Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 41, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

BLOW UP OF MILD SOLUTIONS OF A SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS WITH DISTINCT FRACTIONAL DIFFUSIONS

JOSÉ VILLA-MORALES

Abstract. We give a sufficient condition for blow up of positive mild solutions to an initial value problem for a nonautonomous weakly coupled system with distinct fractional diffusions. The proof is based on the study of blow up of a particular system of ordinary differential equations.

1. Introduction

Let $i \in \{1, 2\}$ and $j = 3 - i$. In this paper we study blow up of positive mild solutions of

$$\begin{gathered}\frac{\partial u_i(t,x)}{\partial t} = g_i(t)\Delta_{\alpha_i} u_i(t,x) + h_i(t) u_j^{\beta_i}(t,x), \quad t > 0,\ x \in \mathbb{R}^d,\\ u_i(0,x) = \varphi_i(x), \quad x \in \mathbb{R}^d,\end{gathered} \tag{1.1}$$

where $\Delta_{\alpha_i} = -(-\Delta)^{\alpha_i/2}$, $0 < \alpha_i \le 2$, is the α_i-Laplacian, $\beta_i \ge 1$ are constants, φ_i are non negative, not identically zero, bounded continuous functions and $h_i, g_i : (0,\infty) \to [0,\infty)$ are continuous functions.

If there exist a solution (u_1, u_2) of (1.1) defined in $[0,\infty) \times \mathbb{R}^d$, we say that (u_1, u_2) is a global solution, on the other hand if there exists a number $t_e < \infty$ such that (u_1, u_2) is unbounded in $[0,t] \times \mathbb{R}^d$, for each $t > t_e$, we say that (u_1, u_2) blows up in finite time.

The associated integral system of (1.1) is

$$\begin{aligned}u_i(t,x) &= \int_{\mathbb{R}^d} p_i(G_i(t), y-x)\varphi_i(y)dy\\ &\quad + \int_0^t \int_{\mathbb{R}^d} p_i(G_i(s,t), y-x) h_i(s) u_j^{\beta_i}(s,y)\, dy\, ds.\end{aligned} \tag{1.2}$$

Here $p_i(t,x)$ denote the fundamental solution of $\frac{\partial}{\partial t} - \Delta_{\alpha_i}$ and

$$G_i(s,t) = \int_s^t g_i(r)dr, \quad 0 \le s \le t,$$

2000 *Mathematics Subject Classification.* 35K55, 35K45, 35B40, 35K20.
Key words and phrases. Blow up; weakly coupled system; mild solution; fractal diffusion; nonautonomous initial value problem.

Submitted June 18, 2013. Published February 5, 2014.
Supported by grants 118294 of CONACyT and PIM13-3N from UAA.

where $G_i(t) = G_i(0,t)$. We say that (u_1, u_2) is a mild solution of (1.1) if (u_1, u_2) is a solution of (1.2).

Our main result reads as follows.

Theorem 1.1. *Assume that $\beta_i\beta_j > 1$ and*

$$\lim_{t\to\infty} G_i(t) = \infty. \tag{1.3}$$

Let $a \in \{1,2\}$ such that

$$\alpha_a = \min\{\alpha_1, \alpha_2\} \quad \text{and} \quad b = 3 - a. \tag{1.4}$$

Define

$$f_i(t) = h_i(t)\Big(\frac{G_b(t)}{(G_j(t)^{\alpha_b/\alpha_j} + G_b(t))^{\beta_i}}\Big)^{d/\alpha_b}, \quad t > 0. \tag{1.5}$$

Then the positive solution of (1.2) blows up in finite time if

$$\int_{\cdot}^{\infty} F(s)ds = \infty, \tag{1.6}$$

where

$$F(t) = \Big(f_i(t)^{1/(\beta_i+1)} f_j(t)^{1/(\beta_j+1)}\Big)^{(\beta_i+1)(\beta_j+1)/(\beta_i+\beta_j+2)}. \tag{1.7}$$

It is well known that a classical solution is a mild solution. Therefore, if we give a sufficient condition for blow up of positive solutions to (1.2) then we have a condition for blow up of classical solutions to (1.1).

Corollary 1.2. *Moreover, assume that $\rho_i > 0$, $\sigma_i > -1$ and*

$$\begin{aligned} &\frac{d\rho_b}{\alpha_b} + \frac{\sigma_i(1+\beta_j) + \sigma_j(1+\beta_i)}{\beta_i+\beta_j+2} + 1 \\ &\geq \frac{d}{\beta_i+\beta_j+2}\big[\beta_i(\beta_j+1)\max\{\frac{\rho_j}{\alpha_j}, \frac{\rho_b}{\alpha_b}\} + \beta_j(\beta_i+1)\max\{\frac{\rho_i}{\alpha_i}, \frac{\rho_b}{\alpha_b}\}\big], \end{aligned} \tag{1.8}$$

then each (classical) solution to

$$\begin{gathered} \frac{\partial u_i(t,x)}{\partial t} = \rho_i t^{\rho_i - 1}\Delta_{\alpha_i} u_i(t,x) + t^{\sigma_i} u_j^{\beta_i}(t,x), \quad t > 0,\ x \in \mathbb{R}^d, \\ u_i(0,x) = \varphi_i(x), \quad x \in \mathbb{R}^d. \end{gathered} \tag{1.9}$$

blow up in finite time.

In applied mathematics it is well known the importance of the study of equations such as (1.1). In fact, for example, they arise in fields like molecular biology, hydrodynamics and statistical physics [13]. Also, notice that generators of the form $g_i(t)\Delta_{\alpha_i}$ arise in models of anomalous growth of certain fractal interfaces [8].

There are many related works. Here are some of them:

- When $\alpha_1 = \alpha_2 = 2$, $\rho_1 = \rho_2 = 1$, $\sigma_1 = \sigma_2 = 0$ and $\varphi_1 = \varphi_2$ in (1.9), Fujita [3] showed that if $d < \alpha_1/\beta_1$, then for any non-vanishing initial condition the solution of (1.9) is infinite for all t large enough.
- When $\alpha_1 = \alpha_2$, $\rho_1 = \rho_2$, $\sigma_1 = \sigma_2$ and $\varphi_1 = \varphi_2$ in (1.9), Pérez and Villa [11] showed that if $\sigma_1 + 1 \geq d\rho_1(\beta_1 - 1)/\alpha_1$, then the solutions of (1.9) blow up in finite time.
- When $\alpha_1 = \alpha_2 = 2$ and $\rho_1 = \rho_2 = 1$ in (1.9), Uda [15] proved that all positive solutions of (1.9) blow up if $\max\{\frac{(\sigma_2+1)\beta_1+\sigma_1+1}{\beta_1\beta_2-1}, \frac{(\sigma_1+1)\beta_2+\sigma_2+1}{\beta_1\beta_2-1}\} \geq \frac{d}{2}$.

• When $\alpha_1 = \alpha_2$, $g_1(t) = g_2(t) = t^{\rho-1}$, $\rho > 0$, and $h_1(t) = h_2(t) = 1$ in (1.1), Pérez [10] proved that every positive solution blows up in finite time if $\min\{\frac{\alpha_1}{\rho(\beta_1-1)}, \frac{\alpha_1}{\rho(\beta_2-1)}\} > d$.

• When $\rho_1 = \rho_2 = 1$ and the nonlinear terms in (1.9) are of the form $h(t,x)u^{\beta_i}$, $h(t,x) = O(t^\sigma |x|^\gamma)$, Guedda and Kirane [5] also studied blow up.

Other related results (when $\alpha_1 = \alpha_2 = 2$) can be found, for example in [1, 2, 6, 9] and references therein.

It is worth while to mention that Guedda and Kirane [5] observed that to reduce the study of blow up of (1.1) to a system of ordinary differential equations we must have a comparison result between $p_i(t,x)$ and $p_j(t,x)$. Therefore, the goal of this paper is to use the comparison result given in [7, Lemma 2.4] to follow the usual approach, see among others [14] or [4].

When $\alpha_1 = \alpha_2 = 2$, $\rho_1 = \rho_2 = 1$ and $\sigma_1 = \sigma_2 = 0$ the Uda condition (1.10), the Pérez condition (1.11) and the condition (1.8) become

$$d \le \frac{2(\max\{\beta_1, \beta_2\} + 1)}{\beta_1\beta_2 - 1} = C_U, \tag{1.10}$$

$$d < \frac{2}{\max\{\beta_1, \beta_2\} - 1} = C_A, \tag{1.11}$$

$$d \le \frac{\beta_1 + \beta_2 + 2}{\beta_1\beta_2 - 1} = C_V, \tag{1.12}$$

respectively. Since $C_A \le C_V \le C_U$ we see that the Uda condition (1.10) is the best. Also, from this we see that C_V, given in (1.12), is not the optimal bound (critical dimension), but we believe that it is the best we can get by constructing a convenient subsolution of the solution of (1.2). In fact, the condition (1.8) coincides with the condition for blow up given by Pérez and Villa [11].

This article is organized as follows. In Section 1 we prove the existence of local solutions for the equation (1.2). In Section 2 we give some preliminary results and discuses a sufficient condition for blow up of a system of ordinary differential equations, finally in Section 3 we prove the main result and its corollary.

2. Existence of local solution

The existence of local solutions for the weakly coupled system (1.2) follows form the fix-point theorem of Banach. We begin introducing some normed linear spaces. By $L^\infty(\mathbb{R}^d)$ we denote the space of all real-valued functions essentially bounded defined on $\mathbb{R}^d$. Let $\tau > 0$ be a real number that we will fix later. Define

$$E_\tau = \{(u_1, u_2) : [0, \tau] \to L^\infty(\mathbb{R}^d) \times L^\infty(\mathbb{R}^d),\ |||(u_1, u_2)||| < \infty\},$$

where

$$|||(u_1, u_2)||| = \sup_{0 \le t \le \tau} \{\|u_1(t)\|_\infty + \|u_2(t)\|_\infty\}.$$

Then E_τ is a Banach space and the sets, $R > 0$,

$$P_\tau = \{(u_1, u_2) \in E_\tau,\ u_1 \ge 0, u_2 \ge 0\},$$
$$B_\tau = \{(u_1, u_2) \in E_\tau,\ |||(u_1, u_2)||| \le R\},$$

are closed subspaces of E_τ.

Theorem 2.1. *There exists a $\tau = \tau(\varphi_1, \varphi_2) > 0$ such that the integral system (1.2) has a local solution in $B_\tau \cap P_\tau$.*

Proof. Define the operator $\Psi : B_\tau \cap P_\tau \to B_\tau \cap P_\tau$, by

$$\begin{aligned}
&\Psi(u_1,u_2)(t,x)\\
&= \Big(\int_{\mathbb{R}^d} p_1(G_1(t),y-x)\varphi_1(y)dy, \int_{\mathbb{R}^d} p_2(G_2(t),y-x)\varphi_2(y)dy\Big)\\
&\quad + \Big(\int_0^t\int_{\mathbb{R}^d} p_1(G_1(s,t),y-x)h_1(s)u_2^{\beta_1}(s,y)\,dy\,ds,\\
&\quad \int_0^t\int_{\mathbb{R}^d} p_2(G_2(s,t),y-x)h_2(s)u_1^{\beta_2}(s,y)\,dy\,ds\Big).
\end{aligned}$$

We choose R sufficiently large such that Ψ is onto $B_\tau \cap P_\tau$. We are going to show that Ψ is a contraction, therefore Ψ has a fix point. Let $(u_1,u_2),(\tilde u_1,\tilde u_2) \in B_\tau \cap P_\tau$ with $u_i(0) = \widetilde{u}_i(0)$,

$$\begin{aligned}
&|||\Psi(u_1,u_2) - \Psi(\tilde u_1,\tilde u_2)|||\\
&= |||\Big(\int_0^t\int_{\mathbb{R}^d} p_1(G_1(s,t),y-x)h_1(s)[u_2^{\beta_1}(s,y)-\tilde u_2^{\beta_1}(s,y)]\,dy\,ds,\\
&\quad \int_0^t\int_{\mathbb{R}^d} p_2(G_2(s,t),y-x)h_2(s)[u_1^{\beta_2}(s,y)-\tilde u_1^{\beta_2}(s,y)]\,dy\,ds\Big)|||\\
&\le \sum_{i=1}^2 \sup_{t\in[0,\tau]}\int_0^t\int_{\mathbb{R}^d} p_i(G_i(s,t),y-x)h_i(s)\|u_j^{\beta_i}(s)-\tilde u_j^{\beta_i}(s)\|_\infty\,dy\,ds.
\end{aligned}$$

Let $w,z>0$ and $p\ge 1$, then

$$|w^p - z^p| \le p(w\vee z)^{p-1}|w-z|.$$

Using the previous elementary inequality we obtain

$$\begin{aligned}
|u_j^{\beta_i}(s,x)-\tilde u_j^{\beta_i}(s,x)| &\le \beta_i(u_j(s,x)\vee \tilde u_j(s,x))^{\beta_i-1}|u_j(s,x)-\tilde u_j(s,x)|\\
&\le \beta_i R^{\beta_i-1}\|u_j-\tilde u_j\|_\infty,
\end{aligned}$$

from this we deduce

$$\begin{aligned}
|||\Psi(u_1,u_2)-\Psi(\tilde u_1,\tilde u_2)||| &\le \sum_{i=1}^2\sup_{t\in[0,\tau]}\int_0^t h_i(s)\beta_i R^{\beta_i-1}\|u_i(s)-\tilde u_i(s)\|_\infty ds\\
&\le (\sum_{i=1}^2 \beta_i R^{\beta_i-1}\int_0^\tau h_i(s)ds)|||(u_1,u_2)-(\tilde u_1,\tilde u_2)|||.
\end{aligned}$$

Since $\lim_{t\to0}\int_0^t h_i(s)ds = 0$, we can choose $\tau>0$ small enough such that Ψ is a contraction. □

3. Preliminary results

Lemma 3.1. *For any $s,t>0$ and any $x,y\in\mathbb{R}^d$, we have*

(i) $p_i(ts,x) = t^{-d/\alpha_i}p_i(s,t^{-1/\alpha_i}x)$.
(ii) $p_i(t,x) \ge (\frac{s}{t})^{d/\alpha_i}p_i(s,x)$, *for $t\ge s$.*
(iii) $p_i(t,\frac{1}{\tau}(x-y)) \ge p_i(t,x)p_i(t,y)$, *if $p_i(t,0)\le 1$ and $\tau\ge 2$.*
(iv) *There exist constants $c_i\in(0,1]$ such that*

$$p_i(t,x) \ge c_i p_b(t^{\alpha_b/\alpha_i},x), \tag{3.1}$$

where b is as in (1.4).

For the proof of (i)-(iii) see [14, Section 2], and for (iv) see [7, Lemma 2.4].

Lemma 3.2. *Let u_i be a positive solution of* (1.2)*, then*
$$u_i(t_0,x) \ge c_i(t_0)p_b\big(2^{-\alpha_b}G_i(t_0)^{\alpha_b/\alpha_i},x\big), \quad \forall x\in\mathbb{R}^d, \tag{3.2}$$
where
$$c_i(t_0)=c_i2^{-d}\int_{\mathbb{R}^d}p_b\big(G_i(t_0)^{\alpha_b/\alpha_i},2y\big)\varphi_i(y)dy$$
and $t_0>1$ is large enough such that
$$p_b(G_i(t_0)^{\alpha_b/\alpha_i},0)\le 1. \tag{3.3}$$

Proof. By (i) of Lemma 3.1 and (1.3) there exist t_0 large enough such that
$$p_b\big(G_i(t_0)^{\alpha_b/\alpha_i},0\big)=G_i(t_0)^{-d/\alpha_i}p_b(1,0)\le 1. \tag{3.4}$$
Using (iii) and (i) of Lemma 3.1, we obtain
$$\begin{aligned} p_b\big(G_i(t_0)^{\alpha_b/\alpha_i},y-x\big) &\ge p_b\big(G_i(t_0)^{\alpha_b/\alpha_i},2x\big)p_b(G_i(t_0)^{\alpha_b/\alpha_i},2y)\\ &=2^{-d}p_b\big(2^{-\alpha_b}G_i(t_0)^{\alpha_b/\alpha_i},x\big)p_b(G_i(t_0)^{\alpha_b/\alpha_i},2y). \end{aligned}$$
From (1.2), (iv) of Lemma 3.1 and the previous inequality we conclude
$$u_i(t_0,x)\ge \big(c_i2^{-d}\int_{\mathbb{R}^d}p_b(G_i(t_0)^{\alpha_b/\alpha_i},2y)\varphi_i(y)dy\big)p_b(2^{-\alpha_b}G_i(t_0)^{\alpha_b/\alpha_i},x).$$
Getting the desired result. ∎

Observe that the semigroup property implies
$$\begin{aligned} &u_i(t+t_0,x)\\ &=\int_{\mathbb{R}^d}p_i(G_i(t_0,t+t_0),y-x)u_i(t_0,y)dy\\ &\quad+\int_0^t\int_{\mathbb{R}^d}p_i(G_i(s+t_0,t+t_0),y-x)h_i(s+t_0)u_j^{\beta_i}(s+t_0,y)\,dy\,ds. \end{aligned} \tag{3.5}$$
Let
$$\bar{u}_i(t)=\int_{\mathbb{R}^d}p_b(G_b(t),x)u_i(t,x)dx, \quad t\ge 0. \tag{3.6}$$

Lemma 3.3. *If $\overline{u}_i$ blow up in finite time, then u_i also does.*

Proof. Let t_0 be given in Lemma 3.1. Take $t_0<t_j<\infty$ the explosion time of $\overline{u}_j$. From (1.3) we can choose $t>t_j$ large enough such that
$$G_i(t_j+t_0,t+t_0)>2^{\alpha_i}G_b(t_j+t_0)^{\alpha_i/\alpha_b}.$$
Thus, for each $0\le s\le t_j$,
$$\begin{aligned} \int_{s+t_0}^{t+t_0}g_i(r)dr &\ge \int_{t_j+t_0}^{t+t_0}g_i(r)dr\\ &>2^{\alpha_i}\Big(\int_0^{t_j+t_0}g_b(r)dr\Big)^{\alpha_i/\alpha_b}\\ &\ge 2^{\alpha_i}\Big(\int_0^{s+t_0}g_b(r)dr\Big)^{\alpha_i/\alpha_b}, \end{aligned}$$

hence

$$\tau_i = \frac{G_i(s+t_0, t+t_0)^{1/\alpha_i}}{G_b(s+t_0)^{1/\alpha_b}} \geq 2.$$

On the other hand, (3.4) implies

$$p_b(G_b(s+t_0), 0) \leq p_b(G_b(t_0), 0) = G_b(t_0)^{-d/\alpha_b} p_b(1, 0) \leq 1.$$

Using (i) and (iii) of Lemma 3.1 we obtain

$$\begin{aligned} p_b(G_i(s+t_0, t+t_0)^{\alpha_b/\alpha_i}, y-x) &= \tau_i^{-d} p_b(G_b(s+t_0), \frac{1}{\tau_i}(y-x)) \\ &\geq \tau_i^{-d} p_b(G_b(s+t_0), x) p_b(G_b(s+t_0), y). \end{aligned}$$

From (3.5), (iv) of Lemma 3.1 and Jensen's inequality we deduce that

$$\begin{aligned} &u_i(t+t_0, x) \\ &\geq c_i \int_0^{t_j} h_i(s+t_0) \int_{\mathbb{R}^d} p_b\big(G_i(s+t_0, t+t_0)^{\alpha_b/\alpha_i}, y-x\big) u_j(s+t_0, y)^{\beta_i}\, dy\, ds \\ &\geq c_i \int_0^{t_j} \tau_i^{-d} h_i(s+t_0) p_b(G_b(s+t_0), x) \overline{u}_j(s+t_0)^{\beta_i} ds. \end{aligned}$$

Then $u_i(t+t_0, x) = \infty$. The definition (3.6) of $\overline{u}_i$ implies that $\overline{u}_i$ blows up in finite time, and working as before we conclude that u_j also blows up in finite time. □

In what follows by c we mean a positive constant that may change from place to place. The following result is interesting in itself.

Proposition 3.4. *Let $v_i, f_i : [t_0, \infty) \to \mathbb{R}$ be continuous functions such that*

$$v_i(t) \geq k + k \int_{t_0}^t f_i(s) v_j(s)^{\beta_i} ds, \quad t \geq t_0,$$

where $k > 0$ is a constant. Then v_i blow up in finite time if

$$\int_{t_0}^{\infty} \Big(f_i(s)^{1/(\beta_i+1)} f_j(s)^{1/(\beta_j+1)} \Big)^{(\beta_i+1)(\beta_j+1)/(\beta_i+\beta_j+2)} ds = \infty.$$

Proof. Consider the system

$$z_i(t) = \frac{k}{2} + k \int_{t_0}^t f_i(s) z_j(s)^{\beta_i} ds, \quad t \geq t_0. \tag{3.7}$$

Let $N_i = \{t > t_0 : z_i(s) < v_i(s), \forall s \in [0, t]\}$. It is clear that $N_i \neq \emptyset$. Let $e_i = \sup N_i$. Without loss of generality suppose that $e_i \geq e_j$. If $e_i < \infty$, then the continuity of $v_j - z_j$, yields

$$0 = (v_j - z_j)(e_j) \geq \frac{k}{2} + k \int_{t_0}^{e_j} f_j(s)[v_i(s)^{\beta_j} - z_i(s)^{\beta_j}] ds \geq \frac{k}{2}.$$

Therefore, $z_i(t) \leq v_i(t)$ for each $t \geq t_0$. Define

$$Z(t) = \log z_i(t) z_j(t), \quad t \geq t_0. \tag{3.8}$$

Then, by (3.7),

$$\begin{aligned} Z'(t) &= \frac{f_i(t) z_j(t)^{\beta_i}}{z_i(t)} + \frac{f_i(t) z_i(t)^{\beta_j}}{z_j(t)} \\ &= \frac{(f_i(t)^{1/(\beta_i+1)} z_j(t))^{\beta_i+1} + (f_j(t)^{1/(\beta_j+1)} z_i(t))^{\beta_j+1}}{z_i(t) z_j(t)}. \end{aligned}$$

From [12, Proposition 1, p.259] we see that for each $x, y > 0$,

$$y^{\beta_i+1} + x^{\beta_j+1} \geq c(xy)^{(\beta_i+1)(\beta_j+1)/(\beta_i+\beta_j+2)}.$$

Using this and (3.8) we obtain

$$\begin{aligned} Z'(t) &\geq c\Big(f_i(t)^{1/(\beta_i+1)} f_j(t)^{1/(\beta_j+1)}\Big)^{(\beta_i+1)(\beta_j+1)/(\beta_i+\beta_j+2)} \\ &\quad \times \big(z_i(t)z_j(t)\big)^{(\beta_i\beta_i-1)/(\beta_i+\beta_j+2)} \\ &= cF(t)\exp\big(\frac{\beta_i\beta_i - 1}{\beta_i+\beta_j+2} Z(t)\big), \end{aligned}$$

where F is like (1.7). Consider the equation

$$H'(t) = cF(t)\exp(cH(t)), \quad t > t_0, \ H(t_0) = 2\log\frac{k}{2}.$$

whose solution is

$$H(t) = \log\Big(e^{-cH(t_0)} - c^2 \int_{t_0}^{t} F(s)ds\Big)^{-1/c}.$$

Since $H \leq Z$ then the result follows from (1.6). □

4. Blow up results

Proof of Theorem 1.1. From (3.5) and (3.1),

$$\begin{aligned} &u_i(t+t_0, x) \\ &\geq \int_{\mathbb{R}^d} c_i p_b(G_i(t_0, t+t_0)^{\alpha_b/\alpha_i}, y-x) u_i(t_0, y) dy \\ &\quad + \int_0^t h_i(s+t_0) \int_{\mathbb{R}^d} c_i p_b(G_i(s+t_0, t+t_0)^{\alpha_b/\alpha_i}, y-x) u_j^{\beta_i}(s+t_0, y)\, dy\, ds. \end{aligned}$$

Multiplying by $p_b(G_b(t+t_0), x)$ and integrating with respect to x we obtain

$$\begin{aligned} \bar{u}_i(t+t_0) &\geq c_i \int_{\mathbb{R}^d} p_b(G_i(t_0, t+t_0)^{\alpha_b/\alpha_i} + G_b(t+t_0), y) u_i(t_0, y) dy \\ &\quad + c_i \int_0^t h_i(s+t_0) \int_{\mathbb{R}^d} p_b(G_i(s+t_0, t+t_0)^{\alpha_b/\alpha_i} + G_b(t+t_0), y) \\ &\quad \times u_j^{\beta_i}(s+t_0, y)\, dy\, ds. \end{aligned}$$

The property (ii) of Lemma 3.1 and Jensen's inequality, rendering

$$\begin{aligned} \bar{u}_i(t+t_0) &\geq c_i \int_{\mathbb{R}^d} p_b\Big(G_i(t_0, t+t_0)^{\alpha_b/\alpha_i} + G_b(t+t_0), y\Big) u_i(t_0, y) dy \\ &\quad + c_i \int_0^t \big(\frac{G_b(s+t_0)}{G_i(s+t_0, t+t_0)^{\alpha_b/\alpha_i} + G_b(t+t_0)}\big)^{d/\alpha_b} \\ &\quad \times h_i(s+t_0)(\bar{u}_j(s+t_0))^{\beta_i} ds. \end{aligned}$$

Moreover, (3.2) and that $G_i(s, \cdot)$ is increasing implies

$$\begin{aligned} &\bar{u}_i(t+t_0) \geq c_i c_i(t_0) p_b(1, 0)(2G_i(t+t_0)^{\alpha_b/\alpha_i} + 2G_b(t+t_0))^{-d/\alpha_b} \\ &\quad + c_i \int_0^t h_i(s+t_0)\big(\frac{G_b(s+t_0)}{2G_i(t+t_0)^{\alpha_b/\alpha_i} + 2G_b(t+t_0)}\big)^{d/\alpha_b} (\bar{u}_j(s+t_0))^{\beta_i} ds. \end{aligned}$$

Let

$$v_i(t+t_0)=\bar{u}_i(t+t_0)(G_i(t+t_0)^{\alpha_b/\alpha_i}+G_b(t+t_0))^{d/\alpha_b},$$

then

$$v_i(t+t_0)\geq c+c\int_0^t f_i(s+t_0)v_j(s+t_0)^{\beta_i}ds,$$

where f_i is defined in (1.5). The result follows from Proposition 3.4 and Lemma 3.3. □

Proof of Corollary 1.2. Let

$$f_i(t)=\frac{t^{\sigma_i+d\rho_b/\alpha_b}}{(t^{\rho_j\alpha_b/\alpha_j}+t^{\rho_b})^{d\beta_i/\alpha_b}},$$

then

$$F(t)=\frac{t^{\theta_1}}{(t^{\theta_2}+t^{\theta_3})^{\theta_4}(t^{\theta_5}+t^{\theta_3})^{\theta_6}}$$

where

$$\theta_1=\frac{d\rho_b}{\alpha_b}+\frac{\sigma_i(1+\beta_j)+\sigma_j(1+\beta_i)}{2+\beta_i+\beta_j},$$

$$\theta_2=\frac{\rho_j\alpha_b}{\alpha_j},\quad \theta_3=\rho_b,\quad \theta_4=\frac{d\beta_i(\beta_j+1)}{\alpha_b(2+\beta_i+\beta_j)},$$

$$\theta_5=\frac{\rho_i\alpha_b}{\alpha_i},\quad \theta_6=\frac{d\beta_j(\beta_i+1)}{\alpha_b(2+\beta_i+\beta_j)}.$$

Using the elementary inequality

$$(t^{\theta_2}+t^{\theta_3})^{\theta_4}(t^{\theta_5}+t^{\theta_3})^{\theta_6}\leq(2t^{\max\{\theta_2,\theta_3\}})^{\theta_4}(2t^{\max\{\theta_5,\theta_3\}})^{\theta_6},\quad t>1,$$

the result follows. □

References

[1] Andreucci, D.; Herrero, M. A.; Velázquez, J. J. L.; *Liouville theorems and blow up behavior in semilinear reaction diffusion systems.* Ann. Inst. Henri Poincaré, Anal. Non Linéaire **14** (1997), 1-53.

[2] Fila, M.; Levine, A.; Uda, Y. A.; *Fujita-type global existence-global non-existence theorem for a system of reaction diffusion equations with differing diffusivities.* Math. Methods Appl. Sci. **17** (1994), 807-835.

[3] Fujita, H.; *On the blowing up of solutions of the Cauchy problem for $u_t=\Delta u+u^{1+\alpha}$.* J. Fac. Sci. Univ. Tokyo Sect. I, **13** (1966), 109-124.

[4] Guedda, M.; Kirane, M.; *A note on nonexistence of global solutions to a nonlinear integral equation.* Bull. Belg. Math. Soc. Simon Stevin **6** (1999), 491-497.

[5] Guedda, M.; Kirane, M.; *Criticality for some evolution equations.* Differential Equations **37** (2001), 540-550.

[6] Kobayashi, Y.; *Behavior of the life span for solutions to the system of reation-diffusion equations.* Hiroshima Math. J. **33** (2003), 167-187.

[7] López-Mimbela, J. A.; Villa J.; *Local time and Tanaka formula for a multitype Dawson-Watanabe superprocess.* Math. Nachr. **279** (2006), 1695-1708.

[8] Mann Jr., J. A.; Woyczyński, W. A.; *Growing Fractal Interfaces in the Presence of Self-similar Hopping Surface Diffusion.* Phys.*A* Vol. **291** (2001), 159-183.

[9] Mochizuki, K.; Huang, Q.; *Existence and behavior of solutions for a weakly coupled system of reaction-diffusion equations.* Methods Appl. Anal. **5** (1998), 109-124.

[10] Pérez, A.; *A blow up condition for a nonautonomous semilinear system.* Electron. J. Diff. Eqns. **2006**, No. 94 (2006), 1-8.

[11] Pérez, A.; Villa, J.; *A note on blow-up of a nonlinear integral equation.* Bull. Belg. Math. Soc. Simon Stevin **17** (2010), 891-897.

[12] Qi, Y.-W.; Levine, H. A.; *The critical exponent of degenerate parabolic systems.* Z. angew Math. Phys. **44** (1993), 549-265.

[13] Shlesinger, M. F.; Zaslavsky, G .M.; Frisch, U. (Eds). *Lévy Fligths and Related Topics in Physics.* Lecture Notes in Physics Vol. **450**. Springer-Verlag, Berlin (1995).

[14] Sugitani, S.; *On nonexistence of global solutions for some nonlinear integral equations.* Osaka J. Math. **12** (1975), 45-51.

[15] Uda, Y.; *The critical exponent for a weakly coupled system of the generalized Fujita type reaction-diffusion equations.* Z. angew Math. Phys. **46** (1995), 366-383.

José Villa Morales

Universidad Autónoma de Aguascalientes, Departamento de Matemáticas y Física Aguascalientes, Aguascalientes, México

E-mail address: jvilla@correo.uaa.mx