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HÖLDER CONTINUITY FOR A PERIODIC 2-COMPONENT μ-B SYSTEM

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Abstract. In this article, we consider the Cauchy problem of a periodic 2-component μ-b system. We show that the date to solution for the periodic 2-component μ-b system is Hölder continuous from bounded set of Sobolev spaces with exponent $s > 5/2$ measured in a weaker Sobolev norm with index $r < s$ for the periodic case.

1. Introduction

In this article, we reconsider the Cauchy problem of the following two-component periodic μ-b system

$$\begin{gathered} \mu(u)_t - u_{txx} = bu_x(\mu(u) - u_{xx}) - uu_{xxx} + \rho\rho_x, \quad t > 0, x \in \mathbb{R}, \\ \rho_t = (\rho u)_x, \quad t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad \rho(0,x) = \rho_0(x), \quad x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), \quad \rho(t,x+1) = \rho(t,x), \quad t \geq 0,\ x \in \mathbb{R}, \end{gathered} \tag{1.1}$$

where $b \in \mathbb{R}$, $\mu(u) = \int_{\mathbb{S}} u\,dx$ and $\mathbb{S} = \mathbb{R}/\mathbb{Z} := (0,1)$.

Recently, Zou [23] introduced the system

$$\begin{gathered} \mu(u)_t - u_{txx} = 2\mu(u)u_x - 2u_xu_{xx} - uu_{xxx} + \rho\rho_x - \gamma_1 u_{xxx}, \quad t > 0,\ x \in \mathbb{R}, \\ \rho_t = (\rho u)_x - 2\gamma_2\rho_x, \quad t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad \rho(0,x) = \rho_0(x), \quad x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), \quad \rho(t,x+1) = \rho(t,x), \quad t \geq 0,\ x \in \mathbb{R}, \end{gathered} \tag{1.2}$$

where $\mu(u) = \int_{\mathbb{S}} u\,dx$, $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ and $\gamma_i \in \mathbb{R}$, $i = 1,2$. By integrating both sides of the first equation in the system (1.2) over the circle $\mathbb{S}$ and using the periodicity of u, one obtains

$$\mu(u_t) = \mu(u)_t = 0,$$

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which implies the following 2-component periodic μ-Hunter-Saxton system

$$\begin{gathered} -u_{txx} = 2\mu(u)u_x - 2u_x u_{xx} - uu_{xxx} + \rho\rho_x - \gamma_1 u_{xxx}, \quad t > 0,\ x \in \mathbb{R}, \\ \rho_t = (\rho u)_x - 2\gamma_2 \rho_x, \quad t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad \rho(0,x) = \rho_0(x), \quad x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), \quad \rho(t,x+1) = \rho(t,x), \quad t \geq 0,\ x \in \mathbb{R}. \end{gathered} \tag{1.3}$$

This system is a 2-component generalization of the generalized Hunter-Saxton equation obtained in [10]. Zou [23] shows that this system is both a bi-Hamiltonian Euler equation and a bi-variational equation. Liu-Yin [14] established the local well-posedness, precise blow-up scenario and global existence result to the system (1.3).

If $b = 2$, then system (1.1) becomes the system (1.3) with $\gamma_1 = \gamma_2 = 0$. Therefore, system (1.1) generalizes system (1.3) in some sense.

If $\rho \equiv 0$, then system (1.1) becomes the system

$$\begin{gathered} \mu(u_t) - u_{xxt} + uu_{xxx} - bu_x(\mu(u) - u_{xx}) = 0, \quad t > 0,\ x \in \mathbb{S}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{S}. \end{gathered} \tag{1.4}$$

The above equation is called μ-b equation. If $b = 2$, then equation (1.4) becomes the well-known μ-CH equation. Lenells, Misiołek and Tiğlay [13] introduced the μ-CH, the μ-DP as well as μ-Burgers equations, and the μ-b equation (see also [11]). In the case $b = 3$, the μ-b equation reduces to the μ-DP equations. In addition, if $\mu(u) = 0$, they reduce to the HS and μ-Burgers equations, respectively. It is remarked that the μ-Hunter-Saxton equation has a very close relation with the periodic Hunter-Saxton and Camassa-Holm equations, that is, (1.4) will reduce to the Hunter-Saxton equation [9, 19, 21] if $\mu(u) = 0$ and $b = 2$.

The local well-posedness of the μ-CH and μ-DP Cauchy problems have been studied in [10] and [13]. Recently, Fu et. al. [3] described precise blow-up scenarios for μ-CH and μ-DP.

When $\rho \not\equiv 0$ and $\gamma_i = 0$ $(i = 1, 2)$, Constanin-Ivanov [2] considered the peakon solutions of the Cauchy problem of system (1.3). In paper [20], Wunsch studied the the Cauchy problem of 2-component periodic Hunter-Saxton system, see also [12]. The local well-posedness of system (1.1) was established in our paper [17].

Recently, some properties of solutions to the Camassa-Holm equation have been studied by many authors. Himonas et al. [5] studied the persistence properties and unique continuation of solutions of the Camassa-Holm equation, see [4, 22] for the similar properties of solutions to other shallow water equation. Himonas-Kenig [6] and Himonas et al. [7] considered the non-uniform dependence on initial data for the Camassa-Holm equation on the line and on the circle, respectively. Lv et al. [16] obtained the non-uniform dependence on initial data for μ-b equation. Lv-Wang [15] considered the system (1.1) with $\rho = \gamma - \gamma_{xx}$ and obtained the non-uniform dependence on initial data. Just recently, Chen et al. [1] and Himonas et al. [8] studied the Hölder continuity of the solution map for shallow water equations. Thompson [18] also studied the Hölder continuity for the CH system, which is obtained from (1.1) by replacing the operator $\mu - \partial_x^2$ with the operator $1 - \partial_x^2$.

Our work has been inspired by [1, 8]. In this paper, we shall study the problem (1.1). We remark that there is significant difference between system (1.1) and CH system because of the two operators $1-\partial_x^2$ and $\mu-\partial_x^2$. Moreover, the properties of u and γ are different, see Proposition 2.1. So the system (1.1) will have the properties unlike the signal equation, for example, μ-b equation. And this is different from the CH system.

This paper is organized as follows. In section 2, we will recall some known results about the well-posedness and then state out our main results. Section 3 is concerned with the proof of the main results.

Notation In this paper, the symbols $\lesssim$, $\approx$ and $\gtrsim$ are used to denote inequality/equality up to a positive universal constant. For example, $f(x) \lesssim g(x)$ means that $f(x) \leq cg(x)$ for some positive universal constant c. In the following, we denote by $*$ the spatial convolution. Given a Banach space Z, we denote its norm by $\|\cdot\|_Z$. Since all space of functions are over $\mathbb{S}$, for simplicity, we drop $\mathbb{S}$ in our notations of function spaces if there is no ambiguity. Let $[A, B] = AB - BA$ denotes the commutator of linear operator A and B. Set $\|z\|^2_{H^s\times H^{s-1}} = \|u\|^2_{H^s} + \|\rho\|^2_{H^{s-1}}$, where $z = (u, \rho)$.

2. Some known results and main result

In this section we first recall the known results, and then state out our main result.

As $\mu(u)_t = 0$ under spatial periodicity, we can re-write (1.1) as follows:

$$\begin{gathered} u_t - uu_x = \partial_x A^{-1}\Big(b\mu(u)u + \frac{3-b}{2}u_x^2 + \frac{1}{2}\rho^2\Big), \quad t > 0,\ x \in \mathbb{S}, \\ \rho_t - u\rho_x = u_x\rho, \quad t > 0,\ x \in \mathbb{S}, \\ u(0, x) = u_0(x), \quad \rho(0, x) = \rho_0(x), \quad x \in \mathbb{S}, \end{gathered} \tag{2.1}$$

where $A = \mu - \partial_x^2$ is an isomorphism between $H^s(\mathbb{S})$ and $H^{s-2}(\mathbb{S})$ with the inverse $v = A^{-1}u$ given by

$$\begin{aligned} v(x) = (\frac{x^2}{2} - \frac{x}{2} + \frac{13}{12})\mu(u) + (x - 1/2)\int_0^1\int_0^y u(s)\mathrm{d}s\mathrm{d}y \\ - \int_0^x u(s)\mathrm{d}s\mathrm{d}y + \int_0^1\int_0^y\int_0^s u(r)\mathrm{d}r\mathrm{d}s\mathrm{d}y. \end{aligned}$$

Since A^{-1} and ∂_x commute, the following identities hold:

$$A^{-1}\partial_x u(x) = (x - 1/2)\int_0^1 u(x)\mathrm{d}x - \int_0^x u(y)\mathrm{d}y + \int_0^1\int_0^x u(y)\mathrm{d}y\mathrm{d}x, \tag{2.2}$$

$$A^{-1}\partial_x^2 u(x) = -u(x) + \int_0^1 u(x)\mathrm{d}x. \tag{2.3}$$

It is easy to show that $\mu(\Lambda^{-1}\partial_x u(x)) = 0$.

Proposition 2.1 ([17, Theorem 2.1]). *Given $z_0 = (u_0, \rho_0) \in H^s \times H^{s-1}$, $s \geq 2$. Then there exists a maximal existence time $T = T(\|z_0\|_{H^s\times H^{s-1}}) > 0$ and a unique solution $z = (u, \rho)$ to system* (2.1) *such that*

$$z = z(\cdot, z_0) \in C([0, T); H^s \times H^{s-1}) \cap C^1([0, T); H^{s-1} \times H^{s-2}).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping

$$z_0 \to z(\cdot, z_0) : H^s \times H^{s-1} \to C([0, T); H^s \times H^{s-1}) \cap C^1([0, T); H^{s-1} \times H^{s-2})$$

is continuous.

Next, an explicit estimate for the maximal existence time T is given.

Proposition 2.2. *Let $s > \frac{5}{2}$. If $z = (u, \rho)$ is a solution of system (2.1) with initial data z_0 described in Proposition 2.1, then the maximal existence time T satisfies*

$$T \geq T_0 := \frac{1}{2C_s\|z_0\|_{H^s\times H^{s-1}}},$$

where C_s is a constant depending only on s. Also, we have

$$\|z(t)\|_{H^s\times H^{s-1}} \leq 2\|z_0\|_{H^s\times H^{s-1}}, \quad 0 \leq t \leq T_0.$$

Now, we state our main result.

Theorem 2.3. *Assume $s > 5/2$ and $3/2 < r < s$. Then the solution map to (2.1) with (2.2) is Hölder continuous with exponent $\alpha = \alpha(s, r)$ as a map from $B(0, h)$ with $H^r(\mathbb{S})$ norm to $C([0, T_0], H^r(\mathbb{S}))$, where T_0 is defined as in Proposition 2.2. More precisely, for initial data $(u(0), \rho(0))$ and $(\hat{u}(0), \hat{\rho}(0))$ in a ball $B(0, h) := \{u \in H^s : \|u\|_{H^s} \leq h\}$ of H^s, the solutions of (2.1) with (2.2) $(u(x, t), \rho(x, t)$ and $\hat{u}(x, t), \hat{\rho}(x, t)$ satisfy the inequality*

$$\begin{aligned} \|u(t) - \hat{u}(t)\|_{C([0,T_0];H^r)} &\leq c\|u(0) - \hat{u}(0)\|^{\alpha}_{H^r}, \\ \|\rho(t) - \hat{\rho}(t)\|_{C([0,T_0];H^r)} &\leq c\|\rho(0) - \hat{\rho}(0)\|^{\alpha}_{H^r}, \end{aligned} \tag{2.4}$$

where α is given by

$$\alpha = \begin{cases} 1 & \text{if } (s, r) \in \Omega_1, \\ s - r & \text{if } (s, r) \in \Omega_2 \end{cases} \tag{2.5}$$

and the regions Ω_1 and Ω_2 are defined by

$$\begin{aligned} \Omega_1 &= \{(s, r) : s > 5/2,\, 3/2 < r \leq s - 1\}, \\ \Omega_2 &= \{(s, r) : s > 5/2,\, s - 1 < r < s\}. \end{aligned}$$

3. Proof of Theorem 2.3

In this section, we prove Theorem 2.3 by using energy method. We shall prove that

$$\|z(t) - \hat{z}(t)\|_{C([0,T_0];H^r\times H^{r-1})} \leq c\|z(0) - \hat{z}(0)\|^{\alpha}_{H^r\times H^{r-1}},$$

where $\|z(t)\|_{H^r\times H^{r-1}} = \|u(t)\|_{H^r} + \|\rho(t)\|_{H^{r-1}}$.

We note that $\|u(0) - \hat{u}(0)\|_{H^r} > 0$ and $\|\rho(0) - \hat{\rho}(0)\|_{H^{r-1}} > 0$. Indeed, due to $r > 3/2$, it follows from Sobolev embedding $H^{\frac{1}{2}+}(\mathbb{S}) \hookrightarrow C^0(\mathbb{S})$ that

$$\|u(0) - \hat{u}(0)\|_{C^0} \lesssim \|u(0) - \hat{u}(0)\|_{H^r}.$$

Hence $u(0) \equiv \hat{u}(0)$ if $\|u(0) - \hat{u}(0)\|_{H^r} = 0$, and it follows from Proposition 2.1 that $u(x, t) = \hat{u}(x, t)$. Therefore, we will assume that $\|u(0) - \hat{u}(0)\|_{H^r} > 0$ and $\|\rho(0) - \hat{\rho}(0)\|_{H^{r-1}} > 0$. To prove Theorem 2.3, we need the following Lemmas.

Lemma 3.1 ([8, Lemma 1])**.** *If $r + 1 \geq 0$, then*

$$\|[\Lambda^r\partial_x, f]v\|_{L^2} \leq c\|f\|_{H^s}\|v\|_{H^r}$$

provided that $s > 3/2$ and $r + 1 \leq s$.

Proof of Theorem 2.3. Let $u_0(x)$, $\rho(0), \hat{u}_0(x)$, $\hat{\rho}(0) \in B(0, h)$ and $(u(x, t), \rho(x, t))$ and $(\hat{u}(x, t), \hat{\rho}(x, t))$ be the two solutions to (2.1) with initial data $(u_0(x), \rho(0))$ and $(\hat{u}_0(x), \hat{\rho}(0))$, respectively. Let

$$v = u - \hat{u}, \quad \sigma = \rho - \hat{\rho},$$

then v and σ satisfy that

$$
\begin{aligned}
v_t - \frac{1}{2}\partial_x[v(u+\hat{u})] &= -\partial_x A^{-1}\big[b\mu(u)v + b\mu(v)\hat{u} \\
&\quad + \frac{3-b}{2}\,(v_x(u+\hat{u})_x) + \frac{1}{2}\sigma(\rho+\hat{\rho})\big], \quad t>0,\ x\in\mathbb{S}, \\
\sigma_t &= (v\rho + \sigma\hat{u})_x, \quad t>0,\ x\in\mathbb{S}, \\
v(0,x) &= u_0(x) - \hat{u}_0(x), \quad x\in\mathbb{S}, \\
\sigma(0,x) &= \rho_0(x) - \hat{\rho}_0(x), \quad x\in\mathbb{S}.
\end{aligned} \tag{3.1}
$$

Let $\Lambda = (1-\partial_x)^{1/2}$. Applying Λ^r and Λ^{r-1} to both sides of the first and second equation of (3.1), then multiplying both sides by $\Lambda^r v$ and $\Lambda^{r-1}\sigma$, respectively, and integrating, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v(t)\|_{H^r}^2 \\
&= \frac{1}{2}\int_{\mathbb{S}} \Lambda^r\partial_x[v(u+\hat{u})]\cdot\Lambda^r v\mathrm{d}x - \int_{\mathbb{S}}\Lambda^r\partial_x A^{-1}\Big[b\mu(u)v + b\mu(v)\hat{u} \\
&\quad + \frac{3-b}{2}\big(v_x(u+\hat{u})_x\big) + \frac{1}{2}\sigma(\rho+\hat{\rho})\Big]\cdot\Lambda^r v\mathrm{d}x,
\end{aligned} \tag{3.2}
$$

and

$$
\frac{1}{2}\frac{d}{dt}\|\sigma(t)\|_{H^{r-1}}^2 = \int_{\mathbb{S}}\Lambda^{r-1}(v\rho+\sigma\hat{u})_x\cdot\Lambda^{r-1}\sigma\mathrm{d}x. \tag{3.3}
$$

It follows from Lemma 3.1 that

$$
\begin{aligned}
&\big|\frac{1}{2}\int_{\mathbb{S}}\Lambda^r\partial_x[v(u+\hat{u})]\cdot\Lambda^r v\mathrm{d}x\big| \\
&= \frac{1}{2}\big|\int_{\mathbb{S}}[\Lambda^r\partial_x, u+\hat{u}]v\cdot\Lambda^r v\mathrm{d}x - \int_{\mathbb{S}}(u+\hat{u})\Lambda^r\partial_x v\cdot\Lambda^r v\mathrm{d}x\big| \\
&\lesssim \big|\int_{\mathbb{S}}[\Lambda^r\partial_x, u+\hat{u}]v\cdot\Lambda^r v\mathrm{d}x\big| + \big|\int_{\mathbb{S}}(u+\hat{u})\Lambda^r\partial_x v\cdot\Lambda^r v\mathrm{d}x\big| \\
&\lesssim \big|\int_{\mathbb{S}}[\Lambda^r\partial_x, u+\hat{u}]v\cdot\Lambda^r v\mathrm{d}x\big| + \big|\int_{\mathbb{S}}\partial_x(u+\hat{u})\cdot(\Lambda^r v)^2\mathrm{d}x\big| \\
&\lesssim \|[\Lambda^r\partial_x, u+\hat{u}]v\|_{L^2}\|v(t)\|_{H^r} + \|\partial_x(u+\hat{u})\|_{L^\infty}\|v(t)\|_{H^r}^2 \\
&\lesssim (\|u+\hat{u}\|_{H^s} + \|\partial_x(u+\hat{u})\|_{L^\infty})\|v(t)\|_{H^r}^2 \\
&\lesssim (\|u+\hat{u}\|_{H^s})\|v(t)\|_{H^r}^2,
\end{aligned} \tag{3.4}
$$

where we have used the facts that $H^{\frac{1}{2}+}\hookrightarrow L^\infty$ and $s>3/2$. It is easy to show that

$$
\begin{aligned}
&\big|-b\int_{\mathbb{S}}\Lambda^r\partial_x A^{-1}[\mu(u)v+\mu(v)\hat{u}]\cdot\Lambda^r v\mathrm{d}x\big| \\
&\lesssim \|\partial_x A^{-1}[\mu(u)v+\mu(v)\hat{u}]\|_{H^r}\cdot\|v(t)\|_{H^r}.
\end{aligned} \tag{3.5}
$$

By (2.2) and (2.3), we have

$$
\begin{aligned}
\|\partial_x A^{-1}u\|_{H^r} &= \big\|\big(x-\frac{1}{2}\big)\int_0^1 u(x)\mathrm{d}x - \int_0^x u(y)\mathrm{d}y + \int_0^1\int_0^x u(y)\mathrm{d}y\mathrm{d}x\big\|_{H^r} \\
&\lesssim \|x-\frac{1}{2}\|_{H^r}\int_0^1|u(x)|\mathrm{d}x + \|u(t)\|_{H^{r-1}} + \int_0^1\int_0^x|u(y)|\mathrm{d}y\mathrm{d}x.
\end{aligned}
$$

Using the above inequality, we have

$$\begin{aligned}
&\|\partial_x A^{-1}[\mu(u)v+\mu(v)\hat{u}]\|_{H^r}\\
&\lesssim |\mu(u)|\Big(\|x-\frac{1}{2}\|_{H^r}\int_0^1|v(x)|\mathrm{d}x+\|v(t)\|_{H^{r-1}}+\int_0^1\int_0^x|v(y)|\mathrm{d}y\mathrm{d}x\Big)\\
&\quad+|\mu(v)|\Big(\|x-\frac{1}{2}\|_{H^r}\int_0^1|\hat{u}(x)|\mathrm{d}x+\|\hat{u}(t)\|_{H^{r-1}}+\int_0^1\int_0^x|\hat{u}(y)|\mathrm{d}y\mathrm{d}x\Big)\\
&\lesssim (\|u\|_{H^s}+\|\hat{u}\|_{H^s})\|v(t)\|_{H^r},
\end{aligned} \tag{3.6}$$

where we have used the inequality

$$|\mu(v)|=\big|\int_{\mathbb{S}}v(x,t)\mathrm{d}x\big|\le\int_{\mathbb{S}}|v(x,t)|\mathrm{d}x\le\|v(t)\|_{H^r}$$

provided that $r\ge 0$. Substituting (3.6) into (3.5), we obtain

$$\big|-b\int_{\mathbb{S}}\Lambda^r\partial_xA^{-1}\left[\mu(u)v+\mu(v)\hat{u}\right]\cdot\Lambda^r v\mathrm{d}x\big|\lesssim(\|u\|_{H^s}+\|w\|_{H^s})\|v(t)\|_{H^r}^2. \tag{3.7}$$

Similarly, integrating by parts, we have

$$\begin{aligned}
&\big|\frac{1}{2}\int_{\mathbb{S}}\Lambda^r\partial_xA^{-1}\left(\sigma(\rho+\hat{\rho})\right)\cdot\Lambda^r v\mathrm{d}x\big|\\
&\lesssim\|\partial_xA^{-1}\sigma(\rho+\hat{\rho})\|_{H^r}\cdot\|v(t)\|_{H^r}\\
&\lesssim\|\sigma(t)\|_{L^2}(\|\rho\|_{H^1}+\|\hat{\rho}\|_{H^1})\cdot\|v(t)\|_{H^r}\\
&\lesssim(\|\rho\|_{H^{s-1}}+\|\hat{\rho}\|_{H^{s-1}})\cdot(\|v(t)\|_{H^r}^2+\|\sigma(t)\|_{H^{r-1}}^2);
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
&\big|-\frac{3-b}{2}\int_{\mathbb{S}}\Lambda^r\partial_xA^{-1}v_x(u+\hat{u})_x\cdot\Lambda^r v\mathrm{d}x\big|\\
&\lesssim\|\partial_xA^{-1}v_x(u+\hat{u})_x\|_{H^r}\cdot\|v(t)\|_{H^r}\\
&\lesssim\|v(t)\|_{L^2}(\|u\|_{H^2}+\|\hat{u}\|_{H^2})\cdot\|v(t)\|_{H^r}\\
&\lesssim(\|u\|_{H^s}+\|\hat{u}\|_{H^s})\cdot\|v(t)\|_{H^r}^2
\end{aligned} \tag{3.9}$$

provided that $s\ge 2$. In the above inequality, we used

$$\big|\int_{\mathbb{S}}v_x(x,t)u_x(x,t)\mathrm{d}x\big|=\big|\int_{\mathbb{S}}v(x,t)u_{xx}(x,t)\mathrm{d}x\big|\le\|v(t)\|_{L^2}\|u\|_{H^2}.$$

It follows from Lemma 3.1 that

$$\begin{aligned}
&\big|\int_{\mathbb{S}}\Lambda^r(v\rho+\sigma\hat{u})_x\cdot\Lambda^r\sigma\mathrm{d}x\big|\\
&\le\|v\rho\|_{H^r}\|v(t)\|_{H^r}+\|[\partial_x\Lambda^{r-1},\hat{u}]\sigma\|_{L^2}\|\sigma(t)\|_{H^{r-1}}+\|\hat{u}_x\|_{L^\infty}\|\sigma(t)\|_{H^{r-1}}^2\\
&\lesssim(\|\hat{u}\|_{H^s}+\|\rho\|_{H^s})(\|v(t)\|_{H^r}^2+\|\sigma(t)\|_{H^{r-1}}^2),
\end{aligned} \tag{3.10}$$

where we used the fact $H^r\hookrightarrow H^s$ $(r\le s)$ again.

Lipschitz continuous Ω_1. Substituting (3.4)-(3.9) and (3.10) into (3.2) and (3.3), respectively, and adding the resulting equalities, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|v(t)\|_{H^r}^2+\|\sigma(t)\|_{H^{r-1}}^2\right)\\
&\lesssim(\|u\|_{H^s}+\|\hat{u}\|_{H^s}+\|\rho\|_{H^{s-1}}+\|\hat{\rho}\|_{H^{s-1}})(\|v(t)\|_{H^r}^2+\|\sigma(t)\|_{H^{r-1}}^2).
\end{aligned}$$

It follows from Proposition 2.2 that

$$\begin{aligned}&\|u\|_{H^s}+\|\hat{u}\|_{H^s}+\|\rho\|_{H^{s-1}}+\|\hat{\rho}\|_{H^{s-1}}\\&\lesssim\|u(0)\|_{H^s}+\|\hat{u}(0)\|_{H^s}+\|\rho(0)\|_{H^{s-1}}+\|\hat{\rho}(0)\|_{H^{s-1}}\lesssim 1\end{aligned}$$

since $u_0,\rho_0,\ \hat{u}_0,\hat{\rho}_0\in B(0,h)$. Consequently, we obtain

$$\frac{1}{2}\frac{d}{dt}\|z(t)\|^2_{H^r\times H^{r-1}}\lesssim c\|z(t)\|^2_{H^r\times H^{r-1}},$$

which implies that

$$\|z(t)\|_{H^r\times H^{r-1}}\le e^{cT_0}\|z(0)\|_{H^r\times H^{r-1}}. \tag{3.11}$$

Or equivalently

$$\|u(t)-\hat{u}(t)\|_{H^r}+\|\rho(t)-\hat{\rho}(t)\|_{H^{r-1}}\le e^{cT_0}(\|u(0)-\hat{u}(0)\|_{H^r}+\|\rho(0)-\hat{\rho}(0)\|_{H^{r-1}}). \tag{3.12}$$

In the beginning of section 3, we obtain that $\|u(0)-\hat{u}(0)\|_{H^r}>0$ and $\|\rho(0)-\hat{\rho}(0)\|_{H^r}>0$. Indeed, if $\|u(0)-\hat{u}(0)\|_{H^r}=0$ or $\|\rho(0)-\hat{\rho}(0)\|_{H^r}=0$, it follows from the Sobolev embedding Theorem and Proposition 2.1 that $u(x,t)\equiv\hat{u}(x,t)$ or $\rho(x,t)\equiv\hat{\rho}(x,t)$, respectively. Thus we can assume that

$$\|u(0)-\hat{u}(0)\|_{H^r}=O(\|\rho(0)-\hat{\rho}(0)\|_{H^{r-1}}).$$

By (3.11), we have

$$\|u(t)-\hat{u}(t)\|_{H^r}\le C(\|u(0)-\hat{u}(0)\|_{H^r}),$$

which is the desired Lipschitz continuity in Ω_1.

Hölder continuous in Ω_2. Since $s-1<r<s$, by interpolating between H^{s-1} and H^s norms, we obtain

$$\|z(t)\|_{H^r\times H^{r-1}}\le\|z(t)\|^{s-r}_{H^{s-1}\times H^{s-2}}\|z(t)\|^{r-s+1}_{H^s\times H^{s-1}}.$$

Moreover, from the Proposition 2.2, we have that

$$\|z(t)\|_{H^s\times H^{s-1}}\lesssim\|u_0\|_{H^s}+\|\hat{u}_0\|_{H^s}+\|\rho_0\|_{H^{s-1}}+\|\hat{\rho}_0\|_{H^{s-1}}\lesssim h,$$

and thus we have

$$\|z(t)\|_{H^r\times H^{r-1}}\lesssim\|z(t)\|^{s-r}_{H^{s-1}\times H^{s-2}}. \tag{3.13}$$

We see that (3.11) is valid for $r=s-1$, $s>5/2$. Therefore, applying (3.11) into (3.13), we obtain

$$\|z(t)\|_{H^r\times H^{r-1}}\lesssim\|z(0)\|^{s-r}_{H^{s-1}\times H^{s-2}},$$

which is the desired Hölder continuity (similar to the discussion in Ω_1). The proof of Theorem 2.3 is completed. □

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