end{cases} \tag{3.6}$$

Hence, $w_\lambda^+(x) = w_\lambda(x) - w_\lambda^-(x)$ for all $x \in \mathbb{R}^N$. It is obvious that $(2\lambda, 0, \dots, 0) \notin \Sigma_\lambda^-$ for λ small, since $\lim_{x\to 0} u(x) = +\infty$.

Lemma 3.3. *For any $\lambda \in (0,1)$ and any $x \in \Sigma_\lambda^-$, we have that*

$$(-\Delta)^\alpha w_\lambda^-(x) \le 0. \tag{3.7}$$

Proof. By direct computation, for $x \in \Sigma_\lambda^-$, we have that

$$\begin{aligned}
(-\Delta)^\alpha w_\lambda^-(x) &= \int_{\mathbb{R}^N} \frac{w_\lambda^-(x) - w_\lambda^-(z)}{|x-z|^{N+2\alpha}} dz \\
&= -\int_{\mathbb{R}^N \setminus \Sigma_\lambda^-} \frac{w_\lambda(z)}{|x-z|^{N+2\alpha}} dz \\
&= -\int_{(\mathcal{B}_1 \setminus (\mathcal{B}_1)_\lambda) \cup ((\mathcal{B}_1)_\lambda \setminus \mathcal{B}_1)} \frac{w_\lambda(z)}{|x-z|^{N+2\alpha}} dz \\
&\quad - \int_{(\Sigma_\lambda \setminus \Sigma_\lambda^-) \cup (\Sigma_\lambda \setminus \Sigma_\lambda^-)_\lambda} \frac{w_\lambda(z)}{|x-z|^{N+2\alpha}} dz - \int_{(\Sigma_\lambda^-)_\lambda} \frac{w_\lambda(z)}{|x-z|^{N+2\alpha}} dz \\
&= -I_1 - I_2 - I_3,
\end{aligned}$$

where $A_\lambda = \{x_\lambda : x \in A\}$ for any set A in $\mathbb{R}^N$. We estimate these integrals separately. Since $u = 0$ in $(\mathcal{B}_1)_\lambda \setminus \mathcal{B}_1$ and $u_\lambda = 0$ in $\mathcal{B}_1 \setminus (\mathcal{B}_1)_\lambda$, we have

$$\begin{aligned}
I_1 &= \int_{(\mathcal{B}_1 \setminus (\mathcal{B}_1)_\lambda) \cup ((\mathcal{B}_1)_\lambda \setminus \mathcal{B}_1)} \frac{w_\lambda(z)}{|x-z|^{N+2\alpha}} dz \\
&= \int_{(\mathcal{B}_1)_\lambda \setminus \mathcal{B}_1} \frac{u_\lambda(z)}{|x-z|^{N+2\alpha}} dz - \int_{\mathcal{B}_1 \setminus (\mathcal{B}_1)_\lambda} \frac{u(z)}{|x-z|^{N+2\alpha}} dz \\
&= \int_{(\mathcal{B}_1)_\lambda \setminus \mathcal{B}_1} u_\lambda(z) (\frac{1}{|x-z|^{N+2\alpha}} - \frac{1}{|x-z_\lambda|^{N+2\alpha}})) dz \ge 0,
\end{aligned}$$

since $u_\lambda \ge 0$ and $|x - z_\lambda| > |x - z|$ for all $x \in \Sigma_\lambda^-$ and $z \in (\mathcal{B}_1)_\lambda \setminus \mathcal{B}_1$.

To decide the sign of I_2 we observe that $w_\lambda(z_\lambda) = -w_\lambda(z)$ for any $z \in \mathbb{R}^N$. Then

$$\begin{aligned}
I_2 &= \int_{(\Sigma_\lambda \setminus \Sigma_\lambda^-) \cup (\Sigma_\lambda \setminus \Sigma_\lambda^-)_\lambda} \frac{w_\lambda(z)}{|x-z|^{N+2\alpha}} dz \\
&= \int_{\Sigma_\lambda \setminus \Sigma_\lambda^-} \frac{w_\lambda(z)}{|x-z|^{N+2\alpha}} dz + \int_{\Sigma_\lambda \setminus \Sigma_\lambda^-} \frac{w_\lambda(z_\lambda)}{|x-z_\lambda|^{N+2\alpha}} dz \\
&= \int_{\Sigma_\lambda \setminus \Sigma_\lambda^-} w_\lambda(z) (\frac{1}{|x-z|^{N+2\alpha}} - \frac{1}{|x-z_\lambda|^{N+2\alpha}}) dz \ge 0,
\end{aligned}$$

since $w_\lambda \ge 0$ in $\Sigma_\lambda \setminus \Sigma_\lambda^-$ and $|x - z_\lambda| > |x - z|$ for all $x \in \Sigma_\lambda^-$ and $z \in \Sigma_\lambda \setminus \Sigma_\lambda^-$.

Finally, since $w_\lambda(z) < 0$ for $z \in \Sigma_\lambda^-$, we deduce that

$$\begin{aligned}
I_3 &= \int_{(\Sigma_\lambda^-)_\lambda} \frac{w_\lambda(z)}{|x-z|^{N+2\alpha}} dz \\
&= \int_{\Sigma_\lambda^-} \frac{w_\lambda(z_\lambda)}{|x-z_\lambda|^{N+2\alpha}} dz \\
&= -\int_{\Sigma_\lambda^-} \frac{w_\lambda(z)}{|x-z_\lambda|^{N+2\alpha}} dz \ge 0.
\end{aligned}$$

The proof is complete. □

Lemma 3.4. *Let the function h satisfy* (H0) *and for $\lambda \in (0,1)$ and $x \in \Sigma_\lambda^-$,*

$$\varphi(x) = -\frac{h(u_\lambda(x)) - h(u(x))}{u_\lambda(x) - u(x)}. \tag{3.8}$$

Then there exists $C > 0$ dependent of λ such that $\|\varphi\|_{L^\infty(\Sigma_\lambda^-)} \le C$.

Proof. For $x \in \Sigma_\lambda^- \subset \Sigma_\lambda \subset \mathbb{R}^N \setminus B_\lambda(0)$, $u_\lambda(x) < u(x)$. Moreover, there exists $M_\lambda > 0$ such that

$$\|u\|_{L^\infty(\mathbb{R}^N \setminus B_\lambda(0))} \le M_\lambda.$$

Since h satisfies (H_0), there exists $C > 0$ depending on λ such that

$$\|\varphi\|_{L^\infty(\Sigma_\lambda^-)} \le C.$$

$\square$

Remark 3.5. We note that $M_\lambda \to \infty$ as $\lambda \to 0$, since $\lim_{x\to 0} u(x) = \infty$.

Proof of Proposition 3.1. We divide the proof into four steps.

Step 1. We prove that if λ is close to 1, then $w_\lambda > 0$ in Σ_λ. First we show that $w_\lambda \ge 0$ in Σ_λ, i.e. Σ_λ^- is empty. By contradiction, we assume that $\Sigma_\lambda^- \ne \emptyset$. Now we apply (3.7) and linearity of the fractional Laplacian to obtain that, for $x \in \Sigma_\lambda^-$,

$$(-\Delta)^\alpha w_\lambda^+(x) \ge (-\Delta)^\alpha w_\lambda(x) = (-\Delta)^\alpha u_\lambda(x) - (-\Delta)^\alpha u(x). \tag{3.9}$$

Combining equation (3.1) with (3.9) and (3.5), for $x \in \Sigma_\lambda^-$, we have

$$\begin{aligned}(-\Delta)^\alpha w_\lambda^+(x) &\ge (-\Delta)^\alpha u_\lambda(x) - (-\Delta)^\alpha u(x)\\ &= h(u_\lambda(x)) + \eta(x_\lambda) - h(u(x)) - \eta(x)\\ &= -\varphi(x) w_\lambda^+(x) + \eta(x_\lambda) - \eta(x).\end{aligned}$$

By Lemma 3.4 and assumption (H1), we have that $\eta(x_\lambda) \ge \eta(x)$ for $x \in \Sigma_\lambda^-$ and then

$$-(-\Delta)^\alpha w_\lambda^+(x) \le \varphi(x) w_\lambda^+(x), \quad x \in \Sigma_\lambda^-. \tag{3.10}$$

Moreover, $w_\lambda^+ = 0$ in $(\Sigma_\lambda^-)^c$. Choosing $\lambda \in (0,1)$ close enough to 1 we have that $|\Sigma_\lambda^-|$ is small and we apply Lemma 3.2 to obtain that

$$w_\lambda = w_\lambda^+ \ge 0 \quad \text{in } \Sigma_\lambda^-,$$

which is impossible. Thus,

$$w_\lambda \ge 0 \quad \text{in } \Sigma_\lambda.$$

To complete Step 1, we claim that for $0 < \lambda < 1$, if $w_\lambda \ge 0$ and $w_\lambda \not\equiv 0$ in Σ_λ, then $w_\lambda > 0$ in Σ_λ. We assume that the claim is true for the moment. Since the function u is positive in $B_1(0)$ and $u = 0$ on $\partial B_1(0)$, w_λ is positive on $\partial B_1(0) \cap \partial\Sigma_\lambda$ and then $w_\lambda \not\equiv 0$ in Σ_λ.

Now we prove the claim. Suppose on the contrary that there exists $x_0 \in \Sigma_\lambda$ such that $w_\lambda(x_0) = 0$, i.e., $u_\lambda(x_0) = u(x_0)$. Then

$$(-\Delta)^\alpha w_\lambda(x_0) = (-\Delta)^\alpha u_\lambda(x_0) - (-\Delta)^\alpha u(x_0) = \eta((x_0)_\lambda) - \eta(x_0).$$

Since $x_0 \in \Sigma_\lambda$, we have $|x_0| > |(x_0)_\lambda|$. By assumption (H1), we have that $\eta((x_0)_\lambda) \ge \eta(x_0)$ and then

$$(-\Delta)^\alpha w_\lambda(x_0) \ge 0. \tag{3.11}$$

On the other hand, let $K_\lambda = \{(x_1, x') \in \mathbb{R}^N : x_1 > \lambda\}$. Noting $w_\lambda(z_\lambda) = -w_\lambda(z)$ for any $z \in \mathbb{R}^N$ and $w_\lambda(x_0) = 0$, we deduce

$$\begin{aligned}(-\Delta)^\alpha w_\lambda(x_0) &= -\int_{K_\lambda} \frac{w_\lambda(z)}{|x_0 - z|^{N+2\alpha}} dz - \int_{\mathbb{R}^N \setminus K_\lambda} \frac{w_\lambda(z)}{|x_0 - z|^{N+2\alpha}} dz \\ &= -\int_{K_\lambda} \frac{w_\lambda(z)}{|x_0 - z|^{N+2\alpha}} dz - \int_{K_\lambda} \frac{w_\lambda(z_\lambda)}{|x_0 - z_\lambda|^{N+2\alpha}} dz \\ &= -\int_{K_\lambda} w_\lambda(z)\Big(\frac{1}{|x_0 - z|^{N+2\alpha}} - \frac{1}{|x_0 - z_\lambda|^{N+2\alpha}}\Big) dz.\end{aligned}$$

The facts $|x_0 - z_\lambda| > |x_0 - z|$ for $z \in K_\lambda$, $w_\lambda(z) \ge 0$ and $w_\lambda(z) \not\equiv 0$ in K_λ yield

$$(-\Delta)^\alpha w_\lambda(x_0) < 0, \tag{3.12}$$

which contradicts (3.11), completing the proof of the claim.

Step 2. We prove $\lambda_0 := \inf\{\lambda \in (0,1) : w_\lambda > 0 \text{ in } \Sigma_\lambda\} = 0$. Were it not true, we would have $\lambda_0 > 0$. Hence, $w_{\lambda_0} \ge 0$ in Σ_{λ_0} and $w_{\lambda_0} \not\equiv 0$ in Σ_{λ_0}. The claim in Step 1 implies $w_{\lambda_0} > 0$ in Σ_{λ_0}.

Next we claim that if $w_\lambda > 0$ in Σ_λ for $\lambda \in (0,1)$, then there exists $\epsilon \in (0, \lambda/4)$ such that $w_{\lambda_\epsilon} > 0$ in $\Sigma_{\lambda_\epsilon}$, where $\lambda_\epsilon = \lambda - \epsilon > 3\lambda/4$. This claim directly implies that $\lambda_0 = 0$, which contradicts the fact $\lambda_0 > 0$.

Now we prove the claim. Let $D_\mu = \{x \in \Sigma_\lambda : \operatorname{dist}(x, \partial\Sigma_\lambda) \ge \mu\}$ for $\mu > 0$ small. Since $w_\lambda > 0$ in Σ_λ and D_μ is compact, there exists $\mu_0 > 0$ such that $w_\lambda \ge \mu_0$ in D_μ. By the continuity of $w_\lambda(x)$, for $\epsilon > 0$ small enough and $\lambda_\epsilon = \lambda - \epsilon$, we have that $w_{\lambda_\epsilon}(x) \ge 0$ in D_μ. Therefore, $\Sigma^-_{\lambda_\epsilon} \subset \Sigma_{\lambda_\epsilon} \setminus D_\mu$ and $|\Sigma^-_{\lambda_\epsilon}|$ is small if ϵ and μ are small. Using (3.7) and proceeding as in Step 1, we have for all $x \in \Sigma^-_{\lambda_\epsilon}$ that

$$\begin{aligned}(-\Delta)^\alpha w^+_{\lambda_\epsilon}(x) &= (-\Delta)^\alpha u_{\lambda_\epsilon}(x) - (-\Delta)^\alpha u(x) - (-\Delta)^\alpha w^-_{\lambda_\epsilon}(x) \\ &\ge (-\Delta)^\alpha u_{\lambda_\epsilon}(x) - (-\Delta)^\alpha u(x) \\ &= -\varphi(x) w^+_{\lambda_\epsilon}(x) + \eta(x_{\lambda_\epsilon}) - \eta(x) \\ &\ge -\varphi(x) w^+_{\lambda_\epsilon}(x).\end{aligned}$$

By Lemma 3.4, if $\lambda_\epsilon > 3\lambda/4$, $\varphi(x)$ is controlled by some constant depending on λ.

Since $w^+_{\lambda_\epsilon} = 0$ in $(\Sigma^-_{\lambda_\epsilon})^c$ and $|\Sigma^-_{\lambda_\epsilon}|$ is small, for ϵ and μ small, Proposition 3.2 implies that $w_{\lambda_\epsilon} \ge 0$ in $\Sigma_{\lambda_\epsilon}$. Combining with $\lambda_\epsilon > 0$ and $w_{\lambda_\epsilon} \not\equiv 0$ in $\Sigma_{\lambda_\epsilon}$, we obtain $w_{\lambda_\epsilon} > 0$ in $\Sigma_{\lambda_\epsilon}$. The proof of the claim complete.

Step 3. By Step 2, we have $\lambda_0 = 0$, which implies that $u(-x_1, x') \ge u(x_1, x')$ for $x_1 \ge 0$. Using the same argument from the other side, we conclude that $u(-x_1, x') \le u(x_1, x')$ for $x_1 \ge 0$ and then $u(-x_1, x') = u(x_1, x')$ for $x_1 \ge 0$. Repeating this procedure in all directions we see that u is radially symmetric.

Finally, we prove $u(r)$ is strictly decreasing in $r \in (0,1)$. Let us consider $0 < x_1 < \widetilde{x}_1 < 1$ and let $\lambda = \frac{x_1 + \widetilde{x}_1}{2}$. As proved above we have

$$w_\lambda(x) > 0 \quad \text{for } x \in \Sigma_\lambda.$$

Then

$$\begin{aligned}0 < w_\lambda(\widetilde{x}_1, 0, \dots, 0) &= u_\lambda(\widetilde{x}_1, 0, \dots, 0) - u(\widetilde{x}_1, 0, \dots, 0) \\ &= u(x_1, 0, \dots, 0) - u(\widetilde{x}_1, 0, \dots, 0);\end{aligned}$$

i.e., $u(x_1, 0, \dots, 0) > u(\widetilde{x}_1, 0, \dots, 0)$. Using the radial symmetry of u, we conclude the monotonicity of u. $\square$

Proof of Theorem 1.1. The existence of solutions was proved in Proposition 2.5, and by Proposition 3.1 in the particular case of $\eta = 0$ and $h(s) = -s^p$ with $p \geq 1$, the solution is radially symmetric. The proof is complete. $\square$

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