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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A DISCRETE NONLINEAR BOUNDARY VALUE PROBLEM

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ABSTRACT. In this article, we show the existence and multiplicity of positive solutions for a discrete nonlinear boundary value problem involving the p-Laplacian. Our approach is based on critical point theorems in finite dimensional Banach spaces.

1. INTRODUCTION

It is well known that in fields of research, such as computer science, mechanical engineering, control systems, artificial or biological neural networks, economics and many others, the mathematical modelling of important questions leads naturally to the consideration of nonlinear difference equations. For this reason, in recent years, many authors have widely developed various methods and techniques, such as fixed point theorems, upper and lower solutions, and Brouwer degree, to study discrete problems (see, e.g., [5, 6, 14, 16, 17, 21, 24] and references therein). Recently, also the critical point theory has aroused the attention of many authors in the study of these problems (see, e.g., [2, 3, 7, 9, 13, 18, 19]).

Let N be a positive integer, denote with $[1, N]$ the discrete interval $\{1, \dots, N\}$ and consider the problem

$$\begin{gathered} -\Delta(\phi_p(\Delta u_{k-1})) + q_k\phi_p(u_k) = \lambda f(k, u_k), \quad k \in [1, N], \\ u_0 = u_{N+1} = 0, \end{gathered} \tag{1.1}$$

where, $f : [1, N] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\Delta u_{k-1} := u_k - u_{k-1}$ is the forward difference operator, $q_k \geq 0$ for all $k \in [1, N]$, $\phi_p(s) := |s|^{p-2}s$, $1 < p < +\infty$ and λ is a positive parameter.

In the present article, first we obtain the existence of at least one solution for problem (1.1). It is worth noticing that, usually, to obtain the existence of one solution, asymptotic conditions both at zero and at infinity on the nonlinear term are requested, while, here, it is assumed only a unique algebraic condition (see (3.5) in Corollary 3.6). As a consequence, by combining with the classical Ambrosetti-Rabinowitz condition (see [4]), the existence of two solutions is obtained (see Theorem 4.1). Subsequently, an existence result of three non-negative solutions is obtained combining two algebraic conditions which guarantee the existence of two

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local minima for the Euler-Lagrange functional and applying the mountain pass theorem as given by Pucci and Serrin (see [20]) to ensure the existence of the third critical point (see Theorem 4.3).

Our approach is variational and the main tool is a local minimum theorem established in [8], of whose two its consequences are here applied (see Theorems 2.1 and 2.2). We also refer the interested reader to the papers [1, 10, 11, 12, 15] and references therein, in which the Ricceri Variational Principle and its variants have been successfully used to obtain the existence and multiplicity of solutions for nonlinear boundary value problems.

Further, we state two special cases of our results. First, combining Remark 3.5 and Theorem 4.1, one has the following theorem.

Theorem 1.1. *Let $g:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $g(0)\neq 0$ and*

$$\lim_{\xi\to 0^+}\frac{g(\xi)}{\xi}=+\infty. \tag{1.2}$$

Let $G(t):=\int_0^t g(\xi)d\xi$ for all $t\in\mathbb{R}$, and assume that

(AR) *there exist constants $\nu>2$ and $R>0$ such that, for all $|\xi|\geq R$, one has*

$$0<\nu G(\xi)\leq \xi g(\xi).$$

Then, for each

$$\lambda\in\Big]0,\frac{2}{N(N+1)}\sup_{\gamma>0}\frac{\gamma^2}{\max_{|\xi|\leq\gamma}G(\xi)}\Big[,$$

the problem

$$\begin{gathered}-\Delta^2 u_{k-1}+q_k u_k=\lambda g(u_k),\quad k\in[1,N],\\ u_0=u_{N+1}=0,\end{gathered}$$

admits at least two non-trivial solutions.

Instead, Theorem 4.3 gives the following theorem.

Theorem 1.2. *Let $g:\mathbb{R}\to\mathbb{R}$ be a non-negative continuous function such that*

$$\begin{gathered}\lim_{\xi\to 0^+}\frac{g(\xi)}{\xi}=+\infty,\quad \lim_{\xi\to+\infty}\frac{g(\xi)}{\xi}=0,\\ \int_0^1 g(x)dx<\frac{1}{2(N+1)}\int_0^2 g(x)dx.\end{gathered}$$

Then, for each

$$\lambda\in\frac{1}{N}\Big]\frac{4}{\int_0^2 g(x)dx},\frac{2}{(N+1)\int_0^1 g(x)dx}\Big[,$$

the problem

$$\begin{gathered}-\Delta^2 u_{k-1}=\lambda g(u_k),\quad k\in[1,N],\\ u_0=u_{N+1}=0,\end{gathered}$$

admits at least three non-negative solutions.

2. Preliminaries

Our main tools are Theorems 2.1 and 2.2, consequences of the existence result of a local minimum theorem [8, Theorem 3.1] which is inspired by the Ricceri Variational Principle [23].

For a given non-empty set X, and two functionals $\Phi, \Psi : X \to \mathbb{R}$, we define the following functions

$$\beta(r_1, r_2) := \inf_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\sup_{u\in\Phi^{-1}(]r_1,r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)},$$
$$\rho_2(r_1, r_2) := \sup_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(]-\infty,r_1])} \Psi(u)}{\Phi(v) - r_1},$$

for all $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, and

$$\rho(r) := \sup_{v\in\Phi^{-1}(]r,+\infty[)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{\Phi(v) - r},$$

for all $r \in \mathbb{R}$.

Theorem 2.1 ([8, Theorem 5.1])**.** *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Put $I_\lambda := \Phi - \lambda\Psi$ and assume that there are $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \rho_2(r_1, r_2). \tag{2.1}$$

Then, for each λ in the interval $]\frac{1}{\rho_2(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)}[$ there is $u_{0,\lambda}$ in $\Phi^{-1}(]r_1, r_2[)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(]r_1, r_2[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Theorem 2.2 ([8, Theorem 5.3])**.** *Let X be a real Banach space; $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X; $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Fix $\inf_X \Phi < r < \sup_X \Phi$ and assume that*

$$\rho(r) > 0, \tag{2.2}$$

and for each $\lambda > \frac{1}{\rho(r)}$ the function $I_\lambda := \Phi - \lambda\Psi$ is coercive. Then, for each $\lambda > \frac{1}{\rho(r)}$ there is $u_{0,\lambda} \in \Phi^{-1}(]r, +\infty[)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(]r, +\infty[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Remark 2.3. It is worth noticing that whenever X is a finite dimensional Banach space, a careful reading of the proofs of Theorems 2.1 and 2.2 shows that regarding to the regularity of the derivative of Φ and Ψ, it is enough to require only that Φ' and Ψ' are two continuous functionals on X^*.

Now, consider the N-dimensional Banach space

$$S := \{u : [0, N + 1] \to \mathbb{R} : u_0 = u_{N+1} = 0\}$$

endowed with the norm

$$\|u\| := \Big(\sum_{k=1}^{N+1} |\Delta u_{k-1}|^p + q_k|u_k|^p \Big)^{1/p}.$$

In the sequel, we will use the inequality

$$\max_{k\in[1,N]}|u_k| \leq \frac{(N+1)^{(p-1)/p}}{2}\|u\| \tag{2.3}$$

for every $u \in S$. It immediately follows, for instance, from [18, Lemma 2.2]. Put

$$\Phi(u) := \frac{\|u\|^p}{p}, \quad \Psi(u) := \sum_{k=1}^{N} F(k,u_k), \quad I_\lambda(u) := \Phi(u) - \lambda\Psi(u) \tag{2.4}$$

for every $u \in S$, where $F(k,t) := \int_0^t f(k,\xi)d\xi$ for every $(k,t) \in [1,N]\times\mathbb{R}$.

Standard arguments show that $I_\lambda \in C^1(S,\mathbb{R})$ as well as that critical points of I_λ are exactly the solutions of problem (1.1). In fact, one has

$$\begin{aligned} I'_\lambda(u)(v) &= \sum_{k=1}^{N+1}[\phi_p(\Delta u_{k-1})\Delta v_{k-1} + q_k|u_k|^{p-2}u_kv_k - \lambda f(k,u_k)v_k] \\ &= -\sum_{k=1}^{N}[\Delta(\phi_p(\Delta u_{k-1}))v_k - q_k|u_k|^{p-2}u_kv_k + \lambda f(k,u_k)v_k] \end{aligned}$$

for all $u, v \in S$ (see [18] for more details).

Finally, for the reader's convenience we recall [9, Theorems 2.2 and 2.3] in order to get positive solutions to problem (1.1), i.e. $u_k > 0$ for all $k \in [1,N]$.

Lemma 2.4. *Let $u \in S$ and assume that one of the following conditions holds:*

(A1) $-\Delta(\phi_p(\Delta u_{k-1})) + q_k\phi_p(u_k) \geq 0$ *for all* $k \in [1,N]$;

(A2) *if* $u_k \leq 0$, *then* $-\Delta(\phi_p(\Delta u_{k-1})) + q_k\phi_p(u_k) = 0$.

Then, either $u > 0$ in $[1,N]$ or $u \equiv 0$.

Remark 2.5. If $f : [1,N]\times\mathbb{R} \to \mathbb{R}$ is a non-negative function, then, owing to Lemma 2.4 part (A1), all solutions of problem (1.1) are either zero or positive. Now, let $f : [1,N]\times\mathbb{R} \to \mathbb{R}$ be such that $f(k,0) = 0$ for all $k \in [0,N]$. Put

$$f^*(k,t) := \begin{cases} f(k,t), & \text{if } t > 0,\\ 0, & \text{if } t \leq 0. \end{cases}$$

Clearly, f^* is a continuous function. Owing to Lemma 2.4 part (A2), all solutions of problem $(P_\lambda^{f^*,q})$ are either zero or positive, and hence are also solutions for (1.1). Hence, we emphasize that when $(P_\lambda^{f^*,q})$ admits non-trivial solutions, then problem (1.1) admits positive solutions, independently of the sign of f.

3. Main results

In this section we establish an existence result of at least one solution, Theorem 3.1, which is based on Theorem 2.1, and we point out some consequences, Theorems 3.2, 3.3 and Corollary 3.6. Finally, we present an other existence result of at least one solution, Theorem 3.7, which is based in turn on Theorem 2.2.

Put $Q := \sum_{k=1}^N q_k$. For every two non-negative constants γ, δ, with

$$(2\gamma)^p \neq (2+Q)(N+1)^{p-1}\delta^p,$$

we set

$$a_\gamma(\delta) := \frac{\sum_{k=1}^N \max_{|t|\leq\gamma} F(k,t) - \sum_{k=1}^N F(k,\delta)}{(2\gamma)^p - (2+Q)(N+1)^{p-1}\delta^p}.$$

Theorem 3.1. *Assume that there exist three real constants γ_1, γ_2 and δ, with*
$$0 \le \gamma_1 < \frac{(2+Q)^{1/p}(N+1)^{(p-1)/p}}{2}\delta < \gamma_2, \tag{3.1}$$
such that
$$a_{\gamma_2}(\delta) < a_{\gamma_1}(\delta). \tag{3.2}$$
Then, for each $\lambda \in \frac{1}{p(N+1)^{p-1}}]\frac{1}{a_{\gamma_1}(\delta)}, \frac{1}{a_{\gamma_2}(\delta)}[$, problem (1.1) admits at least one non-trivial solution $\bar{u} \in S$, such that
$$\frac{2\gamma_1}{(N+1)^{(p-1)/p}} < \|\bar{u}\| < \frac{2\gamma_2}{(N+1)^{(p-1)/p}}.$$

Proof. Take the real Banach space S as defined in Section 2, and put Φ, Ψ, I_λ as in (2.4). Our aim is to apply Theorem 2.1 to function I_λ, since critical points of I_λ are solutions to our problem. An easy computation ensures the regularity assumptions required on Φ and Ψ; see Remark 2.3. Therefore, it remains to verify assumption (2.1). To this end, we put
$$r_1 := \frac{(2\gamma_1)^p}{p(N+1)^{p-1}}, \quad r_2 := \frac{(2\gamma_2)^p}{p(N+1)^{p-1}},$$
and pick $w \in S$, defined as follows:
$$w_k := \begin{cases} \delta, & \text{if } k \in [1, N], \\ 0, & \text{if } k = 0,\ k = N+1. \end{cases}$$
Clearly, one has
$$\Phi(w) = \frac{2+Q}{p}\delta^p.$$
From the condition (3.1), we obtain $r_1 < \Phi(w) < r_2$. For all $u \in S$ such that $\Phi(u) < r_2$, taking (2.3) into account, one has $|u_k| < \gamma_2$ for all $k \in [1, N]$, from which it follows
$$\sup_{u \in \Phi^{-1}(]-\infty, r_2[)} \Psi(u) = \sup_{u \in \Phi^{-1}(]-\infty, r_2[)} \sum_{k=1}^{N} F(k, u_k) \le \sum_{k=1}^{N} \max_{|t| \le \gamma_2} F(k, t).$$
Arguing as before, we obtain
$$\sup_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u) \le \sum_{k=1}^{N} \max_{|t| \le \gamma_1} F(k, t).$$
Therefore, one has
$$\begin{aligned} \beta(r_1, r_2) &\le \frac{\sup_{u \in \Phi^{-1}(]-\infty, r_2[)} \Psi(u) - \Psi(w)}{r_2 - \Phi(w)} \\ &\le \frac{\sum_{k=1}^{N} \max_{|t| \le \gamma_2} F(k, t) - \sum_{k=1}^{N} F(k, \delta)}{\frac{(2\gamma_2)^p}{p(N+1)^{p-1}} - \frac{2+Q}{p}\delta^p} \\ &= p(N+1)^{p-1} \frac{\sum_{k=1}^{N} \max_{|t| \le \gamma_2} F(k, t) - \sum_{k=1}^{N} F(k, \delta)}{(2\gamma_2)^p - (2+Q)(N+1)^{p-1}\delta^p} \\ &= p(N+1)^{p-1} a_{\gamma_2}(\delta). \end{aligned}$$

On the other hand, one has

$$\begin{aligned}
\rho_2(r_1,r_2) &\geq \frac{\Psi(w)-\sup_{u\in\Phi^{-1}(]-\infty,r_1])}\Psi(u)}{\Phi(w)-r_1}\\
&\geq \frac{\sum_{k=1}^N F(k,\delta)-\sum_{k=1}^N \max_{|t|\leq\gamma_1} F(k,t)}{\frac{2+Q}{p}\delta^p-\frac{(2\gamma_1)^p}{p(N+1)^{p-1}}}\\
&= p(N+1)^{p-1}\frac{\sum_{k=1}^N \max_{|t|\leq\gamma_1} F(k,t)-\sum_{k=1}^N F(k,\delta)}{(2\gamma_1)^p-(2+Q)(N+1)^{p-1}\delta^p}\\
&= p(N+1)^{p-1}a_{\gamma_1}(\delta).
\end{aligned}$$

Hence, from the assumption (3.2), one has $\beta(r_1,r_2)<\rho_2(r_1,r_2)$. Therefore, from Theorem 2.1, for each $\lambda\in\frac{1}{p(N+1)^{p-1}}]\frac{1}{a_{\gamma_1}(\delta)},\frac{1}{a_{\gamma_2}(\delta)}[$, the functional I_λ admits at least one critical point $\bar{u}$ such that

$$r_1<\Phi(\bar{u})<r_2,$$

that is

$$\frac{2\gamma_1}{(N+1)^{(p-1)/p}}<\|\bar{u}\|<\frac{2\gamma_2}{(N+1)^{(p-1)/p}},$$

and the conclusion is achieved. □

Now, we point out an immediate consequence of Theorem 3.1.

Theorem 3.2. *Assume that there exist two positive constants γ,δ, with*

$$\delta<\frac{2}{(2+Q)^{1/p}(N+1)^{(p-1)/p}}\gamma,$$

for which

$$\frac{\sum_{k=1}^N \max_{|t|\leq\gamma} F(k,t)}{(2\gamma)^p}<\frac{\sum_{k=1}^N F(k,\delta)}{(2+Q)(N+1)^{p-1}\delta^p}. \tag{3.3}$$

Then, for each

$$\lambda\in\Big]\frac{(2+Q)\delta^p}{p\sum_{k=1}^N F(k,\delta)},\frac{(2\gamma)^p}{p(N+1)^{p-1}\sum_{k=1}^N \max_{|t|\leq\gamma} F(k,t)}\Big[,$$

problem (1.1) *admits at least one non-trivial solution $\bar{u}\in S$, such that $|\bar{u}_k|<\gamma$ for all $k\in[1,N]$.*

Proof. The conclusion follows from Theorem 3.1, by taking $\gamma_1:=0$ and $\gamma_2:=\gamma$. Indeed, owing to the inequality (3.3), one has

$$a_\gamma(\delta)=\frac{\sum_{k=1}^N \max_{|t|\leq\gamma} F(k,t)-\sum_{k=1}^N F(k,\delta)}{(2\gamma)^p-(2+Q)(N+1)^{p-1}\delta^p}<\frac{1}{(2\gamma)^p}\sum_{k=1}^N \max_{|t|\leq\gamma} F(k,t).$$

On the other hand, one has

$$a_0(\delta)=\frac{\sum_{k=1}^N F(k,\delta)}{(2+Q)(N+1)^{p-1}\delta^p}.$$

Now, inequality (3.3) yields $a_\gamma(\delta)<a_0(\delta)$. Hence, taking (2.3) into account, Theorem 3.1 ensures the conclusion. □

Theorem 3.3. *Assume that*
$$\lim_{\xi\to 0^+}\frac{\sum_{k=1}^{N}F(k,\xi)}{\xi^p}=+\infty. \tag{3.4}$$
Furthermore, for each $\gamma>0$, set
$$\lambda_\gamma^\star:=\frac{2^p}{p(N+1)^{p-1}}\frac{\gamma^p}{\sum_{k=1}^{N}\max_{|t|\le\gamma}F(k,t)}.$$
Then, for every $\lambda\in]0,\lambda_\gamma^\star[$, problem (1.1) admits at least one non-trivial solution $\bar u\in S$, such that $|\bar u_k|<\gamma$ for all $k\in[1,N]$.

Proof. Fix $\gamma>0$ and $\lambda\in]0,\lambda_\gamma^\star[$. From (3.4) there exists a positive constant δ with
$$\delta<\frac{2}{(2+Q)^{1/p}(N+1)^{(p-1)/p}}\gamma,$$
such that
$$\frac{(2+Q)\delta^p}{p\sum_{k=1}^{N}F(k,\delta)}<\lambda<\frac{(2\gamma)^p}{p(N+1)^{p-1}\sum_{k=1}^{N}\max_{|t|\le\gamma}F(k,t)}.$$
Hence, owing to Theorem 3.2, for every $\lambda\in]0,\lambda_\gamma^\star[$ problem (1.1) admits at least one non-trivial solution $\bar u\in S$, such that $|\bar u_k|<\gamma$ for all $k\in[1,N]$. The proof is complete. ☐

Remark 3.4. We claim that under the above assumptions, the mapping $\lambda\mapsto I_\lambda(\bar u)$ is negative and strictly decreasing in $]0,\lambda_\gamma^\star[$. Indeed, the restriction of the functional I_λ to $\Phi^{-1}(]0,r_2[)$, where $r_2:=\frac{(2\gamma_2)^p}{p(N+1)^{p-1}}$, admits a global minimum, which is a critical point (local minimum) of I_λ in S. Moreover, since $w\in\Phi^{-1}(]0,r_2[)$ and
$$\frac{\Phi(w)}{\Psi(w)}=\frac{(2+Q)\delta^p}{p\sum_{k=1}^{N}F(k,\delta)}<\lambda,$$
we have
$$I_\lambda(\bar u)\le I_\lambda(w)=\Phi(w)-\lambda\Psi(w)<0.$$
Next, we observe that
$$I_\lambda(u)=\lambda\Big(\frac{\Phi(u)}{\lambda}-\Psi(u)\Big),$$
for every $u\in S$ and fix $0<\lambda_1<\lambda_2<\lambda_\gamma^\star$. Set
$$m_{\lambda_1}:=\Big(\frac{\Phi(\bar u_1)}{\lambda_1}-\Psi(\bar u_1)\Big)=\inf_{u\in\Phi^{-1}(]0,r_2[)}\Big(\frac{\Phi(u)}{\lambda_1}-\Psi(u)\Big),$$
$$m_{\lambda_2}:=\Big(\frac{\Phi(\bar u_2)}{\lambda_2}-\Psi(\bar u_2)\Big)=\inf_{u\in\Phi^{-1}(]0,r_2[)}\Big(\frac{\Phi(u)}{\lambda_2}-\Psi(u)\Big).$$
Clearly, as claimed before, $m_{\lambda_i}<0$ (for $i=1,2$), and $m_{\lambda_2}\le m_{\lambda_1}$ thanks to $\lambda_1<\lambda_2$. Then the mapping $\lambda\mapsto I_\lambda(\bar u)$ is strictly decreasing in $]0,\lambda_\gamma^\star[$ owing to
$$I_{\lambda_2}(\bar u_2)=\lambda_2 m_{\lambda_2}\le\lambda_2 m_{\lambda_1}<\lambda_1 m_{\lambda_1}=I_{\lambda_1}(\bar u_1).$$
This concludes the proof of our claim.

Remark 3.5. In other words, Theorem 3.3 ensures that if the asymptotic condition at zero (3.4) is verified then, for every parameter λ belonging to the real interval $]0, \lambda^\star[$, where

$$\lambda^\star := \frac{2^p}{p(N+1)^{p-1}} \sup_{\gamma>0} \frac{\gamma^p}{\sum_{k=1}^N \max_{|t|\leq\gamma} F(k,t)},$$

problem (1.1) admits at least one non-trivial solution $\bar{u} \in S$.

Corollary 3.6. *Let $\alpha : [1,N] \to \mathbb{R}$ be a non-negative and non-zero function and let $g : [0,+\infty) \to \mathbb{R}$ be a continuous function such that $g(0) = 0$. Assume that there exist two positive constants γ, δ, with*

$$\delta < \frac{2}{(2+Q)^{1/p}(N+1)^{(p-1)/p}}\gamma,$$

for which

$$\frac{\max_{0\leq t\leq\gamma} G(t)}{\gamma^p} < \Big(\frac{2^p}{(2+Q)(N+1)^{p-1}}\Big)\frac{G(\delta)}{\delta^p}, \tag{3.5}$$

where $G(t) := \int_0^t g(\xi)d\xi$ for all $t \in \mathbb{R}$. Then, for each

$$\lambda \in \frac{1}{p\sum_{k=1}^N \alpha_k}\Big]\frac{(2+Q)\delta^p}{G(\delta)}, \frac{(2\gamma)^p}{(N+1)^{p-1}\max_{0\leq t\leq\gamma} G(t)}\Big[,$$

the problem

$$\begin{gathered} -\Delta(\phi_p(\Delta u_{k-1})) + q_k\phi_p(u_k) = \lambda\alpha_k g(u_k), \quad k \in [1,N], \\ u_0 = u_{N+1} = 0, \end{gathered} \tag{3.6}$$

admits at least one positive solution $\bar{u} \in S$, such that $\bar{u}_k < \gamma$ for all $k \in [1,N]$.

Proof. Put

$$f(k,t) := \begin{cases} \alpha_k g(t), & \text{if } t \geq 0, \\ 0, & \text{if } t < 0, \end{cases}$$

for every $k \in [1,N]$ and $t \in \mathbb{R}$. The conclusion follows from Theorem 3.2 owing to (3.5) and taking into account Lemma 2.4 part (A2). $\square$

Finally, we present an application of Theorem 2.2 which we will use in next section to obtain multiple solutions.

Theorem 3.7. *Assume that there exist two real constants $\bar{\gamma}, \bar{\delta}$, with*

$$0 < \bar{\gamma} < \frac{(2+Q)^{1/p}(N+1)^{(p-1)/p}}{2}\bar{\delta},$$

such that

$$\sum_{k=1}^N \max_{|t|\leq\bar{\gamma}} F(k,t) < \sum_{k=1}^N F(k,\bar{\delta}), \tag{3.7}$$

and

$$\limsup_{|\xi|\to+\infty} \frac{F(k,\xi)}{|\xi|^p} \leq 0 \quad \textit{uniformly in } k. \tag{3.8}$$

Then, for each $\lambda > \tilde{\lambda}$, where

$$\tilde{\lambda} := \frac{(2+Q)(N+1)^{p-1}\bar{\delta}^p - (2\bar{\gamma})^p}{p(N+1)^{p-1}\big(\sum_{k=1}^N F(k,\bar{\delta}) - \sum_{k=1}^N \max_{|t|\leq\bar{\gamma}} F(k,t)\big)},$$

problem (1.1) *admits at least one non-trivial solution* $\tilde{u} \in S$*, such that* $\|\tilde{u}\| > \frac{2\bar{\gamma}}{(N+1)^{(p-1)/p}}$.

Proof. Take the real Banach space S as defined in Section 2, and put Φ, Ψ, I_λ as in (2.4). Our aim is to apply Theorem 2.2 to function I_λ. The functionals Φ and Ψ satisfy all regularity assumptions requested in Theorem 2.2; see Remark 2.3. Moreover, by standard computations, the assumption (3.8) implies that I_λ, $\lambda > 0$, is coercive. So, our aim is to verify condition (2.2) of Theorem 2.2. To this end, we put

$$r := \frac{(2\bar{\gamma})^p}{p(N+1)^{p-1}},$$

and pick $w \in S$, defined as

$$w_k := \begin{cases} \bar{\delta}, & \text{if } k \in [1, N], \\ 0, & \text{if } k = 0,\ k = N+1. \end{cases}$$

Arguing as in the proof of Theorem 3.1 we obtain that

$$\rho(r) \geq p(N+1)^{p-1} \frac{\sum_{k=1}^N F(k, \bar{\delta}) - \sum_{k=1}^N \max_{|t| \leq \bar{\gamma}} F(k,t)}{(2+Q)(N+1)^{p-1}\bar{\delta}^p - (2\bar{\gamma})^p}.$$

So, from our assumption it follows that $\rho(r) > 0$.

Hence, from Theorem 2.2 for each $\lambda > \tilde{\lambda}$, the functional I_λ admits at least one local minimum $\tilde{u}$ such that $\|\tilde{u}\| > 2\bar{\gamma}/\big((N+1)^{(p-1)/p}\big)$ and the conclusion is achieved. □

4. Multiplicity results

The main aim of this section is to present multiplicity results. First, as consequence of Theorem 3.1, taking into account the classical theorem of Ambrosetti and Rabinowitz, we have the following multiplicity result.

Theorem 4.1. *Let the assumptions of Theorem 3.1 be satisfied. Assume also that* $f(k, 0) \neq 0$ *for some* $k \in [1, N]$*. Moreover, let*

- (AR) *there exist constants* $\nu > p$ *and* $R > 0$ *such that, for all* $|\xi| \geq R$ *and for all* $k \in [1, N]$*, one has*

$$0 < \nu F(k, \xi) \leq \xi f(k, \xi). \tag{4.1}$$

Then, for each $\lambda \in \frac{1}{p(N+1)^{p-1}}\big]\frac{1}{a_{\gamma_1}(\delta)}, \frac{1}{a_{\gamma_2}(\delta)}\big[$*, problem* (1.1) *admits at least two non-trivial solutions* $\bar{u}_1, \bar{u}_2$*, such that*

$$\frac{2\gamma_1}{(N+1)^{(p-1)/p}} < \|\bar{u}_1\| < \frac{2\gamma_2}{(N+1)^{(p-1)/p}}. \tag{4.2}$$

Proof. Fix λ as in the conclusion. So, Theorem 3.1 ensures that problem (1.1) admits at least one non-trivial solution $\bar{u}_1$ satisfying the condition (4.2) which is a local minimum of the functional I_λ.

Now, we prove the existence of the second local minimum distinct from the first one. To this end, we must show that the functional I_λ satisfies the hypotheses of the mountain pass theorem. Clearly, the functional I_λ is of class C^1 and $I_\lambda(0) = 0$.

We can assume that $\bar{u}_1$ is a strict local minimum for I_λ in S. Therefore, there is $\rho > 0$ such that $\inf_{\|u - \bar{u}_1\| = \rho} I_\lambda(u) > I_\lambda(\bar{u}_1)$, so condition [22, (I_1), Theorem 2.2] is verified.

Integrating condition (4.1) shows that there exist constants $a_1, a_2 > 0$ such that

$$F(k,t) \geq a_1|t|^\nu - a_2$$

for all $k \in [1,N]$ and $t \in \mathbb{R}$. Now, choosing any $u \in S \setminus \{0\}$, one has

$$\begin{aligned} I_\lambda(tu) &= (\Phi - \lambda\Psi)(tu) \\ &= \frac{1}{p}\|tu\|^p - \lambda\sum_{k=1}^N F(k, tu_k) \\ &\leq \frac{t^p}{p}\|u\|^p - \lambda t^\nu a_1 \sum_{k=1}^N |u_k|^\nu + \lambda a_2 N \to -\infty \end{aligned}$$

as $t \to +\infty$, so condition [22, (I_2), Theorem 2.2] is satisfied. So, the functional I_λ satisfies the geometry of mountain pass. Moreover, by standard computations, I_λ satisfies the Palais-Smale condition. Hence, the classical theorem of Ambrosetti and Rabinowitz ensures a critical point $\bar{u}_2$ of I_λ such that $I_\lambda(\bar{u}_2) > I_\lambda(\bar{u}_1)$. So, $\bar{u}_1$ and $\bar{u}_2$ are two distinct solutions of (1.1) and the proof is complete. □

Next, as a consequence of Theorems 3.7 and 3.2, the following theorem of the existence of three solutions is obtained and its consequence for the nonlinearity with separable variables is presented.

Theorem 4.2. *Assume that* (3.8) *holds. Moreover, assume that there exist four positive constants* $\gamma, \delta, \bar{\gamma}, \bar{\delta}$, *with*

$$\frac{(2+Q)^{1/p}(N+1)^{(p-1)/p}}{2}\delta < \gamma \leq \bar{\gamma} < \frac{(2+Q)^{1/p}(N+1)^{(p-1)/p}}{2}\bar{\delta},$$

such that (3.3), (3.7) *and*

$$\frac{\sum_{k=1}^N \max_{|t|\leq\gamma} F(k,t)}{(2\gamma)^p} < \frac{\sum_{k=1}^N F(k,\bar{\delta}) - \sum_{k=1}^N \max_{|t|\leq\bar{\gamma}} F(k,t)}{(2+Q)(N+1)^{p-1}\bar{\delta}^p - (2\bar{\gamma})^p}. \tag{4.3}$$

are satisfied. Then, for each

$$\lambda \in \Lambda = \Big]\max\big\{\tilde{\lambda}, \frac{(2+Q)\delta^p}{p\sum_{k=1}^N F(k,\delta)}\big\}, \frac{(2\gamma)^p}{p(N+1)^{p-1}\sum_{k=1}^N \max_{|t|\leq\gamma} F(k,t)}\Big[,$$

problem (1.1) *admits at least three solutions.*

Proof. First, we observe that $\Lambda \neq \emptyset$ owing to (4.3). Next, fix $\lambda \in \Lambda$. Theorem 3.2 ensures a non-trivial solution $\bar{u}$ such that $\|\bar{u}\| < \frac{2\gamma}{(N+1)^{(p-1)/p}}$ which is a local minimum for the associated functional I_λ, as well as Theorem 3.7 guarantees a non-trivial solution $\tilde{u}$ such that $\|\tilde{u}\| > \frac{2\bar{\gamma}}{(N+1)^{(p-1)/p}}$ which is a local minimum for I_λ. Hence, the mountain pass theorem as given by Pucci and Serrin (see [20]) ensures the conclusion. □

Theorem 4.3. *Assume that* $g : \mathbb{R} \to \mathbb{R}$ *is a non-negative continuous function such that*

$$\limsup_{\xi\to 0^+} \frac{G(\xi)}{\xi^p} = +\infty, \tag{4.4}$$

$$\limsup_{\xi\to +\infty} \frac{G(\xi)}{\xi^p} = 0. \tag{4.5}$$

Further, assume that there exist two positive constants $\bar\gamma, \bar\delta$, with
$$\bar\gamma < \frac{(2+Q)^{1/p}(N+1)^{(p-1)/p}}{2}\bar\delta,$$
such that
$$\frac{G(\bar\gamma)}{\bar\gamma^p} < \Big(\frac{2^p}{(2+Q)(N+1)^{p-1}}\Big)\frac{G(\bar\delta)}{\bar\delta^p}. \tag{4.6}$$
Then, for each
$$\lambda \in \frac{1}{p\sum_{k=1}^N \alpha_k}\Big]\frac{(2+Q)\bar\delta^p}{G(\bar\delta)}, \frac{(2\bar\gamma)^p}{(N+1)^{p-1}G(\bar\gamma)}\Big[,$$
problem (3.6) *admits at least three non-negative solutions.*

Proof. Clearly, (4.5) implies (3.8). Moreover, by choosing δ small enough and $\gamma = \bar\gamma$, simple computations show that (4.4) implies (3.3). Finally, from (4.6) we get (3.7) and also (4.3). Hence, Theorem 4.2 ensures the conclusion. □

Finally, we present two applications of our results.

Example 4.4. Consider the problem
$$\begin{gathered} -\Delta^2 u_{k-1} + q_k u_k = \lambda\Big(\frac{1}{6} + |u_k|^2 u_k\Big), \quad k \in [1, N], \\ u_0 = u_{N+1} = 0. \end{gathered} \tag{4.7}$$
Let $g(t) = \frac{1}{6} + |t|^2 t$ for all $t \in \mathbb{R}$. Obviously, $g(0) \neq 0$. Since
$$\lim_{\xi\to 0^+} \frac{g(\xi)}{\xi} = \lim_{\xi\to 0^+} \big(\frac{1}{6\xi} + |\xi|^2\big) = +\infty,$$
condition (1.2) holds true. Choose $\nu = 3$ and $R = 1$, we have
$$0 < 3G(\xi) \le \xi g(\xi),$$
for all $|\xi| \ge 1$. Moreover, one has
$$\frac{2}{N(N+1)} \sup_{\gamma>0} \frac{\gamma^2}{\max_{|\xi|\le\gamma} G(\xi)} = \frac{2}{N(N+1)} \sup_{\gamma>0} \frac{12\gamma}{2+3\gamma^3} = \frac{6}{N(N+1)}.$$
Then, owing to Theorem 1.1, for each $\lambda \in]0, \frac{6}{N(N+1)}[$, problem (4.7) admits at least two non-trivial solutions.

Example 4.5. Consider the problem
$$\begin{gathered} -\Delta^2 u_{k-1} = \frac{1}{10}\Big(\frac{u_k^8}{e^{u_k}} + 1\Big), \quad k \in [1, 3], \\ u_0 = u_4 = 0. \end{gathered}$$
Then, owing to Theorem 1.2, it admits three positive solutions. In fact, one has
$$\lim_{\xi\to 0^+} \frac{g(\xi)}{\xi} = \lim_{\xi\to 0^+} \frac{\frac{\xi^8}{e^\xi} + 1}{\xi} = +\infty,$$
$$\lim_{\xi\to +\infty} \frac{g(\xi)}{\xi} = \lim_{\xi\to +\infty} \frac{\frac{\xi^8}{e^\xi} + 1}{\xi} = 0.$$
Moreover, taking into account that
$$G(t) = t - \frac{\sum_{i=0}^8 \frac{8!}{i!} t^i}{e^t} + 8!, \quad \forall t \in \mathbb{R},$$

one has $G(1) < \frac{1}{8}G(2)$ and $\frac{4}{3G(2)} < \frac{1}{10} < \frac{1}{6G(1)}$.

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