Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 32, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF SOLUTIONS TO QUASILINEAR ELLIPTIC PROBLEMS WITH NONLINEARITY AND ABSORPTION-REACTION GRADIENT TERM

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Abstract. In this article we study the quasilinear elliptic problem

$$\begin{gathered} -\Delta_p u = \pm|\nabla u|^\nu + f(x,u), \quad \text{in } \Omega, \\ u \geq 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded regular domain, $p > 1$ and $0 < \nu \leq p$. Moreover, f is a nonnegative function verifying suitable hypotheses. The main goal of this work is to analyze the interaction between the gradient term and the function f to obtain existence results.

1. Introduction

In this article we will discuss existence results for a class of quasilinear elliptic problems in the form

$$\begin{gathered} -\Delta_p u = \pm|\nabla u|^\nu + f(x,u) \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain and $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, is the classical p-Laplace operator and $0 < \nu \leq p$.

The function $f : \overline{\Omega} \times [0,+\infty) \to [0,+\infty)$ is assumed to be Hölder continuous, non-decreasing, and such that

$$\text{the function } t \mapsto \frac{f(x,t)}{t^{p-1}} \text{ is non-increasing for all } x \in \overline{\Omega}, \tag{1.2}$$

$$\lim_{t\to 0} \frac{f(x,t)}{t^{p-1}} = +\infty \text{ and } \lim_{t\to+\infty} \frac{f(x,t)}{t^{p-1}} = 0 \quad \text{uniformly for } x \in \overline{\Omega}. \tag{1.3}$$

$$f(x,0) \neq 0 \tag{1.4}$$

Notice that problems with gradient term are widely studied in the literature. We can cite the leading works of Boccardo, Gallouët, Murat and their collaborators, see for instance [7],[9] and [8] and the references therein. For some recent works related to our problem, we can cite [1, 2, 4, 21, 24, 5, 25].

2000 *Mathematics Subject Classification.* 35D05, 35D10, 35J25, 35J70, 46E30, 46E35.
Key words and phrases. Quasi-linear elliptic problems; entropy solution; general growth.

Submitted January 14, 2013. Published Janaury 27, 2014.

In the particular case $p = 2$, problem (1.1) is related to the Lane-Emden-Fowler and Emden-Fowler equations, treated in many papers; we particulary cite the works of Radulescu, and his collaborators [13, 14, 15] and more recently [12, 16] and the references therein. For the case without the absence of the gradient term, we refer to [18].

When the nonlinearity is considered as an absorption term we cite [11] where the authors prove the existence of solution even when Ω is of infinite measure, and in the same direction we cite [10].

The extension to the $p-$laplacian, of the previous results obtained in the case of the laplacian, especially when using a sub-supersolution method, has a major difficulty: no general comparison principle for the operator $-\Delta_p u \pm |\nabla u|^\nu$ exist at our knowledge, and there are only few partial results in this direction. In addition, the behavior of the operator changes when considering the cases $p < 2$ and $p > 2$. We refer the reader to [22] for a general discussion about this fact.

2. Preliminaries

The next comparison principles will be used frequently in this paper, for complete proofs of the first three ones we refer to [22] and we refer to [3] for the last one.

Considering the problem

$$\begin{gathered} -\operatorname{div}(a(x,\nabla u)) + H(x,\nabla u) = f(x) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{2.1}$$

and having in mind the particular case

$$\begin{gathered} -\Delta_p u \pm |\nabla u|^q = f(x) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

with $q \le p$ we have the following result.

Theorem 2.1 ([22]). *Under the hypotheses: $q > \frac{N(p-1)}{N-1}$, $1 < p \le 2$ and*

$$f = f_1(x) + \operatorname{div}(f_2(x)) \quad \textit{where } f_1 \in L^1(\Omega),\ f_2 \in (L^{p'}(\Omega))^N \tag{2.2}$$

$$[a(x,\xi) - a(x,\eta)](\xi - \eta) \ge \alpha(|\xi|^2 - |\eta|^2)^{\frac{p-2}{2}}|\xi - \eta|^2, \quad \alpha > 0 \tag{2.3}$$

$$a(x,0) = 0 \tag{2.4}$$

$$|a(x,\xi)| \le \beta(k(x) + |\xi|^{p-1}), \quad \beta > 0,\ k(x) \in L^{p'}(\Omega) \tag{2.5}$$

$$\begin{gathered} |H(x,\xi) - H(x,\eta)| \le \gamma(b(x) + |\xi|^{q-1} + |\eta|^{q-1})|\xi - \eta|, \\ \gamma > 0, \quad b(x) \in L^r(\Omega), \end{gathered} \tag{2.6}$$

where

$$1 \le q \le p - 1 + \frac{p}{N}, \quad r \ge \frac{N(q-(p-1))}{q-1} \quad (\textit{with } r = \infty \textit{ if } q = 1).$$

If u and v are respectively sub- and super-solution of (2.1), *such as*

$$(1+|u|)^{\overline{q}-1}u \in W_0^{1,p}(\Omega), \quad (1+|v|)^{\overline{q}-1}v \in W_0^{1,p}(\Omega), \quad \overline{q} = \frac{(N-p)(q-(p-1))}{p(p-q)} \tag{2.7}$$

then $u \le v$ in Ω.

Theorem 2.2 ([22]). *Under the hypotheses: $q < \frac{N(p-1)}{N-1}$, $2 - \frac{1}{N} < p \le 2$, (2.2), 2.3, 2.4, 2.5, and*

$$\begin{gathered} |H(x,\xi) - H(x,\eta)| \le \gamma(b(x) + |\xi|^{q-1} + |\eta|^{q-1})|\xi - \eta|, \\ \gamma > 0,\ b(x) \in L^r(\Omega), \\ r > \frac{N(p-1)}{N(p-1)-(N-1)}, \quad 1 \le q < \frac{N(p-1)}{(N-1)}. \end{gathered} \tag{2.8}$$

If u and v are respectively sub- and super-solution of (2.1)*, then $u \le v$ in Ω.*

Theorem 2.3 ([22]). *Under the hypotheses: $p > 2$, $q > \frac{p}{2} + \frac{(p-1)}{N-1}$, (2.4)), (2.5), and*

$$[a(x,\xi) - a(x,\eta)](\xi - \eta) \ge \alpha(1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}|\xi - \eta|^2, \quad \alpha > 0 \tag{2.9}$$

$$|H(x,\xi) - H(x,\eta)| \le \gamma(b(x) + |\xi|^{q-1} + |\eta|^{q-1})|\xi - \eta|, \quad \gamma > 0, \tag{2.10}$$

$$b(x) \in L^N(\Omega) \quad \textit{where } 1 \le q \le \frac{p}{2} + \frac{p}{N}. \tag{2.11}$$

If u and v are respectively sub- and super-solution of (2.1)*, such as*

$$(1+|u|)^{\overline{q}-1}u \in W_0^{1,p}(\Omega), \quad (1+|v|)^{\overline{q}-1}v \in W_0^{1,p}(\Omega), \quad \overline{q} = \frac{(N-p)(q-\frac{p}{2})}{p(\frac{p}{2}+1-q)} \tag{2.12}$$

then $u \le v$ in Ω.

Theorem 2.4 ([3]). *Assume that $1 < p$ and let f be a non-negative continuous function such that $\frac{f(x,s)}{s^{p-1}}$ is decreasing for $s > 0$. Suppose that $u, v \in W_0^{1,p}(\Omega)$ are such that*

$$\begin{gathered} -\Delta_p u \ge f(x,u), \quad u > 0 in\ \Omega, \\ -\Delta_p v \le f(x,v), \quad v > 0 in\ \Omega. \end{gathered} \tag{2.13}$$

Then $u \ge v$ in Ω.

Since we are dealing with a generalized notion of solution, we recall here the definition of entropy solutions for elliptic problems.

Definition 2.5. Let u be a measurable function. We say that $u \in \mathcal{T}_0^{1,p}(\Omega)$ if $T_k(u) \in W_0^{1,p}(\Omega)$ for all $k > 0$, where

$$T_k(s) = \begin{cases} k \operatorname{sgn}(s) & \text{if } |s| \ge k, \\ s & \text{if } |s| \le k. \end{cases} \tag{2.14}$$

Let $H \in L^1(\Omega)$. Then $u \in \mathcal{T}_0^{1,p}(\Omega)$ is an entropy solution to the problem

$$\begin{gathered} -\Delta_p u = H \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{gathered} \tag{2.15}$$

if for all $k > 0$ and all $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, we have

$$\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla(T_k(u-v))\rangle = \int_\Omega H T_k(u-v). \tag{2.16}$$

We refer to [6] and [17] for more properties of entropy solutions. It is clear that if u is an entropy solution to problem (1.1), then u is a distributional solution to (1.1).

3. The absorption case

In this section we consider the problem

$$\begin{gathered} -\Delta_p u + |\nabla u|^\nu = f(x,u) \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.1}$$

Theorem 3.1. *Assume that the assumptions on f hold. If $0 < \nu \le p$, then problem (3.1) has at least one entropy solution $u \in W_0^{1,p}(\Omega)$.*

Proof. We split the proof into several steps.

Step 1: Construction of supersolution and subsolution. To obtain the existence result we will use sub-supersolution argument. Let us consider the problem

$$\begin{gathered} -\Delta_p w = f(x,w) \quad \text{in } \Omega, \\ w > 0 \quad \text{in } \Omega, \\ w = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.2}$$

Then under the hypothesis on f, problem (3.2) possesses a unique solution w which is a supersolution of (3.1). For the subsolution to problem (3.1), we consider $\underline{u} = 0$.

Finally by Theorem 2.4 we reach that $\underline{u} \le w$. To obtain the existence result we use a monotonicity argument. Since no general comparison principle is known for this kind of problems, we will consider different values of p.

The following steps 2, 3 and 4 are devoted to proving the existence of solution in the singular case, namely $p < 2$, but for different ranges of p and ν.

Step 2: Existence result for $\frac{2N}{N+1} \le p < 2$ and $1 \le \nu \le p - 1 + \frac{p}{N}$. In this case, by [22, Theorems 3.1 and 3.2] we know that a comparison principle holds for the operator $-\Delta_p u + |\nabla u|^\nu$ in the space $W_0^{1,p}(\Omega)$.

Then, we define the sequence $\{u_n\}_{n\in\mathbb{N}}$ as follows: $u_0 = \underline{u}$ and for $n \ge 1$, u_n is the solution to problem

$$\begin{gathered} -\Delta_p u_n + |\nabla u_n|^\nu = f(x,u_{n-1}) \quad \text{in } \Omega, \\ u_n > 0 \quad \text{in } \Omega, \\ u_n = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.3}$$

We claim that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is increasing in n and for all $n \ge 0$, $u_n \le w$. Notice that the last statement follows easily from Theorem 2.4. To prove the monotonicity of $\{u_n\}_{n\in\mathbb{N}}$, we will use the comparison result obtained in [22]. It is clear that u_1 solves

$$-\Delta_p u_1 + |\nabla u_1|^\nu = f(x,u_0).$$

By the definition of u_0, we obtain that

$$-\Delta_p u_1 + |\nabla u_1|^\nu \ge -\Delta_p u_0 + |\nabla u_0|^\nu.$$

Thus, by the comparison principle in [22], we reach $u_1 \ge u_0$. Let us show that $u_2 \ge u_1$. As above, u_2 satisfies

$$-\Delta_p u_2 + |\nabla u_2|^\nu = f(x,u_1).$$

Since f is a nondecreasing function, it follows that

$$-\Delta_p u_2 + |\nabla u_2|^\nu \ge -\Delta_p u_1 + |\nabla u_1|^\nu.$$

Hence $u_2 \geq u_1$. Therefore, the result follows by induction and then the claim follows.

Thus, using u_n as a test function in (3.3) and by the non decreasing property of f, we obtain that $\|u_n\|_{W_0^{1,p}(\Omega)} \leq C$. Hence we obtain the existence of $u \in W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and $u_n \to u$ strongly in $L^\sigma(\Omega)$ for all $\sigma < p^*$.

Since $\underline{u} \leq u \leq w \in L^\infty(\Omega)$, it follows that $u \in L^\infty(\Omega)$ and $u_n \to u$ strongly in $L^\sigma(\Omega)$ for all $\sigma \geq 1$.

Therefore, to have the existence result, we just have to prove that $|\nabla u_n|^\nu \to |\nabla u|^\nu$ in $L^1(\Omega)$. By the hypothesis on ν, we can see that $\nu < p$, then using $(u - u_n)$ as a test function in (3.3), it follows that

$$\begin{aligned}
&\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla u dx - \int_\Omega |\nabla u_n|^p dx + \int_\Omega |\nabla u_n|^\nu (u - u_n) dx \\
&= \lambda \int_\Omega f(x, u_{n-1})(u - u_n) dx.
\end{aligned}$$

By the Dominated Convergence Theorem and as f is assumed to be Hölder continuous, we obtain

$$\int_\Omega f(x, u_{n-1})(u - u_n) dx = o(1).$$

Now using Hölder inequality and the fact that $\nu < p$, we obtain

$$\int_\Omega |\nabla u_n|^\nu (u - u_n) dx \leq \Big(\int_\Omega |\nabla u_n|^p dx\Big)^{\nu/p} \Big(\int_\Omega (u - u_n)^{\frac{p}{p-\nu}} dx\Big)^{\frac{p-\nu}{p}} = o(1).$$

We obtain

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla u dx - \int_\Omega |\nabla u_n|^p dx = o(1).$$

Then, using Young inequality there results

$$\begin{aligned}
\int_\Omega |\nabla u_n|^p dx &= \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla u dx + o(1) \\
&\leq \frac{p-1}{p} \int_\Omega |\nabla u_n|^p + \frac{1}{p}\int_\Omega |\nabla u|^p dx + o(1).
\end{aligned}$$

Thus,

$$\int_\Omega |\nabla u_n|^p dx \leq \int_\Omega |\nabla u|^p dx + o(1).$$

It is clear that

$$\int_\Omega |\nabla u|^p dx \leq \liminf \int_\Omega |\nabla u_n|^p dx \leq \limsup \int_\Omega |\nabla u_n|^p dx \leq \int_\Omega |\nabla u|^p dx.$$

Therefore, $\|u_n\|_{W_0^{1,p}(\Omega)} \to \|u\|_{W_0^{1,p}(\Omega)}$ and then $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$. Hence the existence result follows in this case.

Step 3: Existence result for $\frac{2N}{N+1} \leq p < 2$ and $p - 1 + \frac{p}{N} \leq \nu \leq p$. In this case, to get a monotone sequence, we have to change the approximation. Since $\frac{2N}{N+1} \leq p$ then $\nu \geq 1$.

For fixed $n \in \mathbb{N}^*$, we define the sequence $\{v_{n,k}\}_{k\in\mathbb{N}}$ as follow: $v_{n,0} = \underline{u}$ and for $k \geq 1$, $v_{n,k}$ is the solution to problem

$$\begin{gathered} -\Delta_p v_{k,n} + \frac{|\nabla v_{k,n}|^\nu}{1 + \frac{1}{n}|\nabla v_{k,n}|^\nu} = f(x, v_{k-1,n}) \quad \text{in } \Omega, \\ v_{k,n} > 0 \quad \text{in } \Omega, \\ v_{k,n} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.4}$$

Let us begin by proving that the sequence $\{v_{k,n}\}_{k\in\mathbb{N}}$ is increasing in k and that $v_{k,n} \leq w$, for all $k \geq 0$. For simplicity, we set

$$H_n(\xi) = \frac{|\xi|^\nu}{1 + \frac{1}{n}|\xi|^\nu} \quad \text{where } \xi \in \mathbb{R}^N.$$

It is clear that $v_{1,n}$ solves

$$-\Delta_p v_{1,n} + H_n(\nabla v_{1,n}) = f(x, v_{0,n}).$$

By the definition of $v_{0,n}$, we obtain that

$$-\Delta_p v_{1,n} + H_n(\nabla v_{1,n}) \geq -\Delta_p v_{0,n} + H_n(\nabla v_{0,n}).$$

It is clear that H_n satisfies the hypotheses of the comparison principle in [22]. Hence we reach $v_{1,n} \geq v_{0,n}$. In the same way, and using an induction argument, we conclude that $v_{k,n} \geq v_{k-1,n}$ for all $k \in \mathbb{N}^*$.

Now, as in the proof of the previous step, using $v_{k,n}$ as a test function in (3.4) and by the hypotheses on f, we obtain that $\|v_{k,n}\|_{W_0^{1,p}(\Omega)} \leq C$. Thus we obtain the existence of $u_n \in W_0^{1,p}(\Omega)$ such that $v_{k,n} \rightharpoonup u_n$ weakly in $W_0^{1,p}(\Omega)$. As in the previous step, we can show that $v_{k,n} \to u_n$ strongly in $W_0^{1,p}(\Omega)$.

Note that by the previous computation we obtain easily that

$$v_{k,n} \geq v_{k,n+1} \quad \text{for all } k \geq 1.$$

Hence we conclude that u_n is the minimal solution to problem

$$\begin{gathered} -\Delta_p u_n + \frac{|\nabla u_n|^\nu}{1 + \frac{1}{n}|\nabla u_n|^\nu} = f(x, u_n) \quad \text{in } \Omega, \\ u_n > 0 \quad \text{in } \Omega, \\ u_n = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.5}$$

with $u_n \leq u_{n+1}$ for all $n \geq 1$. It is clear that $\underline{u} \leq u_n \leq w \in L^\infty(\Omega)$. Then, as above using u_n as a test function in (3.5), we reach that $\|u_n\|_{W_0^{1,p}(\Omega)} \leq C$ and thus, we obtain the existence of $u \in W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$.

If $\nu < p$, then we follow the above computation to reach that $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$ and the existence result holds.

If $\nu = p$, then as in Step 2, we obtain that

$$f(x, u_{n-1}) \to f(x, u) \quad \text{strongly in } L^1(\Omega).$$

We set $k_n(x) \equiv f(x, u_{n-1})$, then

$$-\Delta_p u_n + |\nabla u_n|^p = k_n(x)$$

with $k_n \to k \equiv f(x, u)$ strongly in $L^1(\Omega)$. Therefore, using the result of [23], we conclude that $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$ and the result follows.

Step 4: Existence result for $\frac{2N}{N+1} \le p < 2$ and $0 < \nu \le 1$. In this case, we adopt a new approximation of the gradient term, namely we set

$$Q_n(\xi) = (|\xi| + \frac{1}{n})^\nu \quad \text{where } \xi \in \mathbb{R}^N.$$

For fixed $n \in \mathbb{N}^*$, we define the sequence $\{v_{n,k}\}_{k\in\mathbb{N}}$ as follows: $v_{n,0} = \underline{u}$ and for $k \ge 1$, $v_{n,k}$ is the solution to problem

$$\begin{gathered} -\Delta_p v_{k,n} + Q_n(\nabla v_{k,n}) = f(x, v_{k-1,n}) \quad \text{in } \Omega, \\ v_{k,n} > 0 \quad \text{in } \Omega, \\ v_{k,n} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.6}$$

As above we have $v_{k,n} \le w$ for all $k \ge 0$. It is clear that Q_n satisfies the condition of [22, Theorems 3.1 and 3.2].

We claim that the sequence $\{v_{k,n}\}_{k\in\mathbb{N}}$ is increasing in k, for all fixed n. To prove the claim, we observe that $v_{1,n}$ solves

$$-\Delta_p v_{1,n} + Q_n(\nabla v_{1,n}) = f(x, v_{0,n}).$$

By the definition of $v_{0,n}$, we obtain that

$$-\Delta_p v_{1,n} + Q_n(\nabla v_{1,n}) \ge -\Delta_p v_{0,n} + Q_n(\nabla v_{0,n}).$$

Hence, using again the comparison principle in [22], we reach that $v_{1,n} \ge v_{0,n}$. In the same way, using an iteration argument, we conclude that $v_{k,n} \ge v_{k-1,n}$ for all $k \in \mathbb{N}^*$ and then the claim follows.

Now for fixed k, we claim that $v_{k,n} \le v_{k,n+1}$. Using the non decreasing property and the regularity of f we see that the claim follows if we can prove that $v_{1,n} \le v_{1,n+1}$.

By the definition of $v_{1,n}$ and $v_{1,n+1}$, we have

$$-\Delta_p v_{1,n} + Q_n(\nabla v_{1,n}) = -\Delta_p v_{1,n+1} + Q_{n+1}(\nabla v_{1,n+1}) \le -\Delta_p v_{1,n+1} + Q_n(\nabla v_{1,n+1}).$$

Thus, using the comparison principle of [22], we conclude that $v_{1,n} \le v_{1,n+1}$. The general result follows by induction.

Now, as in the previous steps, using $v_{k,n}$ as a test function in (3.6) and by the Hölder continuity of f, we obtain that $\|v_{k,n}\|_{W_0^{1,p}(\Omega)} \le C$. Thus, we obtain the existence of $u_n \in W_0^{1,p}(\Omega)$ such that $v_{k,n} \rightharpoonup u_n$ weakly in $W_0^{1,p}(\Omega)$ as $k \to \infty$. The compactness arguments used in the first step allow us to prove that $v_{k,n} \to u_n$ strongly in $W_0^{1,p}(\Omega)$. Hence, we find that u_n is the minimal solution to problem

$$\begin{gathered} -\Delta_p u_n + Q_n(\nabla u_n) = f(x, u_n) \quad \text{in } \Omega, \\ u_n > 0 \quad \text{in } \Omega, \\ u_n = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.7}$$

with $u_n \le u_{n+1}$ for all $n \ge 1$. It is clear that $\underline{u} \le u_n \le w \in L^\infty(\Omega)$. Then, as above, using u_n as a test function in (3.6) we obtain easily that $\|u_n\|_{W_0^{1,p}(\Omega)} \le C$. Thus, we obtain the existence of $u \in W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$. Since $\nu < p$, we conclude that $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$ as above, and the existence result follows.

Step 5: Existence result for $2 < p$ and $\nu \le p$. To deal with the degenerate case $p > 2$, we will make a perturbation in the principal part of the operator, namely for $\varepsilon > 0$, we consider the next approximating problems

$$\begin{gathered} -L_\varepsilon u + |\nabla u|^\nu = f(x,u) \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.8}$$

where

$$-L_\varepsilon u = -\operatorname{div}((\varepsilon + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u).$$

We begin by proving that problem (3.8) has a minimal solution u_ε at least for ε small. Fixed $\varepsilon > 0$, then we define w_ε to be the unique solution of problem

$$\begin{gathered} -L_\varepsilon w_\varepsilon = f(x,w_\varepsilon) \quad \text{in } \Omega, \\ w_\varepsilon > 0 \quad \text{in } \Omega, \\ w_\varepsilon = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.9}$$

(see [19] for the proof of the uniqueness result). It is clear that w_ε is a bounded supersolution to (3.8) and $\|w_\varepsilon\|_{L^\infty} \le C$ for all $\varepsilon \ge 0$. The function $\underline{u} = 0$ is aldo a subsolution of (3.8).

Now, for ε fixed we define the sequence $\{v_{n,k}\}_{k\in\mathbb{N}}$ as follows: $v_{n,0} = \underline{u}$ and for $k \ge 1$, $v_{n,k}$ is the solution to problem

$$\begin{gathered} -L_\varepsilon v_{k,n} + D_n(\nabla v_{k,n}) = f(x, v_{k-1,n}) \quad \text{in } \Omega, \\ v_{k,n} > 0 \quad \text{in } \Omega, \\ v_{k,n} = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.10}$$

where

$$D_n(\xi) = \begin{cases} \frac{|\xi|^\nu}{1+\frac{1}{n}|\xi|^\nu} & \text{if } 1 < \nu \le p \\ (|\xi| + \frac{1}{n})^\nu & \text{if } \nu \le 1. \end{cases}$$

It is clear that $v_{k,n} \le w_\varepsilon$ for all $k \ge 0$.

We claim that the sequence $\{v_{k,n}\}_{k\in\mathbb{N}}$ is increasing in k for every fixed n. To prove the claim, we observe that $v_{1,n}$ solves

$$-L_\varepsilon v_{1,n} + D_n(\nabla v_{1,n}) = f(x, v_{0,n}).$$

By the definition of $v_{0,n}$, we obtain that

$$-L_\varepsilon v_{1,n} + D_n(\nabla v_{1,n}) \ge -L_\varepsilon v_{0,n} + D_n(\nabla v_{0,n}).$$

Hence, using the comparison principle in [22, Theorem 4.1], we reach that $v_{1,n} \ge v_{0,n}$. In the same way, using an induction argument, we conclude that $v_{k,n} \ge v_{k-1,n}$ for all $k \in \mathbb{N}^*$ and then the claim follows.

Using $v_{k,n}$ as a test function in (3.10) we easily get that $\|v_{k,n}\|_{W_0^{1,p}(\Omega)} \le C$. Thus, we obtain the existence of $u_n \in W_0^{1,p}(\Omega)$ such that $v_{k,n} \rightharpoonup u_n$ weakly in $W_0^{1,p}(\Omega)$. By the compactness argument used in the Step 2, we obtain that $v_{k,n} \to u_n$ strongly in $W_0^{1,p}(\Omega)$ and u_n is the minimal solution to the problem

$$\begin{gathered} -L_\varepsilon u_n + D_n(\nabla u_n) = f(x, u_n) \quad \text{in } \Omega, \\ u_n > 0 \quad \text{in } \Omega, \\ u_n = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.11}$$

Now, we pass to the limit in n.

Using u_n as a test function in (3.11) and as f is assumed to be Hölder continuous, we find that $\|u_n\|_{W_0^{1,p}(\Omega)} \le C$. Thus, we obtain the existence of $u_\varepsilon \in W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup u_\varepsilon$ weakly in $W_0^{1,p}(\Omega)$.

If $\nu < p$, then using the compactness arguments of Step 2 and by the result of [23], we obtain that $u_n \to u_\varepsilon$ strongly in $W_0^{1,p}(\Omega)$. Hence it follows that u_ε is the minimal solution to problem

$$\begin{gathered} -L_\varepsilon u_\varepsilon + |\nabla u_\varepsilon|^\nu = f(x,u_\varepsilon) \quad \text{in } \Omega, \\ u_\varepsilon > 0 \quad \text{in } \Omega, \\ u_\varepsilon = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.12}$$

If $\nu = p$, then by the argument of the last part of Step 3 and using the compactness result of [23], we reach the strong convergence of $\{u_n\}_{n\in\mathbb{N}}$ in $W_0^{1,p}(\Omega)$. Thus, we obtain a minimal solution to (3.12) also in this case.

To finish, we just have to pass to the limit in ε. Notice that, in general, the sequence $\{u_\varepsilon\}_\varepsilon$ is not necessarily monotone in ε. Using u_ε as a test function in (3.12) we reach that $\|u_\varepsilon\|_{W_0^{1,p}(\Omega)} \le C$ and then $u_\varepsilon \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$. Since $\underline{u} \le u_\varepsilon \le w_\varepsilon \le C$, then we easily get that

$$f(x,u_\varepsilon) \to f(x,u) \text{ strongly in } L^1(\Omega).$$

Since $\nu < p$, then using a variation of the compactness result of [23], there results that $u_\varepsilon \to u$ strongly in $W_0^{1,p}(\Omega)$. Hence u solves

$$\begin{gathered} -\Delta_p u + |\nabla u|^\nu = f(x,u) \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.13}$$

and the existence result follows. It is clear that $\underline{u} \le u \le w$. ☐

4. The reaction case

In this section, we study the reaction case, namely we consider the problem

$$\begin{gathered} -\Delta_p u = f(x,u) + |\nabla u|^\nu \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{4.1}$$

with $\nu < p-1$. The main existence result reads as follows.

Theorem 4.1. *Suppose that the hypotheses made on f hold. Then, problem* (4.1) *has at least one entropy solution.*

Proof. As in the proof of Theorem 3.1, problem (4.1) has a subsolution $\underline{u} = 0$. To obtain a supersolution, we first consider problem

$$\begin{gathered} -\Delta_p u = f(x,u) + 1 \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.2}$$

By the assumptions on f, we reach that problem (4.2) has a unique positive solution $v \in \mathcal{C}^{1,\sigma}(\overline{\Omega})$ with $\sigma < 1$. Then for $C > 1$ we have

$$-\Delta_p(Cv) = C^{p-1} f(x,v) + C^{p-1}.$$

By hypothesis (1.2), we obtain $-\Delta_p(Cv) \ge f(x,Cv) + C^{p-1}$.

Since $\nu < p-1$, one can always choose C large enough to have $C^{p-1} > C^\nu |\nabla v|^\nu + 1$. Thus
$$-\Delta_p(Cv) \geq f(x, Cv) + |\nabla Cv|^\nu + 1$$
and then $\overline{u} = Cv$ is a supersolution to problem (4.1).

To prove the existence, we follow the arguments used in the previous section. By the comparison principle in Theorem 2.4 we have that $\underline{u} \leq \overline{u}$.

First case: $\frac{2N}{N+1} \leq p < 2$ and $\nu < p-1$. Since $p < 2$, then $\nu < 1$, thus as in the proof of Theorem 3.1, we obtain the existence of u_n, the minimal solution to problem
$$\begin{gathered} -\Delta_p u_n = f(x, u_n) + Q_n(\nabla u_n) \quad \text{in } \Omega, \\ u_n > 0 \quad \text{in } \Omega, \\ u_n = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{4.3}$$
where
$$Q_n(\xi) = (|\xi| + \frac{1}{n})^\nu, \quad \text{for } \xi \in \mathbb{R}^N.$$
It is clear that $\underline{u} \leq u_n \leq \overline{u}$. Using u_n as a test function in (4.3) and by the fact that $\nu < p-1$, it follows that $\|u_n\|_{W_0^{1,p}(\Omega)} \leq C$.

Then we obtain the existence of $u \in W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$. Notice that $\nu < p$, hence by the previous compactness arguments we can prove that $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$ and the existence result follows.

Second case: $2 < p$ and $\nu < p-1$. For fixed $\varepsilon > 0$ small, we claim that problem
$$\begin{gathered} -L_\varepsilon u = f(x, u) + |\nabla u|^\nu \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{4.4}$$
where
$$-L_\varepsilon u = -\operatorname{div}((\varepsilon + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u),$$
has a minimal solution u_ε, at leat for ε small such that $\underline{u} \leq u_\varepsilon \leq \overline{u}$.

Since $\underline{u}, \overline{u} \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$, then for ε small we reach that $\underline{u}$ (respectively $\overline{u}$) is a subsolution (respectively supersolution) to (4.4).

Fix an ε small enough so that the previous statement still holds true, and define
$$D_n(\xi) = \begin{cases} \frac{|\xi|^\nu}{1 + \frac{1}{n}|\xi|^\nu} & \text{if } 1 < \nu < p-1, \\ (|\xi| + \frac{1}{n})^\nu & \text{if } \nu \leq 1. \end{cases}$$
Let u_n be the minimal solution to problem
$$\begin{gathered} -L_\varepsilon u_n = f(x, u_n) + D_n(\nabla u_n) \quad \text{in } \Omega, \\ v_{k,n} > 0 \quad \text{in } \Omega, \\ v_{k,n} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.5}$$
Notice that $u_n = \lim_{k\to\infty} v_{n,k}$ where the sequence $\{v_{n,k}\}_{k\in\mathbb{N}}$ is defined as follows: $v_{n,0} = \underline{u}$ and for $k \geq 1$, $v_{k,n}$ is the solution to problem
$$\begin{gathered} -L_\varepsilon v_{k,n} = f(x, v_{k-1,n}) + D_n(\nabla v_{k,n}) \quad \text{in } \Omega, \\ v_{k,n} > 0 \quad \text{in } \Omega, \\ v_{k,n} = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

Using u_n as a test function in (4.5) and as f is a nondecreasing Hölder continuous function, we reach $\|u_n\|_{W_0^{1,p}(\Omega)} \leq C$. Thus, we obtain the existence of $u_\varepsilon \in W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup u_\varepsilon$ weakly in $W_0^{1,p}(\Omega)$. By the compactness argument in Step 2 of Theorem 3.1 we obtain that $u_n \rightharpoonup u_\varepsilon$ strongly in $W_0^{1,p}(\Omega)$ and u_ε is the minimal solution to (4.4). It is clear that $\underline{u} \leq u_\varepsilon \leq \overline{u}$, and the claim follows.

The last step is to pass to the limit in ε. Using u_ε as a test function in (4.4), we reach that $\|u_\varepsilon\|_{W_0^{1,p}(\Omega)} \leq C$ and then $u_\varepsilon \to u$ weakly in $W_0^{1,p}(\Omega)$.

Since $\nu < p$, a modification of the arguments used in the proof of Theorem 3.1, allows us to obtain that $u_\varepsilon \to u$ strongly in $W_0^{1,p}(\Omega)$. Thus u solves

$$\begin{gathered} -\Delta_p u = f(x,u) + |\nabla u|^\nu \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.6}$$

□

Remark 4.2. Observe that the condition 1.4 imposed on f to ensure that 0 is a strict subsolution, is not necessary, indeed one can drope it, and consider as subsolution the function introduced in [12], in [19] and in [20], defined by $\underline{u} = Mh(c\varphi_1)$ where M and c are positive constants to be chosen, φ_1 is the first eigenfunction of the p-laplacian and h is the solution to the differential equation

$$\begin{gathered} h''(t) = q(h(t))g(h(t)), \\ h > 0, \quad h' > 0, \\ h(0) = h'(0) = 0. \end{gathered}$$

where $q : (0,+\infty) \to (0,+\infty)$ is a non-increasing and Hölder continuous function, and $g(s)$ behaves like $\frac{1}{s^\beta}$, for some $\beta > 0$.

Acknowledgments. I am deeply grateful to Professors B. Abdellaoui and V. Radulescu, and to the anonymous referees for providing constructive comments that help in improving the contents of this article.

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