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ON VANISHING AT SPACE INFINITY FOR A SEMILINEAR HEAT EQUATION WITH ABSORPTION

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ABSTRACT. We consider a Cauchy problem for a semilinear heat equation with absorption. The initial datum of the problem is bounded and its infimum is positive. We study solutions which do not vanish in the total space at the vanishing time; they vanish only at space infinity.

1. Introduction

We consider the semilinear heat equation (with absorption)

$$u_t = \Delta u - u^{-p}, \quad x \in \mathbb{R}^d,\ t > 0 \tag{1.1}$$

supplemented by initial data

$$u(x,0) = u_0(x) > 0, \quad x \in \mathbb{R}^d, \tag{1.2}$$

with $d \geq 1$ and $p > -1$. The function u_0 is assumed to satisfy

$$u_0 \text{ is bounded and continuous in } \mathbb{R}^d, \tag{1.3}$$

$$m := \inf_{x\in\mathbb{R}^d} u_0(x) > 0. \tag{1.4}$$

In Theorem 5.5 we prove that the Cauchy problem (1.1)-(1.2) has a unique positive classical solution under the hypotheses (1.3)-(1.4). However, this solution need not exist globally in time. For a given initial datum u_0 we define

$$T(u_0) = \sup\big\{t > 0;\ \inf_{x\in\mathbb{R}^d} u(x,t) > 0\big\} < \infty$$

and call it the maximal existence time of the positive solution or the vanishing time for (1.1)-(1.2). It is clear that

$$\lim_{t\to T(u_0)} \inf_{x\in\mathbb{R}^d} u(x,t) = 0. \tag{1.5}$$

If this happens, we say that the solution vanishes at $t = T(u_0)$. Usually, *quenching* happens in the case $p > 0$ (see [18, 10, 11, 12]), while *dead-core* occurs when $-1 < p < 0$ (see [15, 16, 14, 13, 27]). Here we study *vanishing* in the case $p > -1$.

Let v be a space independent solution of (1.1) with an initial datum $m = \inf_{x\in\mathbb{R}^d} u_0(x)$. It is easily seen that the solution of the problem

$$v' = -v^{-p}, \quad \text{for } t > 0, \quad v(0) = m \tag{1.6}$$

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is expressed by

$$v(t) = \{(p+1)(T(m)-t)\}^{1/(p+1)} \quad \text{with} \quad T(m) = \frac{m^{p+1}}{p+1} \tag{1.7}$$

and

$$v(t) = \{m^{p+1} - (p+1)t\}^{1/(p+1)}. \tag{1.8}$$

It is immediate that $T(u_0) \geq T(m)$ by the comparison principle (see Theorem 5.3). Next we study the case $T(u_0) = T(m)$ in the following theorem (see [3, 4, 6, 7, 9, 26, 27, 28].

Theorem 1.1. *Assume* (1.3)-(1.4). *If there exists a sequence* $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R}^d$ *such that*

$$u_0(x + a_k) \to m \quad \textit{a.e. in } \mathbb{R}^d \textit{ as } k \to \infty, \tag{1.9}$$

then $T(u_0) = T(m)$. *Moreover, if* $u_0 \not\equiv m$, *then the solution of* (1.1)-(1.2) *does not vanish in* $\mathbb{R}^d$ *at* $t = T(m)$. *(It vanishes only at space infinity.)*

Remark 1.2. If $u_0 \not\equiv m$, then $|a_k| \to \infty$ as $k \to \infty$.

The next theorem describes the behavior of the limit of the solution of (1.1)-(1.2) as $t \to T(m)$. We prove that $u(x, T(m)) = \lim_{t \to T(m)} u(x,t)$ for every $x \in \mathbb{R}^d$ (see Lemma 3.1).

Theorem 1.3. *Under the hypotheses as Theorem 1.1,*

$$\lim_{k \to \infty} u(a_k, T(m)) = 0.$$

Finally, we consider the relation between the form of initial data and the profile of a vanishing solution at time $T(m)$ (see [5]). In [20, §2b], for the equation

$$u_t = \Delta u + f(u),$$

a subsolution and a supersolution of the form $\varphi(T(m)-t+h(x,t))$ were constructed. Here we construct a subsolution and a supersolution of the form $\varphi(T(m)-t+g(x,t))$ where $g(x,t)$ decays to zero at space infinity and

$$\varphi(s) = v(T(m) - s) = \{(p+1)s\}^{1/(p+1)}, \tag{1.10}$$

to estimate the profile at the vanishing time for (1.1)-(1.2). It is clear that

$$\varphi' = \varphi^{-p}, \quad \varphi(T(m)) = m, \quad \lim_{s \to 0} \varphi(s) = 0. \tag{1.11}$$

Let ψ be a positive function satisfying the following conditions:

$$\psi(x) \text{ is bounded and continuous in } \mathbb{R}^d, \tag{1.12}$$

$$\psi(x) > 0 \quad \text{for } x \in \mathbb{R}^d, \tag{1.13}$$

there exists a constant $C_1 > 0$ such that

$$\sup_{x \in \mathbb{R}^d} \Big[\inf_{y \in B(0,1)} \Big\{ \sup_{z \in B(y,1)} \frac{\psi(x)}{\psi(x+z)} \Big\} \Big] \leq C_1, \tag{1.14}$$

and there exist constants $a \in (0, 1/(4T(m)))$ and $C_2 > 0$ such that

$$\sup_{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d} \frac{\psi(x-y)}{\psi(x) e^{a|y|^2}} \leq C_2. \tag{1.15}$$

Here $B(x,r)$ denotes the open ball of radius r centered at x.

Theorem 1.4. *Let the hypotheses in Theorem 1.1 hold. If ψ satisfies* (1.12)-(1.15), *and*

$$C_I\psi(x) \le u_0^{p+1}(x) - m^{p+1} \le C_{II}\psi(x) \tag{1.16}$$

for some constants $C_I > 0$ and C_{II}, then there exist constants

$$C = C(C_1, C_2, a, T(m), C_I) > 0 \quad \text{and} \quad C' = C'(C_1, C_2, a, T(m), C_{II}) > 0 \tag{1.17}$$

such that the solution to (1.1)-(1.2) *satisfies*

$$C\psi^{1/(p+1)}(x) \le u(x, T(m)) \le C'\psi^{1/(p+1)}(x).$$

Remark 1.5. Theorem 1.4, we may be restated as follows. If

$$u_0^{p+1}(x) - m^{p+1} \ge C_I\psi(x) \quad (\text{or } \le C_{II}\psi(x)),$$

for some constant $C_I > 0$ (or $C_{II} > 0$), then there exists a constant $C > 0$ (or $C' > 0$) such that the solution of (1.1)-(1.2) satisfies

$$u(x, T(m)) \ge C\{\psi(x)\}^{1/(p+1)} \quad (\text{or } \le C'\{\psi(x)\}^{1/(p+1)}).$$

If ψ is a positive constant, then it satisfies (1.12)-(1.15), and the initial datum with this ψ satisfies (1.16). However, it does not satisfy the hypothesis of Theorem 1.1. In fact, the solution of (1.1)-(1.2) with such an initial datum does not vanish at $t = T(m)$. We shall show examples of ψ satisfying these hypothesis.

Example 1.6. Let f satisfy

$$f(r) = (r^2 + 1)^{-b/2},\ f(r) = e^{-br} \text{ or } f(r) = \{\log(r + e)\}^{-b} \text{ for } r \ge 0 \text{ and } b > 0. \tag{1.18}$$

Assume that $\psi(x)$ satisfies one of the following three conditions:

(1) $\psi(x) = f(|x|)$.

(2) $\psi(x) = \Theta(x/|x|) + \{1 - \Theta((x/|x|)\}\tilde{f}(|x|)$, where $\Theta(\theta) \in C^\infty(S^{d-1})$ satisfies

$$\Theta(\theta) \begin{cases} = 0, & \theta \in S^{d-1} \cap \overline{B(\theta_0, r_1)}, \\ \in (0,1), & \theta \in S^{d-1} \cap B(\theta_0, r_2) \backslash \overline{B(\theta_0, r_1)}, \\ = 1, & \theta \in S^{d-1} \backslash B(\theta_0, r_2) \end{cases}$$

with some direction $\theta_0 \in S^{d-1}$, some constants r_1, r_2 satisfying $0 < r_1 < r_2$ and

$$\tilde{f}(r) = \begin{cases} 1, & r \in [0,1), \\ f(r-1), & r \ge 1. \end{cases}$$

(3) $\psi(x) = \inf_{i\in\mathbf{N}} f(\max\{0, |r_i| - |x - a_i|\})$ with the sequence $\{a_i\}_{i=1}^\infty \subset \mathbb{R}^d$ and $\{r_n\}_{i=1}^\infty \in \mathbb{R}^+$ satisfying $\lim_{i\to\infty} |a_i| = \infty$, $r_1 < r_2 < \dots \to \infty$.

Then $\psi(x)$ satisfies (1.12)–(1.15). Moreover the solution of (1.1)-(1.2) with the u_0 satisfying (1.16) vanishes only at space infinity at $t = T(u_0)$. Here S^{d-1} denotes the $(d-1)$-dimensional unit sphere and $\overline{B}$ does the closure of B.

This article is organized as follows. In Section 2 we prove Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.3. The proof of Theorem 1.4 will be given in Section 4. In Section 5 (Appendix A) we show the existence and the uniqueness for the classical solution (1.1)-(1.2) with the initial data satisfying (1.3)-(1.4) (Theorem 5.5), and we also prove the comparison principle of the problem (Theorem 5.3). In the last section (Appendix B), we will show Lemma 3.1 about the existence and the regularity for the solution at $t = T(m)$.

2. Vanishing only at space infinity

In this section we prove Theorem 1.1. First, we show that $T(u_0) = T(m)$ (see also [30, Theorem 1]).

Lemma 2.1. *Assume* (1.3)-(1.4). *Let $p > -1$ and $d \geq 1$. If there exist sequences $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R}^d$ and $\{r_k\}_{k=1}^{\infty}$ satisfying $0 < r_1 < r_2 < \ldots \to \infty$ such that*

$$\lim_{k\to\infty} \|u_0 - m\|_{L^\infty(B(a_k,r_k))} = 0, \tag{2.1}$$

then the solutions u and v of (1.1)-(1.2) *with initial data u_0 and m satisfy*

$$\lim_{k\to\infty} \|u(\cdot,t) - v(t)\|_{L^\infty(B(a_k,r_k/2))} = 0$$

for any $t \in (0, T(m))$. Moreover $T(u_0) = T(m)$.

Proof. Put $\tilde{u} = u - v$ and $\tilde{u}_0 = u_0 - m$. By Theorem 5.3, $\tilde{u} \geq 0$ for $(x,t) \in \mathbb{R}^d \times (0, T(m))$. It is clear that $\tilde{u}$ satisfies

$$\begin{gathered} \tilde{u}_t = \Delta\tilde{u} - (u^{-p} - v^{-p}), \quad x \in \mathbb{R}^d,\ 0 < t < T(m), \\ \tilde{u}(x,0) = \tilde{u}_0(x), \quad x \in \mathbb{R}^d. \end{gathered} \tag{2.2}$$

From (2.1) for any $\varepsilon > 0$ there exists $k_0 > 0$ such that for any $k \geq k_0$,

$$\|\tilde{u}_0\|_{L^\infty(B(a_k,r_k))} < \varepsilon^2. \tag{2.3}$$

Take $t_0 \in (0, T(m))$. By the mean value theorem we have

$$-(u^{-p} - v^{-p}) = \int_0^1 p\{\theta u + (1-\theta)v\}^{-p-1}\tilde{u}\,d\theta \leq \max\{0,p\}\{v(t_0)\}^{-p-1}\tilde{u}$$

for $t \in (0, t_0)$. Put $K = K(t_0) = \max\{0,p\}(v(t_0))^{-p-1}$. Thus

$$\begin{gathered} \tilde{u}_t \leq \Delta\tilde{u} + K\tilde{u}, \quad x \in \mathbb{R}^d, 0 < t < t_0, \\ \tilde{u}(x,0) = \tilde{u}_0(x), \quad x \in \mathbb{R}^d. \end{gathered}$$

The solution of

$$\begin{gathered} \bar{u}_t = \Delta\bar{u} + K\bar{u}, \quad x \in \mathbb{R}^d, 0 < t < t_0, \\ \bar{u}(x,0) = \tilde{u}_0(x), \quad x \in \mathbb{R}^d \end{gathered}$$

is a supersolution of (2.2). The solution $\bar{u}$ is

$$\begin{aligned} \bar{u}(x,t) &= e^{Kt}\int_{\mathbb{R}^d} G(x-y,t)\tilde{u}_0(y)dy \\ &= e^{Kt}\Big(\int_{\mathbb{R}^d\setminus B(x,r_k/2)} + \int_{B(x,r_k/2)}\Big)G(x-y,t)\tilde{u}_0(y)dy \\ &= I + II, \end{aligned}$$

where $G(x,t)$ is the Green kernel of the heat equation given by

$$G(x,t) = \frac{1}{(4\pi t)^{d/2}}e^{-|x|^2/4t}. \tag{2.4}$$

From the definition of G we see that

$$\begin{aligned} I &\leq e^{Kt_0}\int_{\mathbb{R}^d\setminus B(x,r_k/2)} G(x-y,t)\tilde{u}_0(y)dy \\ &\leq e^{Kt_0}\|\tilde{u}_0\|_{L^\infty(\mathbb{R}^d)}\int_{\mathbb{R}^d\setminus B(0,r_k/2)} G(y,t)dy < \frac{\varepsilon}{2} \end{aligned} \tag{2.5}$$

for any k large enough, where $B(x,r)$ denotes the open ball of radius r centered at x. Note that $y \in B(a_k, r_k)$ whenever $x \in B(a_k, r_k/2)$ and $y \in B(x, r_k/2)$. Thus by (2.3) we obtain

$$\begin{aligned} II &\le e^{Kt_0} \int_{B(x,r_k/2)} G(x-y,t)\tilde{u}_0(y)dy \\ &\le e^{Kt_0} \|\tilde{u_0}\|_{L^\infty(B(x,r_k/2))} \int_{B(x,r_k/2)} G(x-y,t)dy \\ &\le e^{Kt_0} \|\tilde{u_0}\|_{L^\infty(B(a_k,r_k))} \int_{\mathbb{R}^d} G(x-y,t)dy < \frac{\varepsilon}{2} \end{aligned} \tag{2.6}$$

for any $x \in B(a_k, r_k/2)$ and any $\varepsilon \in (0, e^{-Kt_0}/2)$ with k large enough. We thus have

$$\lim_{k\to\infty} \|\bar{u}(\cdot,t)\|_{L^\infty(B(a_k,r_k/2))} = 0, \quad \text{for } t \in (0,t_0).$$

Hence, by Theorem 5.3,

$$\lim_{k\to\infty} \|\tilde{u}(\cdot,t)\|_{L^\infty(B(a_k,r_k/2))} = 0, \quad \text{for } t \in (0,t_0).$$

Since $t_0 \in (0, T(m))$ is arbitrary,

$$\lim_{k\to\infty} \|\tilde{u}(\cdot,t)\|_{L^\infty(B(a_k,r_k/2))} = 0, \quad \text{for } t \in (0,T(m)). \tag{2.7}$$

Next we show that $T(u_0) = T(m)$. Let us assume, to the contrary, that there exists a constant $L > 0$ such that

$$\inf_{t\in(0,T(m))} \Big[\inf_{k\in\mathbf{N}} \Big\{ \operatorname{ess\,sup}_{x\in B(a_k,r_k/2)} u(x,t) \Big\} \Big] \ge L. \tag{2.8}$$

Since $v_t \le 0$ and $\lim_{t\to T(m)} v(t) = 0$, there exists $T_0 \in [0, T(m))$ such that

$$v(T_0) \le \frac{L}{3}.$$

From (2.7) there exists a constant $k_0 \ge 0$ such that

$$\sup_{k\ge k_0} \|u(\cdot,T_0) - v(T_0)\|_{L^\infty(B(a_k,r_k/2))} \le \frac{L}{3}.$$

From (2.8), we see that

$$\begin{aligned} &\sup_{k\ge k_0} \{\|u(\cdot,T_0) - v(T_0)\|_{L^\infty(B(a_k,r_k/2))}\} \\ &= \sup_{k\ge k_0} \Big\{ \operatorname{ess\,sup}_{x\in B(a_k,r_k/2)} u(x,T_0) - v(T_0) \Big\} \\ &\ge L - \frac{L}{3} = \frac{2L}{3} > \frac{L}{3}. \end{aligned}$$

This is a contradiction. We thus conclude that

$$\inf_{t\in[0,T(m))} \Big[\inf_{k\in\mathbf{N}} \Big\{ \operatorname{ess\,sup}_{x\in B(a_k,r_k/2)} u(x,t) \Big\} \Big] = 0 \tag{2.9}$$

and $T(u_0) \le T(m)$. By Theorem 5.3, we see that $T(u_0) \ge T(m)$. We thus obtain $T(u_0) = T(m)$. □

The next lemma shows that the solution of (1.1)-(1.2) does not vanish in $\mathbb{R}^d$ even at the vanishing time. The lemma is shown by using the argument in [22, Lemma 2.3] (see also [30]).

Lemma 2.2. *Let $u(x,t)$ be a solution of (1.1)-(1.2) in $\mathbb{R}^d \times [0, T(m))$ with m defined in (1.4). Suppose that there exist $t_0 \in (0, T(m))$, $a \in \mathbb{R}^d$, $r_0 > 0$ and $\theta > 1$ such that*

$$u(x,t) \ge \theta\varphi(T(m) - t) \quad \text{in } |x-a| < r_0,\ t_0 \le t < T,$$

where φ is defined in (1.10). Then u does not vanish at $t = T(m)$ in a neighborhood of a.

Proof. For convenience we let $T = T(m)$. We shall construct a suitable supersolution. Put $\varepsilon > 0$ and $\tilde\theta = \tilde\theta(\varepsilon) \in (1, \theta)$ satisfy

$$\tilde\theta\varphi(T - t_0 + \varepsilon/2) \le \theta\varphi(T - t_0). \tag{2.10}$$

Define

$$\omega(x,t) = \tilde\theta\varphi(T - t + h(r)),$$

where $r = |x-a|$ and

$$h(r) = \varepsilon\Big(\frac{1 + \cos\frac{\pi r}{r_0}}{2}\Big) = \varepsilon\Big\{\cos\Big(\frac{\pi r}{2r_0}\Big)\Big\}^2.$$

Thus, from (1.10) and (1.11) we have

$$\begin{aligned}\omega_t - \Delta\omega + \omega^{-p} &= -\tilde\theta\varphi' - \tilde\theta\varphi'\Delta h - \tilde\theta\varphi''|\nabla h|^2 + (\tilde\theta\varphi)^{-p} \\ &= \tilde\theta(-\varphi^{-p})\Big\{1 + \Delta h + \frac{\varphi''}{\varphi'}|\nabla h|^2 - \tilde\theta^{-p-1}\Big\},\end{aligned}$$

where

$$\nabla h = h_r\nabla r = h_r\frac{x-a}{r},$$
$$\Delta h = \operatorname{div}(\nabla h) = h_{rr} + \frac{d-1}{r}h_r.$$

Since $\varphi'' = -p\varphi^{-p-1}\varphi'$, there exists $t_0 \in (0, T)$ such that for $t \in (t_0, T)$

$$\begin{aligned}1 + \Delta h + \frac{\varphi''}{\varphi'}|\nabla h|^2 - \tilde\theta^{-p-1} &= 1 + \Delta h - p\varphi^{-p-1}|\nabla h|^2 - \tilde\theta^{-p-1} \\ &\ge (1 - \tilde\theta^{-p-1}) + \Delta h - \frac{p|\nabla h|^2}{(p+1)(T-t+h)} \\ &\ge (1 - \tilde\theta^{-p-1}) + \Big(h_{rr} + \frac{d-1}{r}h_r\Big) - \frac{p|\nabla h|^2}{(p+1)h}.\end{aligned} \tag{2.11}$$

We thus conclude that

$$\begin{gathered}\omega_t \le \Delta\omega - \omega^{-p}, \quad |x-a| < r_0, t_0 \le t < T, \\ \omega(x,t_0) \le u(x,t_0), \quad |x-a| < r_0, \\ \omega(x,t) \le u(x,t), \quad |x-a| = r_0, t_0 \le t < T\end{gathered} \tag{2.12}$$

for any $\varepsilon > 0$ sufficient small.

By Theorem 5.3, for $x \in B(a, r_0)$ and $t \in [t_0, T)$ we have $u(x,t) \ge \omega(x,t)$. Since φ is an increasing function, we obtain

$$u(x,t) \ge \tilde\theta\varphi\Big(T - t + h\big(\frac{r_0}{2}\big)\Big) = \tilde\theta\varphi\big(T - t + \frac{\varepsilon}{2}\big) \quad \text{for } (x,t) \in B(a, r_0/2) \times [t_0, T).$$

From (2.10), we see that

$$\tilde\theta\varphi(T - t + \varepsilon/2) \le \theta\varphi(T - t) \quad \text{for } t \in (t_0, T).$$

Since

$$u(x,t) \ge \theta\varphi(T-t) \quad \text{for } (x,t) \in B(a,r_0/2) \times [t_0,T),$$

we have

$$u(x,t) \ge \tilde{\theta}\varphi(T-t+\frac{\varepsilon}{2}) \quad \text{for } (x,t) \in B(a,r_0/2) \times [t_0,T)$$

and u does not vanish at $t = T$ in $B(a,r_0/2)$. □

Next we show that the condition on u_0 in Theorem 1.1 is equivalent to the one in Lemma 2.1.

Lemma 2.3. *Assume* (1.3)-(1.4). *Condition* (1.9) *is equivalent to condition* (2.1) *for sequences* $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R}^d$ *and* $\{r_k\}_{k=1}^{\infty}$ *satisfying* $0 < r_1 < r_2 < \ldots \to \infty$.

Proof. If (1.9) is assumed, since $B(0,r_k) \subset \mathbb{R}^d$ for any $k \in \mathbf{N}$, we see that

$$\lim_{k \to\infty} \|u_0(x+a_k) - m\|_{L^\infty(B(0,r_k))} = 0,$$

which gives (2.1).

Assuming that (2.1) holds, for $x_0 \in \mathbb{R}^d$ we let $k_0 = k_0(x_0) > 0$ be such that $B(x_0,1) \subset B(0,r_k)$ for $k \ge k_0$. Since

$$\lim_{k\to\infty} \|u_0 - m\|_{L^\infty(B(a_k,r_k))} = 0,$$

we have

$$u_0(x+a_k) \to m \quad \text{a.e. in } B(x_0,1) \text{ as } k \to \infty.$$

Since $x_0 \in \mathbb{R}^d$ is arbitrary, we obtain (1.9). □

Finally, we shall prove that the vanishing occurs only at space infinity by using Lemma 2.2.

Proof of Theorem 1.1. Lemmas 2.1 and 2.3 yield $T(u_0) = T(m)$. Let $T = T(u_0) = T(m)$. We need to show that for any $a \in \mathbb{R}^d$ there exist $t_0 \in (0,T)$, $r_0 > 0$ and $\theta > 1$ such that for $x \in B(a,r_0)$ and $t \in [t_0,T)$

$$u(x,t) \ge \theta\varphi(T-t).$$

From the strong maximum principle (or Theorem 5.4), we obtain

$$u(x,t) > v(t) \quad \text{for } (x,t) \in D \times (0,T)$$

for any compact set $D \subset \mathbb{R}^d$. We thus may let $u_0(x) > m$ for $x \in B(a,r_0)$ without loss of generality. Let $w(x,t)$ be a solution of

$$\begin{gathered} w_t = \Delta w, \quad x \in B(a,r_0), t > 0, \\ w(x,t) = 1, \quad x \in \partial B(a,r_0), t \ge 0, \\ 1 \le w(x,0) \le u_0(x)/m, \quad x \in B(a,r_0), \\ w(x,0) \not\equiv 1, \quad x \in B(a,r_0). \end{gathered} \tag{2.13}$$

It is clear that $vw \le u$ on $\partial B(a,r_0) \times (0,T)$ and $B(a,r_0) \times \{0\}$. From (2.13) we obtain

$$(vw)_t = -v^{-p}w + v\Delta w \le -(vw)^{-p} + \Delta(vw).$$

Then for any $a \in \mathbb{R}^d$ and any $r_0 > 0$, vw is a subsolution of (1.1)-(1.2) in $B(a,r_0)$. Thus, by the strong maximum principle, for any $(x,t) \in B(a,r_0) \times (0,T)$, we see that $w(x,t) > 1$. In particular, for any $\tilde{r}_0 \in (0,r_0)$ there exist $\theta > 1$ and $t_0 \in (0,T)$ such that

$$w(x,t) \ge \theta, \quad |x-a| < \tilde{r}_0,\ t_0 \le t < T.$$

This implies

$$u(x,t) \geq \theta\varphi(T-t), \quad |x-a| < \tilde{r}_0,\ t_0 \leq t < T$$

by the comparison principle. By Lemma 2.2, u does not vanish in a neighborhood of a. Since $a \in \mathbb{R}^d$ is arbitrary, it does not do in $\mathbb{R}^d$. □

3. Behavior at vanishing time

In this section we prove Theorem 1.3. The proof for the theorem uses the argument of the proof in [30, Theorem 3]. First we introduce a lemma on the existence for the solution to (1.1)-(1.2) at $t = T(m)$.

Lemma 3.1. *Assume the same hypotheses as in Theorem 1.1. Then $u(x,T) = \lim_{t\to T} u(x,t)$ exists for any $x \in \mathbb{R}^d$ with $T = T(m)$. Moreover $u(x,T) \in C^\infty(\mathbb{R}^d)$.*

The proof of this lemma shall be shown in Appendix B. Now we proceed with the proof of Theorem 1.3.

Proof of Theorem 2.2. It is clear in the case $u_0 \equiv m$. We should only consider the case $u_0 \not\equiv m$. Let $\{r_k\}_{k=1}^\infty$ be as defined in Lemma 2.1. Let $\varepsilon > 0$ be sufficiently small so that $\varepsilon < b - m$, where $b = \sup_{x\in\mathbb{R}^d} u_0(x)$. Let

$$u_0^{k,\varepsilon}(x) = \begin{cases} m+\varepsilon, & |x-a_k| < r_k - 1, \\ (b-m-\varepsilon)(|x-a_k| - r_k) + b, & r_k - 1 \leq |x-a_k| < r_k, \\ b, & |x-a_k| \geq r_k, \end{cases} \tag{3.1}$$

and $u^{k,\varepsilon}$, v^ε be solutions of (1.1)-(1.2) with initial data $u_0^{k,\varepsilon}$ and $m+\varepsilon$. We write $T^\varepsilon = T(m+\varepsilon)$ for simplicity.

From Lemma 2.1 for any $\varepsilon > 0$ there exists a natural number $k_0 \in \mathbf{N}$ such that for any $k > k_0$, if $x \in B(a_k, r_k/2), t \in (0, T^\varepsilon)$, then

$$v^\varepsilon(t) + \varepsilon \geq u^{k,\varepsilon}(x,t). \tag{3.2}$$

By the comparison principle (see Theorem 5.3) for any $x \in \mathbb{R}^d$ and any $t \in (0, T^\varepsilon)$,

$$u^{k,\varepsilon}(x,t) \geq u(x,t). \tag{3.3}$$

Since $T(m) < T^\varepsilon$, by Lemma 3.1, (3.2) and (3.3), for any $\varepsilon > 0$ there exists $k_0 = k_0(\varepsilon) \in \mathbf{N}$ such that for any $k > k_0$

$$v^\varepsilon(T(m)) + \varepsilon \geq u(a_k, T(m)). \tag{3.4}$$

Since $\varepsilon > 0$ can be chosen arbitrarily small and $\lim_{\varepsilon\to 0} v^\varepsilon(T(m)) = 0$, (3.4) implies

$$\lim_{k\to\infty} u(a_k, T(m)) = 0.$$

□

4. Profile at vanishing

To prove Theorem 1.4, we construct a subsolution and a supersolution of the form $\varphi(T - t + g(x,t))$ with $T = T(m)$, as we have explained before. This is a modification of the method employed in [20] and [29] to study blow-up profile for a semilinear heat equation. Let

$$g^\gamma_{\alpha,\beta}(x,t) = g^{\gamma,\psi}_{\alpha,\beta}(x,t) = \int_{\mathbb{R}^d} G^\gamma_{\alpha,\beta}(x-y,t)\psi(y)dy, \tag{4.1}$$

where

$$G^{\gamma}_{\alpha,\beta}(x,t)=\frac{|x|^{\beta}}{(t+\gamma)^{\alpha-d/2}}G(x,t+\gamma)=\frac{|x|^{\beta}}{(t+\gamma)^{\alpha}}\exp\Big(-\frac{|x|^2}{4(t+\gamma)}\Big)$$

with $\alpha\in\mathbb{R}$, $\beta\geq 0$, $\gamma>0$ are constants (see [5]). Note that $g^{\gamma}_{\alpha,\beta}$ also may be expressed as

$$g^{\gamma}_{\alpha,\beta}(x,t)=\int_{\mathbb{R}^d}G^{\gamma}_{\alpha,\beta}(y,t)\psi(x-y)dy.$$

It is easily seen that:

$$|\nabla g^{\gamma}_{\alpha,0}|\leq\frac{\sqrt{d}g^{\gamma}_{\alpha+1,1}}{2},\tag{4.2}$$

$$\Delta g^{\gamma}_{\alpha,0}=\frac{g^{\gamma}_{\alpha+2,2}}{4}-\frac{dg^{\gamma}_{\alpha+1,0}}{2},\tag{4.3}$$

$$\partial_t g^{\gamma}_{\alpha,0}=\frac{g^{\gamma}_{\alpha+2,2}}{4}-\alpha g^{\gamma}_{\alpha+1,0},\tag{4.4}$$

$$g^{\gamma}_{\alpha,\beta}(x,t)=\frac{g^{\gamma}_{0,\beta}}{(t+\gamma)^{\alpha}}.\tag{4.5}$$

Before proving Theorem 1.4 we prove the next two propositions.

Proposition 4.1. *Assume that $p>-1$. Let ψ be a positive bounded continuous function satisfying* (1.12)-(1.15) *and*

$$\gamma\in\Big(0,\frac{1}{4a}-T\Big).\tag{4.6}$$

Then for any $C>0$ the function

$$W(x,t)=\varphi(T-t+Cg^{\gamma}_{-\alpha,0}(x,t))\tag{4.7}$$

is a supersolution of (1.1) *in $\mathbf{R}^d\times(0,T)$ for α satisfies $\alpha\geq\alpha_1$ with some constant $\alpha_1=\alpha_1(p,d,C_1,C_2,a,T,\gamma)>0$, where φ is defined in* (1.10).

Proposition 4.2. *Assume the same hypotheses as in Proposition 4.1. Then, for each constant $C>0$, the function*

$$w(x,t)=\varphi(T-t+Cg^{\gamma}_{\alpha,0}(x,t))\tag{4.8}$$

is a subsolution of (1.1) *in $\mathbb{R}^d\times(0,T)$ provided that α satisfies $\alpha\geq\alpha_2$ with some constant $\alpha_2=\alpha_2(p,d,C_1,C_2,a,T,\gamma)>0$.*

Before proving Propositions 4.1 and 4.2, we need one lemma on estimates for $g^{\gamma}_{0,\beta}$.

Lemma 4.3. *Assume the same hypotheses as in Propositions 4.1 and 4.2. Then for $\beta=0$, 1, 2, there exist constants $C_3=C_3(C_1,\gamma)>0$ and $C_4=C_4(C_2,a,T,\gamma)>0$ such that*

$$C_3\psi(x)\leq g^{\gamma}_{0,\beta}(x,t)\leq C_4\psi(x)\quad\text{in }\mathbb{R}^d\times[0,T],$$

where C_1 and C_2 are in (1.14) *and* (1.15)*, respectively.*

Proof. First we show $g^{\gamma}_{0,\beta}(x,t)\geq C_3\psi(x)$ with some $C_3>0$. From (1.14) we see that there exists a vector $q\in B(0,\min\{1,2/\sqrt{d}\})$ such that

$$\psi(x)\leq 2C_1\inf_{z\in B(q+x,1)}\psi(z)\tag{4.9}$$

for each $x \in \mathbb{R}^d$. If (4.9) holds, then we see that

$$\begin{aligned} g^\gamma_{0,\beta}(x,t) &\geq \inf_{z\in B(q+x,1)} \psi(z) \times \int_{B(q,1)} |y|^\beta \exp\Big(-\frac{|y|^2}{4\gamma}\Big) dy \\ &\geq \psi(x) \frac{1}{2C_1} \int_{B(q,1)} |y|^\beta \exp\Big(-\frac{|y|^2}{4\gamma}\Big) dy. \end{aligned}$$

Since $|y|^\beta \exp(-|y|^2/4\gamma)$ is radially symmetric and $|q| < 2/\sqrt{d}$, we have

$$\begin{aligned} \int_{B(q,1)} |y|^\beta \exp\Big(-\frac{|y|^2}{4\gamma}\Big) dy &= \int_{B(\tilde{q},1)} |y|^\beta \exp\Big(-\frac{|y|^2}{4\gamma}\Big) dy \\ &\geq \int_{B(0,1)\cap[0,1]^d} |y|^\beta \exp\Big(-\frac{|y|^2}{4\gamma}\Big) dy, \end{aligned}$$

where $\tilde{q} = \big(\frac{|q|}{\sqrt{d}}, \frac{|q|}{\sqrt{d}}, \dots, \frac{|q|}{\sqrt{d}}\big)$. We thus obtain

$$\begin{aligned} g^\gamma_{0,\beta}(x,t) &\geq \psi(x) \frac{1}{2C_1} \int_{B(0,1)\cap[0,1]^d} |y|^\beta \exp\Big(-\frac{|y|^2}{4\gamma}\Big) dy \\ &\geq \psi(x) \frac{1}{2^{d+1}C_1} \int_{B(0,1)} |y|^\beta \exp\Big(-\frac{|y|^2}{4\gamma}\Big) dy. \end{aligned}$$

Let

$$\begin{aligned} C_3 &= \min_{\beta=0,1,2} \frac{2^{-1-d}}{C_1} \int_{B(0,1)} |y|^\beta \exp\Big(-\frac{|y|^2}{4\gamma}\Big) dy \\ &= \frac{2^{-1-d}}{C_1} \int_{B(0,1)} |y|^2 \exp\Big(-\frac{|y|^2}{4\gamma}\Big) dy. \end{aligned} \tag{4.10}$$

Then we have

$$g^\gamma_{0,\beta}(x,t) \geq C_3 \psi(x)$$

for any $x \in \mathbb{R}^d$.

We next prove $g^\gamma_{0,\beta}(x,t) \leq C_4\psi(x)$ with some $C_4 > 0$. In addition to satisfying (4.6) we may assume that

$$\frac{1}{4(T+\gamma)} - a > 0.$$

We thus see that from (1.15),

$$g^\gamma_{0,\beta}(x,t) \leq C_2\psi(x) \int_{\mathbb{R}^d} |y|^\beta \exp\Big\{-\Big(\frac{1}{4(T+\gamma)} - a\Big)|y|^2\Big\} dy$$

for $t \in [0,T]$. By putting

$$C_4 = \max_{\beta=0,1,2} C_2 \int_{\mathbb{R}^d} |y|^\beta \exp\Big\{-\Big(\frac{1}{4(T+\gamma)} - a\Big)|y|^2\Big\} dy,$$

we obtain $g^\gamma_{0,\beta}(x,t) \leq C_4\psi(x)$ for $t \in [0,T]$. $\square$

Proof of Proposition 4.1. By a direct calculation we have

$$\begin{aligned} W_t - \Delta W + W^{-p} &= -\varphi' + C\varphi' \partial_t g^\gamma_{-\alpha,0} \\ &\quad - C\varphi' \Delta g^\gamma_{-\alpha,0} - |C\nabla g^\gamma_{-\alpha,0}|^2 \varphi'' + \varphi^{-p}. \end{aligned} \tag{4.11}$$

From (4.3)-(4.4) we have

$$(\partial_t - \Delta) g^\gamma_{-\alpha,0} = \Big(\alpha + \frac{d}{2}\Big) g^\gamma_{-\alpha+1,0}. \tag{4.12}$$

Since

$$\varphi' = \varphi^{-p}, \tag{4.13}$$

$$\varphi'' = -p\varphi^{-2p-1} \tag{4.14}$$

and

$$\varphi^{p+1} = (p+1)(T - t + Cg^{\gamma}_{-\alpha,0}) \geq (p+1)(Cg^{\gamma}_{-\alpha,0}),$$

we obtain

$$|C\nabla g^{\gamma}_{-\alpha,0}|^2\varphi'' \leq \frac{C|p||\nabla g^{\gamma}_{-\alpha,0}|^2}{(p+1)g^{\gamma}_{-\alpha,0}}\varphi'. \tag{4.15}$$

Lemma 4.3, (4.2) and (4.5) yield

$$|C\nabla g^{\gamma}_{-\alpha,0}|^2\varphi'' \leq \frac{C|p|d(g^{\gamma}_{-\alpha+1,1})^2}{4(p+1)g^{\gamma}_{-\alpha,0}}\varphi' \leq \frac{C|p|dC_4^2(t+\gamma)^{\alpha-2}\psi}{4C_3(p+1)}\varphi'. \tag{4.16}$$

From (4.13), (4.12), (4.16) and Lemma 4.3, we have

$$\begin{aligned} W_t - \Delta W + W^{-p} &\geq \Big\{C_3(t+\gamma)^{\alpha-1}\Big(\alpha + \frac{d}{2}\Big) - \frac{C|p|dC_4^2(t+\gamma)^{\alpha-2}}{4C_3(p+1)}\Big\}\varphi' \\ &\geq \Big\{C_3\gamma\Big(\alpha + \frac{d}{2}\Big) - \frac{C|p|dC_4^2}{4C_3(p+1)}\Big\}(t+\gamma)^{\alpha-2}\varphi'. \end{aligned}$$

If α satisfies

$$\alpha \geq \alpha_1 \equiv \max\Big\{\frac{CC_4^2|p|d}{4C_3^2\gamma(p+1)} - \frac{d}{2}, \frac{1}{2}\Big\} > 0, \tag{4.17}$$

then W is a supersolution of (1.1) in $\mathbb{R}^d \times (0,T)$. □

Proof of Proposition 4.2. As before, for $\varphi = \varphi(T - t + Cg^{\gamma}_{\alpha,0}(x,t))$ we have

$$\begin{aligned} w_t - \Delta w + w^{-p} &= -\varphi' + C\varphi'\partial_t g_{\alpha,0} - C\varphi'\Delta g^{\gamma}_{\alpha,0} - |C\nabla g^{\gamma}_{\alpha,0}|^2\varphi'' + \varphi^{-p} \\ &\leq \frac{C\partial_t g^{\gamma}_{\alpha,0}}{\varphi^p} + \frac{C|\Delta g^{\gamma}_{\alpha,0}|}{\varphi^p} + \frac{|Cp\nabla g^{\gamma}_{\alpha,0}|^2}{\varphi^{2p+1}} \end{aligned} \tag{4.18}$$

by (4.13)-(4.14). It is easily seen that

$$\varphi^{p+1} = (p+1)(T - t + Cg^{\gamma}_{\alpha,0}) \geq (p+1)(Cg^{\gamma}_{\alpha,0}). \tag{4.19}$$

From Lemma 4.3, (4.2), (4.5) and (4.19), it follows that

$$\Big|\frac{\nabla g^{\gamma}_{\alpha,0}}{\varphi^{p+1}}\Big| \leq \frac{\sqrt{d}g^{\gamma}_{0,1}}{2(p+1)(t+\gamma)g^{\gamma}_{0,0}} \leq \frac{C_4\sqrt{d}}{2\gamma(p+1)CC_3}. \tag{4.20}$$

Substituting (4.20) for (4.18), and using (4.3)-(4.5), we have

$$\begin{aligned} &w_t - \Delta w + w^{-p} \\ &\leq \frac{C(p+1)}{(t+\gamma)^{\alpha+2}\varphi}\Big[g^{\gamma}_{0,2} + (t+\gamma)\Big\{-2\alpha g^{\gamma}_{0,0} + g^{\gamma}_{0,0} + \frac{C_4\sqrt{d}|p|g^{\gamma}_{0,1}}{4C_3(p+1)}\Big\}\Big] \\ &\leq \frac{C(p+1)\psi}{(t+\gamma)^{\alpha+2}\varphi}\Big[-2\alpha\gamma C_3 + C_4\Big\{1 + (T+\gamma)\Big(1 + \frac{C_4d|p|}{4C_3(p+1)}\Big)\Big\}\Big] \end{aligned}$$

in $\mathbb{R}^d \times [0,T]$. If α satisfies

$$\alpha \geq \alpha_2 \equiv \frac{C_4}{2\gamma C_3}\Big\{1 + (T+\gamma)\Big(1 + \frac{C_4d|p|}{4C_3(p+1)}\Big)\Big\}, \tag{4.21}$$

then w is a subsolution of (1.1) in $\mathbb{R}^d \times (0,T)$. □

Proof of Theorem 1.4. Let $c_1 = c_1(C_2, a, \gamma, \alpha)$ and $c_2 = c_2(C_1, \gamma, \alpha)$ be positive constants such that

$$g^{\gamma}_{\alpha,0}(x,0) \le c_1\psi(x), \quad g^{\gamma}_{-\alpha,0}(x,0) \ge c_2\psi(x)$$

with $\alpha > 0$ as in Lemma 4.3 and (4.5). Hence

$$m^{p+1} + C_l g^{\gamma}_{\alpha,0}(x,0) \le u_0^{p+1}(x) \le m^{p+1} + C_h g^{\gamma}_{-\alpha,0}(x,0)$$

with $C_l = C_I/c_1$ and $C_h = C_{II}/c_2$.

Since $m^{p+1} = (p+1)T$ by (1.7), we have

$$\begin{aligned} w(x,0) &= \{(p+1)T + C_l g^{\gamma}_{\alpha,0}(x,0)\}^{1/(p+1)} \le u_0(x) \\ &\le \{(p+1)T + C_h g^{\gamma}_{-\alpha,0}(x,0)\}^{1/(p+1)} = W(x,0), \end{aligned}$$

where W and w are defined in (4.7) with $C = C_h/(p+1)$ and (4.8) with $C = C_l/(p+1)$. Propositions 4.1, 4.2 and the comparison principle (see Theorem 5.3) yield

$$w(x,t) \le u(x,t) \le W(x,t) \quad \text{in } \mathbb{R}^d \times [0,T).$$

We thereby get

$$\{C_l g^{\gamma}_{\alpha,0}(x,T)\}^{1/(p+1)} \le u(x,T) \le \{C_h g^{\gamma}_{-\alpha,0}(x,T)\}^{1/(p+1)}.$$

Taking $C = (C_l C_3)^{1/(p+1)}$ and $C' = (C_h C_4)^{1/(p+1)}$, by Lemma 4.3

$$C\psi^{1/(p+1)}(x) \le u(x,T) \le C'\psi^{1/(p+1)}(x).$$

Choosing

$$\gamma = \frac{1}{8a} - \frac{T}{2}, \quad \alpha = \max\{\alpha_1, \alpha_2\}$$

with α_1 in (4.17) and α_2 as in (4.21), we see that C depends only on C_1, C_2, a, T, C_I, and C' does only on C_1, C_2, a, T, C_{II}. $\square$

5. Appendix A: Existence and uniqueness of the classical solution and comparison principle

In this section we prove that Cauchy problem (1.1)-(1.2) with conditions (1.3)-(1.4) has a unique positive classical solution, as well as a comparison principle.

First, we consider the local existence and uniqueness of the classical solution for the problem in time. We know that the solution of (1.1)-(1.2) satisfies the integral equation:

$$u(x,t) = e^{t\Delta}u_0(x) - \int_0^t e^{(t-s)\Delta}u^{-p}(x,s)ds, \tag{5.1}$$

where

$$e^{t\Delta}\xi(x) = \int_{\mathbb{R}^d} G(x-y,t)\xi(y)dy \tag{5.2}$$

with $G(x,t)$ defined in (2.4) and a measurable function ξ. The function $\Xi(x,t) = e^{t\Delta}\xi(x)$ is the unique solution to

$$\begin{gathered} \Xi_t = \Delta\Xi, \quad x \in \mathbb{R}^d, t > 0, \\ \Xi(x,0) = \xi(x), \quad x \in \mathbb{R}^d. \end{gathered}$$

Now we consider the existence in time of a local solutions to (5.1).

Lemma 5.1. *Assume that u is the solution of* (5.1), *where $p > -1$, $p \neq 0$, u_0 is bounded continuous, $\inf_{x\in\mathbb{R}^d} u_0(x) = m > 0$ and $\sup_{x\in\mathbb{R}^d} u_0(x) = M < \infty$. Then the solution satisfying $u \geq m/q$ exists in $\mathbb{R}^d \times (0,T)$ with*

$$T < T_{m,M}(q) \equiv \begin{cases} \min\{q-1, \frac{1}{p}\}(\frac{m}{q})^{p+1}, & p > 0,\\ \min\{(q-1)M^p, \frac{1}{|p|}(\frac{m}{q})^p\}\frac{m}{q}, & -1 < p < 0, \end{cases} \tag{5.3}$$

where $q = \bar{q} = \bar{q}(m,M)$ is a positive solution of

$$\begin{gathered} q - 1 = \frac{1}{p}, \quad p > 0,\\ (q-1)M^p = \frac{1}{|p|}(\frac{m}{q})^p, \quad -1 < q < 0. \end{gathered} \tag{5.4}$$

Moreover, the solution is a unique classical solution of (1.1)-(1.2).

Proof. First we show the existence and the uniqueness of the local solution of (5.1) by a fixed-point theorem. Define

$$\Psi(u) = e^{t\Delta}u_0(x) - \int_0^t e^{(t-s)\Delta}u^{-p}(x,s)ds. \tag{5.5}$$

Set

$$E_T = \{u : [0,T] \to L^\infty(\mathbb{R}^d); \|u\|_{E_T} < \infty\}$$

with the norm

$$\|u\|_{E_T} = \sup_{t\in[0,T]} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)}.$$

Define

$$B^M_{m/q} = \big\{u \in E_T; \inf_{(x,t)\in\mathbb{R}^d\times[0,T]} u(x,t) \geq m/q,\ \|u\|_{E_T} \leq M\big\}.$$

with some $q > 1$. Let $u \in B^M_{m/q}$. Then, for $p > 0$,

$$\begin{aligned} \Psi(u) &= e^{t\Delta}u_0(x) - \int_0^t e^{(t-s)\Delta}u^{-p}(x,s)ds\\ &\geq m - \int_0^T \big(\frac{q}{m}\big)^p ds = m - \big(\frac{q}{m}\big)^p T, \end{aligned}$$

and for $-1 < p < 0$,

$$\Psi(u) \geq m - \int_0^T M^{-p}ds = m - M^{-p}T.$$

If T satisfies (5.3), then $\Psi(u) > m/q$ and Ψ is a mapping from $B^M_{m/q}$ to itself.

Rest of the proof we should show that

$$\|\Psi(u_1) - \Psi(u_2)\|_{E_T} \leq C\|u_1 - u_2\|_{E_T}$$

with some $C \in (0,1)$. For $u_1, u_2 \in B^M_{m/q}$,

$$|\Psi(u_1) - \Psi(u_2)| \leq \int_0^t e^{(t-s)\Delta}|(u_1)^{-p}(x,s) - (u_2)^{-p}(x,s)|ds.$$

By the mean value theorem

$$u_1^{-p} - u_2^{-p} = -\int_0^1 p\{\theta u_1 + (1-\theta)u_2\}^{-p-1}d\theta(u_1 - u_2).$$

Since u_1, $u_2 \geq m/q$, we obtain

$$\|\Psi(u_1) - \Psi(u_2)\|_{E_T} \leq T|p|\big(\frac{q}{m}\big)^{p+1}\|u_1 - u_2\|_{E_T} < C\|u_1 - u_2\|_{E_T}$$

with $C \in (0,1)$ for

$$T < \frac{m^{p+1}}{|p|q^{p+1}}. \tag{5.6}$$

We thus see that if T satisfies (5.3) and (5.6), then Ψ is a contraction map in $B^M_{m/q}$ and have one fixed point in $B^M_{m/q}$ for $(x,t) \in \mathbb{R}^d \times (0,T)$. Thus (5.1) has a unique solution in $\mathbb{R}^d \times (0,T)$. Since $q > 1$ is arbitrary, we may let $q = \bar{q}$.

To complete the proof of Lemma 5.1, let $u(x,t)$ be the nonnegative and bounded solution of (5.1) that has been obtained in $\mathbb{R}^d \times [0,T)$ for some $T > 0$. By (5.1), $u(x,t)$ is continuous in $\mathbb{R}^d \times [0,T)$. Moreover, by considering the difference quotients $\{u(x_1, x_2, \dots, x_{j-1}, x_j + h, x_{j+1}, \dots, x_d, t) - u(x,t)\}/h$ with $h \to 0$, one easily sees that $\partial u(x,t)/\partial x_j$ is locally bounded in $\mathbb{R}^d \times [\tau, T)$ for $j = 1, 2, \dots, d$ and any τ such that $0 < \tau < T$. Then, since $u \geq m/q > 0$, u^{-p} are locally Hölder continuous functions in space uniformly with respect to time. It then follows from the representation formula (5.1) that u is a classical solution of (1.1)-(1.2) in $\mathbb{R}^d \times (0,T)$ with (1.3)-(1.4) (see [1, Chapter 1, Theorem 10]). □

Remark 5.2. Note that $T_{m,M}(q)$ defined in (5.3) has a maximum number at $q = \bar{q}$. For $p > 0$ it is clear that $\bar{q} = (p+1)/p$. On the other hand, for $-1 < p < 0$, $\bar{q} = \lim_{n\to\infty} q_n$ $(< \infty)$ with the sequence $\{q_n\}_{n=1}^{\infty}$ satisfying

$$q_1 = 1, \qquad q_{n+1} = 1 + \frac{1}{|p|}\big(\frac{M}{m}\big)^{-p} q_n^{-p} \quad \text{for } n \in \mathbf{N}.$$

Next, we recall a comparison result for the solution existing locally in time.

Theorem 5.3. *Let $D \subset \mathbb{R}^d$. Assume $u_3(x,t)$, $u_4(x,t) \in C^{2,1}(D \times (0,T))$ satisfy the partial differential inequalities*

$$(u_3)_t \geq \Delta u_3 - u_3^{-p}, \tag{5.7}$$

$$(u_4)_t \leq \Delta u_4 - u_4^{-p} \tag{5.8}$$

with $u_3(x,0) = u_{3,0}(x)$ and $u_4(x,0) = u_{4,0}(x)$, where $u_{3,0}$ and $u_{4,0}$ are continuous in D. Assume that u_4 is bounded in $D \times [0,T')$ for any $T' < T$. Assume that u_3 is bounded from below in $D \times [0,T')\}$ for any $T' < T$. For $D \neq \mathbb{R}^d$, if $u_{3,0} \geq u_{4,0}$ in D and $u_3 \geq u_4$ on ∂D, then $u_3 \geq u_4$ in $D \times (0,T)$. On the other hand, for $D = \mathbb{R}^d$, if $u_{3,0} \geq u_{4,0}$ in $\mathbb{R}^d$, then $u_3 \geq u_4$ in $\mathbb{R}^d \times (0,T)$.

Proof. The proof is based on an maximum principle for a parabolic equation (see [23]) and is standard (see [17], [2], [25] and [8]). □

Next we introduce more strong result for the comparison principle.

Theorem 5.4. *Assume the same hypotheses as in Theorem 5.3 with $D = \mathbb{R}^d$. If $u_{3,0} \not\equiv u_{4,0}$ in $\mathbb{R}^d$, then $u_3 > u_4$ in $\mathbb{R}^d \times (0,T)$.*

Proof. Let $\tilde{w} = u_3 - u_4$. From Theorem 5.3, we see that $\tilde{w} \geq 0$. For $t \in [0,T')$ with $T' < T$,

$$\tilde{w}_t = \Delta\tilde{w} - u_3^{-p} - u_4^{-p} = \Delta\tilde{w} + b(x,t)\tilde{w}$$

with b defined by

$$b(x,t) = \int_0^1 p\{u_3(x,t) + \theta(u_4(x,t) - u_3(x,t))\}^{-p-1} d\theta. \tag{5.9}$$

Note that $b(x,t)$ is bounded in $\mathbb{R}^d \times (0,T')$. Put $\tilde{b}(t) = \inf_{x\in\mathbb{R}^d} b(x,t)$ and

$$\tilde{W}(x,t) = \exp\big\{-\int_0^t \tilde{b}(s)ds\big\}\tilde{w}(x,s).$$

Since $b(x,t) \geq \tilde{b}(t)$ and $\tilde{W} \geq 0$, we see that

$$\tilde{W}_t = \Delta\tilde{W} + (b(x,t) - \tilde{b}(t))\tilde{W} \geq \Delta\tilde{W}$$

and

$$\tilde{W}(x,0) = u_0(x) - m \ (\geq 0,\ \not\equiv 0). \tag{5.10}$$

By the basic comparison principle and (5.10), we get

$$\tilde{W}(x,t) \geq e^{t\Delta}(u_0(x) - m) > 0 \quad \text{in } \mathbb{R}^d \times (0,T'),$$

where $e^{t\Delta}$ is defined in (5.2). Since $T' < T$ is arbitrary, we obtain

$$\tilde{W}(x,t) > 0 \quad \text{in } \mathbb{R}^d \times (0,T).$$

We thus see that

$$\tilde{w}(x,t) > 0 \text{ in } \mathbb{R}^d \times (0,T),$$
$$u_3(x,t) > u_4(x,t) \text{ in } \mathbb{R}^d \times (0,T).$$

□

Finally, we show the existence of solutions for problem (1.1)-(1.2) in $\mathbb{R}^d \times (0,T(m))$.

Theorem 5.5. *Problem* (1.1)-(1.2) *with initial data satisfying* (1.3)–(1.4) *has a unique classical solution in* $\mathbb{R}^d \times (0,T(m))$.

Proof. It is clear for the case $p = 0$ and the case u_0 is a constant (in the case $m = M$). We consider the other case such as

$$p > -1,\ p \neq 0, \tag{5.11}$$
$$m < M, \tag{5.12}$$

where $m = \inf_{x\in\mathbb{R}^d} u_0(x)$ and $M = \sup_{x\in\mathbb{R}^d} u_0(x)$.

From Lemma 5.1, (5.3), (5.4) and Remark 5.2, the problem has a unique classical solution in $\mathbb{R}^d \times (0,T_1]$, where

$$T_1 = \begin{cases} (1-\varepsilon)(\tilde{q}-1)\big(\frac{m}{\tilde{q}}\big)^{p+1}, & p > 0, \\ \frac{\tilde{q}-1}{\tilde{q}} M^p m, & -1 < p < 0, \end{cases} \tag{5.13}$$

with

$$\tilde{q} = 1 + \frac{1}{|p|} \tag{5.14}$$

and some $\varepsilon \in (0,1)$. Note that the fact $\tilde{q} \leq \bar{q}$ means $T_1 < T_{m,M}(\tilde{q})$, where $T_{m,M}$ and $\bar{q}$ are defined in (5.3) and (5.4).

Let $v_m(t)$ and $v_M(t)$ be solutions of (1.6) with initial data $v_m(0) = m$ and $v_M(0) = M$. Theorem 5.3 yields

$$v_m(t) \leq u(x,t) \leq v_M(t) \quad \text{in } \mathbb{R}^d \times (0,T_1]. \tag{5.15}$$

Let $\{m_n\}_{n=0}^{\infty}$ and $\{M_n\}_{n=0}^{\infty}$ be such that $m_0 = m$, $M_0 = M$,

$$m_{n+1} = \begin{cases} m_n\big\{1 - \frac{p^p(1-\varepsilon)}{(p+1)^p}\big\}^{1/(p+1)}, & p > 0, \\ m_n\big\{1 - \big(\frac{m_n}{M_n}\big)^{-p}\frac{1+p}{1-p}\big\}^{1/(p+1)}, & -1 < p < 0 \end{cases} \tag{5.16}$$

and

$$M_{n+1} = \begin{cases} M_n\big\{1 - \frac{p^p(1-\varepsilon)}{(p+1)^p}\big(\frac{m_n}{M_n}\big)^{p+1}\big\}^{1/(p+1)}, & p > 0, \\ M_n\big\{1 - \frac{m_n}{M_n}\frac{1+p}{1-p}\big\}^{1/(p+1)}, & -1 < p < 0. \end{cases} \tag{5.17}$$

Then, (1.8), (5.13) and (5.14) yield $v_m(T_1) = m_1$ and $v_M(T_1) = M_1$. Moreover, from (5.15) we obtain $m_1 \le u(x, T_1) \le M_1$.

Next, we consider

$$\begin{gathered} (u_1)_t = \Delta u_1 - u_1^{-p}, \quad x \in \mathbb{R}^d, t > 0, \\ u_1(x, 0) = u(x, T_1), \quad x \in \mathbb{R}^d. \end{gathered}$$

By the same argument as above, we see that the problem has a unique classical solution in $\mathbb{R}^d \times (0, T_2]$, where the sequence $\{T_n\}_{n=1}^{\infty}$ is defined by

$$T_n = \begin{cases} (1-\varepsilon)(\tilde{q}-1)\big(\frac{m_{n-1}}{\tilde{q}}\big)^{p+1}, & p > 0, \\ \frac{\tilde{q}-1}{\tilde{q}}M_{n-1}^p m_{n-1}, & -1 < p < 0 \end{cases} \tag{5.18}$$

with $\tilde{q}$ defined in (5.14).

It is known that $u_1(x, t) = u(x, T_1 + t)$. We thus see that (1.1)-(1.2) has a unique classical solution in $\mathbb{R}^d \times (0, \tilde{T}_2]$ with

$$\tilde{T}_n = \sum_{k=1}^{n} T_k \quad \text{for } k \in \mathbf{N}.$$

Moreover we have $v_m(\tilde{T}_2) = m_2$, $v_M(\tilde{T}_2) = M_2$ and $m_2 \le u(x, \tilde{T}_2) \le M_2$.

For $n \in \mathbf{N}$, by using same argument as above $n - 2$ more times, we see that

$$\text{(1.1)-(1.2) has a unique classical solution in } \mathbb{R}^d \times (0, \tilde{T}_n]. \tag{5.19}$$

Moreover we have $v_m(\tilde{T}_n) = m_n$, $v_M(\tilde{T}_n) = M_n$ and $m_n \le u(x, \tilde{T}_n) \le M_n$. Note that $n \in \mathbf{N}$ is arbitrary for these results.

Finally we should show that

$$\lim_{n\to\infty} \tilde{T}_n = T(m). \tag{5.20}$$

Since $v_m(t)$ is decay function with respect to t and $v_m(\tilde{T}_n) = m_n > 0$ for any $n \in \mathbf{N}$, we only should prove

$$\lim_{n\to\infty} v_m(\tilde{T}_n) = \lim_{n\to\infty} m_n = 0. \tag{5.21}$$

Put $\mu_n = m_n/M_n$. From (5.12) we get

$$\sup_{n\in\mathbf{N}} \mu_n = \sup_{n\in\mathbf{N}} \frac{v_m(\tilde{T}_n)}{v_M(\tilde{T}_n)} \le \sup_{t\in[0,T(m))} \frac{v_m(t)}{v_M(t)} < 1. \tag{5.22}$$

From (5.16) and (5.17) we see that

$$\mu_{n+1} = \begin{cases} \mu_n\Big\{\frac{1-\frac{p^p(1-\varepsilon)}{(p+1)^p}}{1-\frac{p^p(1-\varepsilon)}{(p+1)^p}\mu_n^{p+1}}\Big\}^{1/(p+1)}, & p > 0, \\ \mu_n\Big\{\frac{1-\mu_n^{-p}\frac{1+p}{1-p}}{1-\mu_n\frac{1+p}{1-p}}\Big\}^{1/(p+1)}, & -1 < p < 0. \end{cases} \tag{5.23}$$

Since $\{\mu_n\}_{n=0}^{\infty}$ is a decrasing sequence with respect to n, by the monotone convergence theorem and (5.22) we have $\lim_{n\to\infty}\mu_n = b$ with some $b \in [0,1)$. We cleam that $\lim_{n\to\infty}\mu_n = 0$. By contraries, assume that $\lim_{n\to\infty}\mu_n = b > 0$. Then from (5.11) we obtain that $b = 1$ and a contradiction occurs. We thus obtain $\lim_{n\to\infty}\mu_n = 0$. This means (5.21) and (5.20). Since (5.19) holds for any $n \in \mathbf{N}$, (1.1)-(1.2) has a unique classical solution in $\mathbb{R}^d \times (0, T(m))$. $\square$

6. Appendix B: Existence and regularity of the solution at $t = T(m)$

In this section we prove Lemma 3.1. Let $T = T(m)$.

Proof of Lemma 3.1. From Theorem 1.1, for any $a \in \mathbb{R}^d$ and any $R \in (0, \sqrt{T})$, there exist constants $C_1 = C_1(a,R) > 0$ and $C_2 = C_2(a,R) > C_1$ such that

$$C_1 < u(x,t) < C_2 \quad \text{for } (x,t) \in Q(R), \tag{6.1}$$

where $Q(r) = Q(r,a) = B(a,r) \times (T - r^2, T)$. Then there exists constant $C_3 = C_3(a,R,p)$ such that

$$\|u\|_{L^q(Q(R))} \le C_3, \quad \|u^{-p}\|_{L^q(Q(R))} \le C_3.$$

From [19, Theorem 6.4.2], we have

$$\|u\|_{W_q^{2,1}(Q(r_1))} \le C_4\left(\|u^{-p}\|_{L_q(Q(R))} + \|u\|_{L_q(Q(R))}\right) \le C_5$$

with constants $q > (d+2)/2$, $C_4 = C_4(d,q) < \infty$, $C_5 = C_5(a,R,p,d,q) < \infty$, and $r_1 = R/2$. From [21, II, Lemma 3.3] we obtain

$$\|u\|_{C^{\alpha}(Q(r_1))} \le C_6(\|u\|_{W_q^{2,1}(Q(r_1))} + \|u\|_{L_q(Q(r))}) \le C_7$$

with $\alpha \in (0,1)$, $C_6 = C_6(d,q,\alpha) < \infty$ and $C_7 = C_7(a,R,p,d,q,\alpha) < \infty$. From (6.1) we see that

$$\|u^{-p}\|_{C^{\alpha}(Q(r_1))} \le C_8$$

with $C_8 = C_8(a,R,p,d,q,\alpha) < \infty$. From [1, Chapter 3, Theorem 5] we get

$$\|u\|_{C^{2+\alpha}(Q(r_2))} \le C_9(\|u\|_{C^{\alpha}(Q(r_1))} + \|u^{-p}\|_{C^{\alpha}(Q(r_1))}) \le C_{10} \tag{6.2}$$

with $r_2 = r_1/2 = R/4$, $C_9 = C_9(a,R,d,\alpha)$ and $C_{10} = C_{10}(a,R,p,d,q,\alpha)$.

Next put $u_1 = \Delta u$. Then we see that

$$(u_1)_t = \Delta u_1 + pu^{-p-1}u_1 - p(p+1)u^{-p-2}|\nabla u|^2.$$

From (6.1) and (6.2) we have

$$\|u_1\|_{C^{\alpha}(Q(r_2))} \le C_{10},$$
$$\|pu^{-p-1}u_1 - p(p+1)u^{-p-2}|\nabla u|^2\|_{C^{\alpha}(Q(r_2))} \le C_{11}$$

with $C_{11} = C_{11}(a,R,p,d,q,\alpha)$. From [1] again,

$$\|u_1\|_{C^{2+\alpha}(Q(r_3))} = \|u\|_{C^{4+\alpha}(Q(r_3))} \le C_{12}$$

with $r_3 = r_2/2 = R/8$ and $C_{12} = C_{12}(a,R,p,d,q,\alpha)$.

Iterating this argument $n-1$ times, we have

$$\|u\|_{C^{2n+2+\alpha}(Q(r_n))} \le C_{13}$$

with $r_n = R/2^n$ and $C_{13} = C_{13}(a,R,p,d,q,\alpha,n)$. We thus see that

$$\Delta^n u_t(x,t) \le C_{14} \text{ for } (x,t) \in Q(r_n)$$

with $C_{14} = C_{14}(a, R, p, d, q, n)$. By integrating $\Delta^n u_t(a, t)$ from $T - r_n/2$ to T with respect to t and subtracting $\Delta^n u_t(a, T - r_n/2)$, we obtain $\Delta^n u(a, T)$. Thus we see that $\Delta^n u(a, T)$ exists.

Since $a \in \mathbb{R}^d$ and $n \in \mathbf{N}$ are arbitrary, we obtain $u(\cdot, T) \in C^\infty(\mathbb{R}^d)$. $\square$

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