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PRACTICAL STABILITY OF LINEAR SWITCHED IMPULSIVE SYSTEM WITH TIME DELAY

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ABSTRACT. This article concerns the study of practical stability of linear switched impulsive systems with time delay. By using Lyapunov functions and the extended Halanay inequality, we establish sufficient conditions for the practical stability and uniform practical stability of a linear switched impulsive system with time delay. The last section provides some illustrative examples.

1. INTRODUCTION

Recently, there has been considerable research on switched impulsive systems with time delay. However, most of them is about Lyapunov stability [10, 11], but not practical stability. Li [5] clarified the different definitions of practical stability, and gave some criteria for the practical stability of switched impulsive system without time delay. The book [13] provides conditions on practical stability of various systems, including ordinary differential equations, impulsive differential equations, functional differential equations. But there has been little study on practical stability of switched impulsive systems with time delay. In this article we fill this gap.

First, we introduce Halanay inequality (see Lemma 3.1). From this inequality, we gain of a upper estimate on a function $u(t)$, which decrease exponentially with time. Then this estimate can be applied to the study of exponential stability, boundedness, practical stability, etc. As in [1, 2, 6, 7, 8, 9], utilizing an extended Halanay inequality, we study Lyapunov stability and attractivity for delay differential systems, impulsive systems with delay, switched systems with delay and difference equations.

To adapt the extended Halanay inequality to linear switched impulsive system with time delay, we establish multiple Lyapunov functions and revise some conditions of the extended Halanay inequality. Also by utilizing the comparison method and the method of segmentation, we settle the problem of discontinuity caused by impulses and switches. Then we give sufficient conditions for practical stability of linear switched impulsive system with time delay, where the influence of delays, impulses, and switches is considered. We strive to conclude the coupling relation of the delay, impulses and the dwell-time. What is more, we distinguish between

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the restriction on the dwell-time of every activation for good subsystems and that for bad subsystems, where the good subsystem denotes the one which is practically stable, and the bad one just the opposite. Lastly, we provide some illustrative examples and the simulations.

2. Preliminaries

It is convenient to establish some notation here. Let $\mathbb{R}^+$ denote the set of all nonnegative real numbers, $\mathbb{R}^n$ the n-dimensional real space equipped with Euclidean norm $|\cdot|$. Denote by $\mathbb{N}_+$ the set of all positive integers, and $\mathbb{N} = \mathbb{N}_+ \cup \{0\}$. Let $\Lambda = \{1, 2, \dots, m\}$, where $m \in \mathbb{N}_+$. If $M = (m_{ij})_{n\times m}$ is a matrix, we write the norm of M as $|M| = \sqrt{\sum_{1\le i\le n, 1\le j\le m} m_{ij}^2}$, and the transposition of M as M^T. Denote by $\lambda_{\max}(M)$ the greatest eigenvalue of M, and $\lambda_{\min}(M)$ the minimum eigenvalue. Set $x(t^+) = \lim_{s\to t^+} x(s)$. Let $r > 0$, and $PC([-r, 0])$ be the Banach space of piecewise continuous functions with supremum norm $\|\cdot\|$. If $x \in PC([t_0, +\infty))$, let

$$\dot{x}(t) = \lim_{h\to 0^-} \frac{x(t+h) - x(t)}{h}.$$

Consider m subsystems with delay,

$$\begin{gathered} \dot{x}(t) = f_i(t, x(t), x(t-r)), \quad i = 1, 2, \dots, m, \;\; m \in \mathbb{N}_+, \\ x_{t_0} = \varphi \end{gathered} \tag{2.1}$$

the switches

$$S = \{(\tau_k, i_k) : i_k \in \Lambda = \{1, 2, \dots, m\}, \tau_k > 0, \; k \in \mathbb{N}_+\}, \tag{2.2}$$

and the impulses

$$x(t_k^+) = I_k(x(t_k)), \quad k = 1, 2, \dots, \tag{2.3}$$

where $x \in \mathbb{R}^n$, $\varphi \in C([-r, 0], \mathbb{R})$, $f_i \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $i \in \Lambda$, $I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$, $k \in \mathbb{N}_+$. Here $\tau_k > 0$ denotes switching intervals. For any $t_0 \in \mathbb{R}^+$, $t_k = t_0 + \sum_{i=1}^k \tau_i$ denotes switching instants, which satisfies $\lim_{k\to+\infty} t_k = +\infty$. We assume that $f_i(t, 0) = 0$, $I_k(0) = 0$ for any $t \ge 0, k \in \mathbb{N}_+, i \in \Lambda$.

According to (2.1)-(2.3), we write switched impulsive systems with time delay as:

$$\begin{gathered} \dot{x}(t) = f_{i_k}(t, x(t), x(t-r)), \quad t \in (t_{k-1}, t_k] \\ x(t_k^+) = I_k(x(t_k)), \quad k = \mathbb{N}_+ \\ x_{t_0} = \varphi. \end{gathered} \tag{2.4}$$

Remark 2.1. We assume throughout this paper that solution of (2.4) is unique and of global existence [4, 11].

Definition 2.2. Given (λ, A) with $0 < \lambda < A$, system (2.4) is said to be

- (i) λ-A-practically stable, if $\|\varphi\| < \lambda$ implies $|x(t, t_0^+, x_0)| < A$ for all $t \ge t_0$, and some $t_0 \in \mathbb{R}^+$.
- (ii) λ-A-uniformly practically stable, if $\|\varphi\| < \lambda$ implies $|x(t, t_0^+, x_0)| < A$, for all $t \ge t_0$, and every $t_0 \in \mathbb{R}^+$.

3. Lemmas

In this section, we provide some results that are needed in Section 4.

Lemma 3.1 (Halanay inequality [2]). *If $r \geq 0$, $a > b > 0$, $u(t)$ is a continuous function satisfying $u(t) \geq 0$, and*

$$D^+u(t) \leq -au(t) + b \sup_{-r\leq\theta\leq 0} u(t+\theta), \quad t \geq t_0,$$

then $u(t) \leq \sup_{-r\leq\theta\leq 0} u(t_0+\theta)e^{-\mu(t-t_0)}$, $t \geq t_0$, where $\mu > 0$ and $\mu - a + be^{\mu r} = 0$.

Lemma 3.2. *Consider (λ, A) with $0 < \lambda < A$. If $2a + b^2 < -1$, then the system*

$$\begin{gathered} \dot{x}(t) = ax(t) + bx(t-r) \\ x_{t_0} = \varphi \end{gathered} \tag{3.1}$$

is λ-A-practically stable.

Proof. Let $x(t)$ denote the solution of (3.1), and define $V(t) = x^2(t)$. Then

$$\begin{aligned} \dot{V}(t) &= 2x(t)\dot{x}(t) \\ &= 2x(t)[ax(t) + bx(t-r)] \\ &\leq (2a+b^2)x^2(t) + x^2(t-r) \\ &\leq (2a+b^2)V(x(t)) + \sup_{\theta\in[-r,0]} V(t+\theta). \end{aligned}$$

By the Halanay inequality and $2a + b^2 < -1$,

$$V(x(t)) \leq \sup_{\theta\in[-r,0]} V(t_0+\theta)\, e^{-u(t-t_0)}, \quad t \geq t_0,$$

where $u > 0$ and $u + 2a + b^2 + e^{ur} = 0$. Consequently, when $\|\varphi\| < \lambda$,

$$|x(t)| = V^{1/2}(t) \leq \sup_{\theta\in[-r,0]} V^{1/2}(t_0+\theta) < \lambda < A.$$

The proof is complete. $\square$

Lemma 3.3 ([6]). *Consider the system*

$$\begin{gathered} D^+f(t) \leq pf(t) + q\bar{f}(t), \quad t \in [t_0, +\infty)\backslash\{t_k, k \in \mathbb{N}_+\} \\ f(t_k) \leq d_k f(t_k^-), \quad k \in \mathbb{N}_+, \end{gathered} \tag{3.2}$$

where

$$\begin{gathered} t_k \in \mathbb{R}^+, \quad t_{k+1} > t_k,\ k \in \mathbb{N},\ \lim_{k\to+\infty} t_k = +\infty, \\ p \in \mathbb{R}, \quad q \geq 0,\ r > 0,\ \delta > 1, \\ f \in PC(\mathbb{R}, \mathbb{R}^+), \quad \bar{f}(t) = \sup\{f(s) : t-r \leq s \leq t\}. \end{gathered} \tag{3.3}$$

Assume that

$$p + q\delta < \frac{\ln\delta}{\sigma}, \quad \text{where } \sigma = \sup\{t_{n+1} - t_n : n \in \mathbb{N}\} < \infty; \tag{3.4}$$

$$0 < \lambda < \frac{\ln\delta}{\sigma} - p - q\delta e^{\lambda r}. \tag{3.5}$$

Then let $f \in PC(\mathbb{R}, \mathbb{R}^+)$ be the solution of (3.2), and define

$$g(t) = \begin{cases} f(t)e^{\lambda(t-t_0)}, & t > t_0, \\ f(t), & t_0 - r \leq t \leq t_0. \end{cases} \tag{3.6}$$

If $t_n \le t_* < t^* < t_{n+1}$ for some $n \in \mathbb{N}$, and $\delta g(t) \ge g(s)$ for any $s \in [t_0-\tau, t^*]$ and $t \in [t_*, t^*]$, then $\delta > g(t^*)/g(t_*)$.

Now, we adapt the conclusion of Lemma 3.3 to the linear switched impulsive system with time delay.

Lemma 3.4 (Extended inequality)**.** *Replace* (3.2) *in Lemma 3.3 by*

$$\begin{gathered} D^- f(t) \le p_{i_k} f(t) + q_{i_k} \bar{f}(t), \quad t \in (t_{k-1}, t_k] \\ f(t_k^+) \le d_k f(t_k), \quad k \in \mathbf{N}_+, \end{gathered} \tag{3.7}$$

where $i_k \in \Lambda, k \in \mathbb{N}_+$, $p_{i_b} = p, q_{i_b} = q$, and b is a given positive integer. Let $f(t)$ be the solution of (3.7)*. And suppose there is a $\lambda > 0$ such that* (3.3)*,* (3.5) *and* (3.6) *hold. If $t_{b-1} \le t_* < t^* \le t_b$, and $\delta g(t) \ge g(s)$ for all $s \in [t_b - r, t^*], t \in (t_*, t^*]$. Then $\delta > g(t^*)/g(t_*^+)$.*

Proof. For any $t \in [t_0, +\infty)$, we can find a $u_t \in [t-r, t]$ such that $\bar{f}(t) = f(u_t)$. Note that $e^{\lambda(t-t_0)} \le 1$, for every $t \in [t_0 - r, t_0]$. Then,

$$f(t)e^{\lambda(t-t_0)} \le g(t), \quad t \in [t_0 - r, +\infty). \tag{3.8}$$

Consider $t \in (t_*, t^*]$, then

$$\begin{aligned} D^- g(t) &= (D^- f(t))e^{\lambda(t-t_0)} + \lambda f(t)e^{\lambda(t-t_0)} \\ &\le (p_{i_b} f(t) + q_{i_b} \bar{f}(t))e^{\lambda(t-t_0)} + \lambda f(t)e^{\lambda(t-t_0)} \\ &= (pf(t) + q\bar{f}(t))e^{\lambda(t-t_0)} + \lambda f(t)e^{\lambda(t-t_0)} \\ &= (\lambda + p)f(t)e^{\lambda(t-t_0)} + qf(u_t)e^{\lambda(u_t-t_0)}e^{\lambda(t-u_t)}, \quad t \in (t_*, t^*]. \end{aligned} \tag{3.9}$$

From (3.8), (3.9) and the assumption in the lemma, we have

$$\begin{aligned} D^- g(t) &\le (\lambda + p)f(t)e^{\lambda(t-t_0)} + qg(u_t)e^{\lambda(t-u_t)} \\ &\le (\lambda + p)g(t) + q\delta g(t)e^{\lambda\tau} \\ &= (\lambda + p + q\delta e^{\lambda\tau})g(t), \quad t \in (t_*, t^*]. \end{aligned} \tag{3.10}$$

By (3.10) and (3.5), we have

$$\int_{t_*}^{t^*} \frac{dg(t)}{g(t)} \le \int_{t_*}^{t^*} (\lambda + p + q\delta e^{\lambda\tau})dt = (\lambda + p + q\delta e^{\lambda\tau})(t^* - t_*) < \ln\delta.$$

Note that $g(t_*) \ne g(t_*^+)$, if $t_* = t_b$. Then

$$\int_{t_*}^{t^*} \frac{dg(t)}{g(t)} = \ln g(t^*) - \ln g(t_*^+) = \ln(\frac{g(t^*)}{g(t_*^+)}).$$

It follows that $\delta > g(t^*)/g(t_*^+)$. □

Lemma 3.5 (Comparison theorem)**.** *Consider two systems:*

$$\begin{gathered} \dot{x}(t) = f_{i_k}(t, x(t), x(t-r)), \quad t \in (t_{k-1}, t_k] \\ x(t_k^+) = c_k x(t_k), \quad k \in \mathbb{N}_+ \\ x_{t_0} = \varphi_1, \end{gathered} \tag{3.11}$$

and

$$\begin{gathered}\dot{y}(t)=g_{i_k}(t,y(t),y(t-r)):=a_{i_k}y(t)+b_{i_k}y(t-r),\quad t\in(t_{k-1},t_k]\\ y(t_k^+)=d_ky(t_k),\quad k\in\mathbb{N}_+\\ y_{t_0}=\varphi_2,\end{gathered}\tag{3.12}$$

where $a_{i_k}\in\mathbb{R}$*;* $x,y,r,c_k,d_k,b_{i_k}\in\mathbb{R}^+$*;* $f_{i_k}(t,u,v),g_{i_k}(t,u,v):\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ *are continuous functions with* $k\in\mathbb{N}_+$ *and* $i_k\in\Lambda$*. Here* $\varphi_1,\varphi_2\in C([-r,0],\mathbb{R}^+)$*,* $f_i(t,0,0)=0$*,* $i\in\Lambda$*. If for every* $u,v\in\mathbb{R}^+$*,* $t\in(t_k,t_{k+1}]$ *and* $s\in[-r,0]$*, we have*

$$f_{i_k}(t,u,v)\le g_{i_k}(t,u,v),\quad c_k\le d_k,\quad \varphi_1(s)\le\varphi_2(s),$$

then, $x(t)\le y(t)$ *for each* $t>t_0$*, where* $x(t)$ *and* $y(t)$ *are the solutions of* (3.11) *and* (3.12) *respectively.*

Proof. (I) when $t\in(t_0,t_1]$, we have $x(t)\le y(t)$ by comparison theorem of functional differential equation [7].

(II) Assume that $x(t)\le y(t)$, where $t\in(t_j,t_{j+1}]$ an $j=0,1,2,\dots,k-1$. Then, we need to prove

$$x(t)\le y(t),\quad t\in(t_k,t_{k+1}].\tag{3.13}$$

Firstly, we claim that $x(t)\le y(t)$ for each $t\in(t_k,t_{k+1}]$, if

$$f_{i_{k+1}}(t,u,v)<g_{i_{k+1}}(t,u,v),\quad c_k\le d_k,\quad x(s)\le y(s),$$

for $t\in(t_k,t_{k+1}]$, $s\in(t_k-r,t_k]$. Otherwise, there is $\bar{t}\in(t_k,t_{k+1}]$, such that $x(\bar{t})>y(\bar{t})$. Define

$$t^*=\inf\{t:x(t)>y(t),t\in(t_k,t_{k+1}]\}.$$

Because $x(t)$ and $y(t)$ are continuous on $(t_k,t_{k+1}]$, and $x(t_k^+)=c_kx(t_k)\le d_ky(t_k)=y(t_k^+)$, we have

$$x(t^*)=y(t^*),\quad x(t)\le y(t),\quad t\in[t_k-r,t^*],$$

where $t^*\in[t_k,t_{k+1})$. Hence, $\dot{x}(t^*)\ge\dot{y}(t^*)$. On the other hand, if $t^*\in(t_k,t_{k+1})$,

$$f_{i_{k+1}}(t^*,x(t^*),x(t^*-r))<g_{i_{k+1}}(t^*,x(t^*),x(t^*-r))\le g_{i_{k+1}}(t^*,y(t^*),y(t^*-r)).$$

Namely, $\dot{x}(t^*)<\dot{y}(t^*)$, if $t^*\in(t_k,t_{k+1})$. Also if $t^*=t_k$, we can have $\dot{x}(t^*)<\dot{y}(t^*)$ similarly. This contradiction proves the result in this case.

Then we need to study system (3.12) on $(t_k,t_{k+1}]$. Taking t_k as the initial time, we rewrite system (3.12) as

$$\begin{gathered}\dot{y}(t)=g(t,y(t),y(t-r)),\quad t\in(t_k,t_{k+1}]\\ y(t_k^+)=d_ky(t_k)\\ y_{t_k}=\varphi_3,\end{gathered}\tag{3.14}$$

where $g(t,y(t),y(t-r))=a_{i_{k+1}}y(t)+b_{i_{k+1}}y(t-r)$, $\varphi_3(s)=y(t_k+s),s\in[-r,0]$. Denote by $\tilde{y}(t)$ the solution of (3.14). Obviously, $y(t)=\tilde{y}(t)$ if $t\in[t_k-r,t_{k+1}]$. If g in (3.14)is replaced by $g+\frac{1}{n}$ for any $n\in\mathbb{N}_+$, then we rewrite the solution as $y_n(t)$ respectively. By the results of previous paragraph, we conclude that

$$x(t)\le y_n(t),\quad t\in(t_k,t_{k+1}].$$

So, to prove (3.13), we need only to prove that

$$y_n(t)\to y(t),\quad \text{as } n\to+\infty,\tag{3.15}$$

for every $t \in (t_k, t_{k+1}]$. Define $z_n(t) = y_n(t) - \tilde{y}(t)$, then

$$\begin{gathered}\dot{z}_n(t) = a_{i_{k+1}} z_n(t) + b_{i_{k+1}} z_n(t-r) + \frac{1}{n}, \quad t \in (t_k, t_{k+1}] \\ z_{n_{t_k}} = \varphi_4(s),\end{gathered}$$

where $\varphi_4(s) = 0$, $s \in [-r, 0]$. By [3, Theorem 2.2 Chap. 2]], we have $z_n(t) \to 0$ as $n \to +\infty$, for each $t \in (t_k, t_{k+1}]$. Namely, (3.15) is true. So, $x(t) \le y(t)$, for each $t \in (t_k, t_{k+1}]$. By mathematical induction, $x(t) \le y(t)$, if $t > t_0$. □

Lemma 3.6 ([8]). *Consider the system*

$$\begin{gathered}\dot{x}(t) = a(t)x(t) + b(t)x(t-r), \\ x_{t_0} = \phi,\end{gathered} \tag{3.16}$$

where $a(t), b(t) \in C(\mathbb{R}^+, R)$, $r > 0$ *is a constant. Assume* $-\frac{1}{2r} \le a(t) + b(t+r) \le -rb^2(t+r)$. *Let* $x(t) = x(t, t_0, \phi)$ *be the solution of* (3.16) *on* $[t_0, +\infty)$. *Then*

$$\begin{gathered}|x(t)| \le \|\phi\| \Big(1 + \int_{t_0}^{t_0+r/2} |b(u)| du\Big) e^{\int_{t_0}^t a(s)ds}, \quad t \in (t_0, t_0 + r/2); \\ |x(t)| \le \sqrt{6V(t_0)} e^{\frac{1}{2}\int_{t_0}^{t-r/2}[a(s)+b(s+r)]ds}, \quad t \in [t_0 + r/2, +\infty).\end{gathered}$$

where $V(t_0) = \big[x(t_0) + \int_{t_0-r}^{t_0} b(s+r)x(s)ds\big]^2 + \int_{-r}^0 \int_{t_0+s}^{t_0} b^2(z+r)x^2(z)\,dz\,ds.$

Corollary 3.7. *If we add an impulse* $x(t_0^+) = d_0 x(t_0)$ *at the initial time* t_0, *and replace the initial function* ϕ *by* $\varphi \in PC(-r, 0)$ *in Lemma 3.5, then the conclusion becomes*

$$\begin{gathered}|x(t)| \le \max\{|x(t_0^+)|, \|\varphi\|\} \Big(1 + \int_{t_0}^{t_0+r/2} |b(u)| du\Big) e^{\int_{t_0}^t a(s)ds}, \quad t \in (t_0, t_0 + r/2); \\ |x(t)| \le \sqrt{6V(t_0^+)} e^{\frac{1}{2}\int_{t_0}^{t-r/2}[a(s)+b(s+r)]ds}, \quad t \in [t_0 + r/2, +\infty),\end{gathered}$$

where

$$V(u) = \big[x(u) + \int_{u-r}^{u} b(s+r)x(s)ds\big]^2 + \int_{-r}^0 \int_{u+s}^{u} b^2(z+r)x^2(z)\,dz\,ds.$$

Remark 3.8. Since the proof of Lemma 3.6 is not dependent on the continuity of the initial function, we can prove Corollary 3.7 similarly.

4. Practical stability results

Now, we are ready to give results on practical stability of the systems, including one-dimensional systems and n-dimensional ones. Firstly, consider the one-dimensional system with constant coefficients. That is, $f_{i_k}(t, x(t), x(t-r)) = a_{i_k} x(t) + b_{i_k} x(t-r)$ in (2.4).

$$\begin{gathered}\dot{x}(t) = a_{i_k} x(t) + b_{i_k} x(t-r), \quad t \in (t_{k-1}, t_k] \\ x(t_k^+) = d_k x(t_k), \quad k \in \mathbb{N}_+ \\ x_{t_0} = \varphi,\end{gathered} \tag{4.1}$$

where $a_{i_k}, b_{i_k} \in \mathbb{R}$, $r, d_k, t_0 \in \mathbb{R}^+$, $i_k \in \Lambda$, $k \in \mathbb{N}_+$, $\varphi \in C([-r, 0], \mathbb{R})$.

The subsystem of (4.1) is

$$\begin{aligned} \dot{x}(t) &= a_i x(t) + b_i x(t-r) \\ x_{t_0} &= \varphi, \end{aligned} \tag{4.2}$$

where $i \in \Lambda$. If $2a_i + b_i^2 < -1$, then the subsystem is practical stable by Lemma 3.2, and we call it a good subsystem. Otherwise, we can not guarantee practical stability of it. So we call it a bad subsystem. In order to guarantee practical stability of (4.1), it is sensible that there would be stricter restriction on the dwell time of bad subsystems than on that of good ones, as Theorem 4.1 shows. For convenience, we assume that the first m_1 subsystems are good subsystems, and the rest are bad ones.

Theorem 4.1. *Consider (λ, A) with $0 < \lambda < A$. If there exist constants $\delta_1, \delta_2 > 1$ and $\beta > 0$ which satisfy*

$$\tilde{\delta}_k = \begin{cases} \delta_1, & \text{if } i_k \le m_1 \\ \delta_2, & \text{if } i_k > m_1; \end{cases} \quad \beta < \frac{\ln \delta_i}{\sigma_i} - p_i - \delta_i e^{\beta r}, \quad i = 1, 2;$$

$$\prod_{j=0}^{k} (\tilde{\delta}_{j+1} \tilde{d}_j^2) e^{-\beta(t_k - t_0)} \le (\frac{A}{\lambda})^2, \quad k \in \mathbb{N},$$

where $m_1 \in \Lambda$, $\Lambda_1 := \{1, 2, \dots, m_1\}$, $\Lambda_2 := \{m_1 + 1, m_1 + 2, \dots, m\}$,

$$2a_i + b_i^2 < -1,\ i \in \Lambda_1; \quad 2a_i + b_i^2 \ge -1,\ i \in \Lambda_2;$$

$$p_1 = \max\{2a_i + b_i^2 : i \in \Lambda_1\}, \quad p_2 = \max\{2a_i + b_i^2 : i \in \Lambda_2\};$$

$$\sigma_1 = \sup\{t_k - t_{k-1} : i_k \in \Lambda_1\}, \quad \sigma_2 = \sup\{t_k - t_{k-1} : i_k \in \Lambda_2\};$$

$$\tilde{d}_0 = 1, \quad \tilde{d}_k = \max\{d_k, (\tilde{\delta}_{k+1})^{-1/2}\},\ k \in \mathbb{N}_+,$$

then system (4.1) is λ-A-uniformly practically stable.

Proof. Let $x(t)$ be the solution of (4.2) and set the function $V(t) = x^2(t)$. Then the derivative of $V(t)$ with respect to each subsystem is:

$$\begin{aligned} \dot{V(t)} &= 2x(t)\dot{x(t)} \\ &= 2x(t)[ax(t) + bx(t-r)] \\ &\le (2a_i + b_i^2)x(t)^2 + x(t-r)^2 \\ &\le (2a_i + b_i^2)V(t) + \sup_{\theta \in [-r,0]} V(t_0 + \theta). \end{aligned}$$

For any $t_0 \in \mathbb{R}^+$, $\|\varphi\| < \lambda$, we have:

$$\sup_{t \in [t_0 - r, t_0]} V(t) = \|\varphi\|^2 < \lambda^2;$$

$$V(t_k^+) = x^2(t_k^+) = d_k^2 x^2(t_k) = d_k^2 V(t_k), \quad k \in \mathbb{N}_+.$$

Define

$$g_1(t) = \begin{cases} V(t)e^{\beta(t - t_0)}, & t \in (t_0, +\infty] \\ V(t), & t \in [t_0 - r, t_0]. \end{cases}$$

Case 1: For any $k \in \mathbb{N}_+$, $d_k \ge (\tilde{\delta}_{k+1})^{-1/2}$.

(I) Consider the condition $t \in (t_0, t_1]$. Then we have

$$V(t_0^+) = d_0^2 V(t_0) < \tilde{\delta}_1 d_0^2 \sup_{t_0 - r \le s \le t_0} V(s) := \alpha_0,$$

where $d_0 = 1$. Note that

$$g_1(t) < \tilde{\delta}_1 d_0^2 \sup_{t_0 - r \le s \le t_0} V(s) = \alpha_0$$

for each $t \in [t_0 - r, t_0]$ and $g_1(t)$ is continuous on $[t_0 - r, t_1]$. We claim that $g_1(t) \le \alpha_0$, for any $t \in [t_0 - r, t_1]$. If not, there is a $\tilde{t}_1 \in (t_0, t_1]$ such that $g_1(\tilde{t}_1) > \alpha_0$. Define

$$\begin{gathered} t_1^* = \inf\{t \in (t_0, \tilde{t}_1] : g_1(t) > \alpha_0\}, \\ t_{1*} = \sup\{t \in [t_0, t_1^*] : g_1(t) \le d_0^2 \sup_{t_0 - r \le s \le t_0} V(s)\} \end{gathered}$$

Hence, $t_0 \le t_{1*} < t_1^* < t_1$ and $\tilde{\delta}_1 g_1(t) \ge g_1(s)$ for all $s \in [t_0 - r, t_1^*]$, $t \in [t_{1*}, t_1^*]$. By Lemma 3.4, $\tilde{\delta}_1 > \frac{g(t_1^*)}{g(t_{1*})} = \tilde{\delta}_1$. This contradiction proves that

$$g_1(t) \le \alpha_0, \quad V(t) \le \alpha_0 e^{-\beta(t - t_0)}, \quad t \in (t_0, t_1].$$

(II) Assume that $V(t) \le \alpha_i e^{-\beta(t - t_0)}$ for each $t \in (t_i, t_{i+1}]$, where

$$\alpha_i = \prod_{j=0}^{i} (\tilde{\delta}_{j+1} d_j^2) \sup_{t_0 - r \le s \le t_0} V(s), \quad i = 0, 1, \dots, k-1.$$

Below we prove $V(t) \le \alpha_k e^{-\beta(t - t_0)}$ for each $t \in (t_k, t_{k+1}]$. Note that

$$V(t_k^+) = d_k^2 V(t_k) \le d_k^2 \alpha_{k-1} e^{-\beta(t_k - t_0)} < \tilde{\delta}_{k+1} d_k^2 \alpha_{k-1} e^{-\beta(t_k - t_0)} := \alpha_k e^{-\beta(t_k - t_0)}.$$

Thus, $g_1(t_k^+) \le d_k^2 \alpha_{k-1} < \alpha_k$. Because $\{\alpha_k\}$ is nondecreasing, we have $g_1(t) \le \alpha_k$ for each $t \in [t_k - r, t_k]$. We claim that $g_1(t) \le \alpha_k$, if $t \in [t_k - r, t_{k+1}]$. Otherwise, by the continuity of $g_1(t)$ on $(t_k, t_{k+1}]$, there is a $\tilde{t}_k \in (t_k, t_{k+1}]$ such that $g_1(\tilde{t}_k) > \alpha_k$. Define

$$\begin{gathered} t_k^* = \inf\{t \in (t_k, \tilde{t}_k] : g_1(t) > \alpha_k\}, \\ E_k = \{t \in (t_k, t_k^*] : g_1(t) \le d_k^2 \alpha_{k-1}\}, \\ t_{k*} = \begin{cases} t_k, & \text{if } E_k = \emptyset \\ \sup E_k, & \text{if } E_k \ne \emptyset. \end{cases} \end{gathered}$$

Hence, $t_k \le t_{1*} < t_1^* \le t_{k+1}$ and $\tilde{\delta}_{k+1} g_1(t) \ge g_1(s)$ for any $s \in [t_k - r, t_k^*]$ and $t \in (t_{k*}, t_k^*]$. By Lemma 3.4, $\tilde{\delta}_{k+1} > \frac{g(t_k^*)}{g(t_{k*}^+)} = \tilde{\delta}_{k+1}$. This leads to a contradiction. So,

$$g_1(t) \le \alpha_k, \quad V(t) \le \alpha_k e^{-\beta(t - t_0)}, \quad t \in (t_k, t_{k+1}].$$

By mathematical induction, $V(t) \le \alpha_k e^{-\beta(t - t_0)}$ for every $t \in (t_k, t_{k+1}]$ and $k \in \mathbb{N}$. Since $d_k = \tilde{d}_k = \max\{d_k, (\tilde{\delta}_{k+1})^{-1/2}\}$, we have

$$|x(t)| = V^{1/2}(t) \le [\alpha_k e^{-\beta(t_k - t_0)}]^{1/2} < \Big[\lambda^2 \prod_{j=0}^{k} (\tilde{\delta}_{j+1} \tilde{d}_j^2) e^{-\beta(t_k - t_0)}\Big]^{1/2} \le A,$$

for every $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}$.

Case 2: There is some $k_0 \in \mathbb{N}_+$, such that $d_{k_0} < (\tilde{\delta}_{k_0+1})^{-1/2}$. We establish a new system

$$\begin{gathered} \dot{y}(t) = (2a_{i_k} + b_{i_k}^2) y(t) + y(t - r), \quad t \in (t_{k-1}, t_k] \\ y(t_k^+) = \tilde{d}_k^2 y(t_k), \quad k \in \mathbb{N}_+ \\ y_{t_0} = \varphi^2, \end{gathered} \tag{4.3}$$

where $\tilde{d}_k = \max\{d_k, (\tilde{\delta}_{k+1})^{-1/2}\}$. By Lemma 3.5 and the results of Case 1,
$$|x(t)| = V^{1/2}(t) \le y^{1/2}(t) < A.$$
So, system (4.1) is λ-A-uniformly practically stable. $\square$

Corollary 4.2. *Consider (λ, A) with $0 < \lambda < A$. Assume that there exist constants $\delta > 1$ and $\beta > 0$ satisfying:*
$$\beta < \frac{\ln \delta}{\sigma} - p - \delta e^{\beta r};$$
$$\delta^{k+1} \prod_{j=0}^{k} \tilde{d}_j^2 e^{-\beta(t_k - t_0)} \le (\frac{A}{\lambda})^2, \quad k \in \mathbb{N},$$
where $p = \max\{2a_i + b_i^2 : i \in \Lambda\}$, $d_0 = 1$, $\tilde{d}_k = \max\{d_k, (\tilde{\delta}_{k+1})^{-1/2}\}$, $\sigma = \sup\{t_{k+1} - t_k : k \in \mathbb{N}\}$. Then system (4.1) is λ-A-uniformly practically stable.

Below we study the one-dimensional system with variable coefficients. Namely, $f_{i_k}(t, x(t), x(t-r)) = a_{i_k}(t)x(t) + b_{i_k}(t)x(t-r)$ in (2.4).
$$\begin{gathered} \dot{x}(t) = a_{i_k}(t)x(t) + b_{i_k}(t)x(t-r), \quad t \in (t_{k-1}, t_k] \\ x(t_k^+) = d_k x(t_k), \quad k \in \mathbb{N}_+ \\ x_{t_0} = \varphi, \end{gathered} \tag{4.4}$$
where $a_{i_k}(t), b_{i_k}(t) : \mathbb{R}^+ \to \mathbb{R}$ are continuous functions, $d_k, r, t_0 \in \mathbb{R}^+$, $k \in \mathbb{N}_+$.

Theorem 4.3. *Consider (λ, A) with $0 < \lambda < A$. If there exist $t_0 \in \mathbb{R}^+$ and $\sigma > 0$ such that*
$$2a_{i_k}(t) + b_{i_k}^2(t) + 1 \le -\sigma < 0,\, t \in (t_{k-1}, t_k],$$
$$\prod_{i=1}^{k} \tilde{d}_i \le \frac{A}{\lambda}, \quad k \in \mathbb{N}_+,$$
where $\tilde{d}_i = \max\{d_i, 1\}$, then system (4.4) is λ-A-practically stable.

Proof. For any $\varphi \in C([-r, 0], \mathbb{R})$ and $\|\varphi\| < \lambda$, denote by $x(t)$ the solution of (4.4). Set the function $V(t) = x^2(t)$. Then the derivative of $V(t)$ with respect to each subsystem is:
$$\begin{aligned} \dot{V}(t) &= 2x(t)\dot{x(t)} \\ &= 2x(t)[a_i(t)x(t) + b_i(t)x(t-r)] \\ &\le [2a_i(t) + b_i^2(t)]x(t)^2 + x(t-r)^2 \\ &\le [2a_i(t) + b_i^2(t)]V(t) + \sup\{V(s) : s \in [t-r, t]\}. \end{aligned}$$
Hence, $\dot{V}(t) \le (2a_{i_k}(t) + b_{i_k}^2(t))V(t) + \sup\{V(s) : s \in [t-r, t]\}$, for each $t \in (t_{k-1}, t_k]$, $k \in \mathbb{N}_+$. Define
$$G = \sup\{V(s) : s \in [t_0 - r, t_0]\} = \sup\{\varphi^2(s) : s \in [-r, 0]\} < \lambda^2.$$
For any given $\varepsilon \in (1, 2)$, we have:

(I) Note that $V(t)$ is continuous on $(t_0 - r, t_1]$ and $V(t_0) \le G < \varepsilon G := \alpha_0$. Then we are to prove that $V(t) < \alpha_0$, for each $t \in (t_0, t_1]$. If not, there is a $\bar{t}_0 \in (t_0, t_1]$

such that $V(t) < \alpha_0$ for each $t \in (t_0, \bar{t}_0)$ and $V(\bar{t}_0) = \alpha_0$. Hence, $\dot{V}(\bar{t}_0) \geq 0$. But,

$$\begin{aligned}\dot{V}(\bar{t}_0) &\leq (2a_{i_1}(\bar{t}_0) + b_{i_1}^2(\bar{t}_0))V(\bar{t}_0) + \sup_{s\in[\bar{t}_0 - r, \bar{t}_0]} V(s)\\ &= [2a_{i_1}(\bar{t}_0) + b_{i_1}^2(\bar{t}_0) + 1]\alpha_0\\ &\leq -\sigma\alpha_0 < 0.\end{aligned}$$

This contradiction proves $V(t) < \alpha_0$ for $t \in (t_0, t_1]$.

(II) Assume that

$$V(t) < \prod_{i=0}^{j-1} \tilde{d}_i^2 \varepsilon G := \alpha_{j-1}$$

for each $t \in (t_{j-1}, t_j]$ and $j = 1, 2, \dots, k$, where $\tilde{d}_0 = 1$. Note that $V(t)$ is continuous on $(t_k, t_{k+1}]$ and $V(t_k^+) \leq (\tilde{d}_k x(t_k))^2 = \tilde{d}_k^2 V(t_k) < \tilde{d}_k^2 \alpha_{k-1} = \alpha_k$. Then we need to prove

$$V(t) < \prod_{i=1}^{k} \tilde{d}_i^2 \varepsilon G := \alpha_k, \quad t \in (t_k, t_{k+1}]. \tag{4.5}$$

If not, there exists a $\bar{t}_k \in (t_k, t_{k+1}]$ such that $V(t) < \alpha_k$ for each $t \in (t_k, \bar{t}_k)$ and $V(\bar{t}_k) = \alpha_k$. Hence, $\dot{V}(\bar{t}_k) \geq 0$. $\{\alpha_k\}$ is nondecreasing, so $V(t) \leq \alpha_k$ for each $t \in [t_0 - r, \bar{t}_k]$. It follows that

$$\begin{aligned}\dot{V}(\bar{t}_k) &\leq \Big(2a_{i_{k+1}}(\bar{t}_k) + b_{i_{k+1}}^2(\bar{t}_k)\Big)V(\bar{t}_k) + \sup_{s\in[\bar{t}_k - r, \bar{t}_k]} V(s)\\ &= [2a_{i_{k+1}}(\bar{t}_k) + b_{i_{k+1}}^2(\bar{t}_k) + 1]\alpha_k\\ &\leq -\sigma\alpha_k < 0.\end{aligned}$$

This contradiction proves (4.5). By mathematical induction, we have

$$V(t) < \alpha_k = \prod_{i=0}^{k} d_i^2 \cdot \varepsilon G.$$

for any $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}$. Furthermore,

$$V(t) \leq \prod_{i=0}^{k} d_i^2 G < \prod_{i=0}^{k} d_i^2 \lambda^2 \leq A^2, \quad \forall t \in (t_k, t_{k+1}],\ k \in \mathbb{N}.$$

Namely, $|x(t)| < A$ for $t \in [t_0, +\infty)$. So system (4.4) is λ-A-practically stable. □

Remark 4.4. We can loosen the restriction on d_k in Theorem 4.3, if the dwell time τ_k is big enough, $k \in \mathbb{N}_+$. As shown is Theorem 4.5.

Theorem 4.5. *Consider (λ, A) with $0 < \lambda < A$. Assume that there are constants $t_0 \in \mathbb{R}^+$ and $\sigma > 0$ such that*

$$2a_{i_k}(t) + b_{i_k}^2(t) + 1 \leq -\sigma < 0, \quad t \in (t_{k-1}, t_k];$$

and suppose

$$\tau_k > r, \quad \prod_{i=1}^{k} \tilde{d}_i^2 e^{-u(t_k - t_0 - kr)} \leq \frac{A^2}{\lambda^2}, \quad k \in \mathbb{N}_+,$$

where $\tau_k = t_k - t_{k-1}$, $\tilde{d}_i = max\{d_i, 1\}$, $u - (1+\sigma) + e^{ur} = 0$. Then system (4.4) is λ-A-practically stable.

Proof. For any $\varphi \in C([-r,0],\mathbb{R})$ and $\|\varphi\| < \lambda$, denote by $x(t)$ the solution of (4.4). Set the function $V(t) = x^2(t)$. Then the derivative of $V(t)$ with respect to each subsystem is:

$$\begin{aligned}\dot{V}(t) &= 2x(t)\dot{x(t)}\\ &= 2x(t)[a_i(t)x(t) + b_i(t)x(t-r)]\\ &\le [2a_i(t) + b_i^2(t)]x(t)^2 + x(t-r)^2\\ &= [2a_i(t) + b_i^2(t)]V(t) + V(t-r).\end{aligned}$$

Obviously, we have

$$\begin{gathered}V(t_k^+) = [d_k x(t_k)]^2 = d_k^2 V(t_k), \quad k \in \mathbb{N}_+,\\ 2a_{i_k}(t) + b_{i_k}^2(t) + 1 \le -\sigma < 0,\ t \in (t_{k-1}, t_k].\end{gathered}$$

Hence,

$$\begin{gathered}\dot{V}(t) \le -(1+\sigma)V(t) + V(t-r), \quad t \in (t_{k-1}, t_k]\\ V(t_k^+) = d_k^2 V(t_k), \quad k \in \mathbb{N}_+\\ V_{t_0} = \varphi^2.\end{gathered}$$

Define

$$G = \sup\{V(s) : s \in [t_0 - r, t_0]\} = \sup\{\varphi^2(s) : s \in [-r, 0]\} < \lambda^2.$$

Below we prove that $V(t) \le \prod_{i=0}^{k} \tilde{d}_i^2 G e^{-u(t-t_0-kr)}$ for $t \in (t_k, t_{k+1}]$, where $\tilde{d}_0 = 1$, and $u - (1+\sigma) + e^{ur} = 0$.

(I) If $t \in (t_0, t_1]$, we establish a comparison system:

$$\begin{gathered}\dot{W}_0(t) = -(1+\sigma)W_0(t) + W_0(t-r), \quad t \in (t_0, t_1]\\ W_0(t_0^+) = \tilde{d}_0^2 V(t_0)\\ W_{0t_0} = \tilde{d}_0^2 \varphi^2,\end{gathered}$$

where $\tilde{d}_0 = 1$. From Lemma 3.5, we have $V(t) \le W_0(t)$ for $t \in (t_0, t_1]$. Furthermore,

$$W_0(t) \le \tilde{d}_0^2 G e^{-u(t-t_0)}, \quad t \in (t_0, t_1],$$

by Halanay inequality. So, $V(t) \le \tilde{d}_0^2 G e^{-u(t-t_0)}$ for each $t \in (t_0, t_1]$.

(II) Assume that $V(t) \le \prod_{i=0}^{j-1} \tilde{d}_i^2 G e^{-u(t-t_0-(j-1)r)}$ for each $t \in (t_{j-1}, t_j]$ and $j = 1, 2, \dots, k$. Below we prove that

$$V(t) \le \prod_{i=1}^{k} \tilde{d}_i^2 G e^{-u(t-t_0-kr)},\ t \in (t_k, t_{k+1}]. \tag{4.6}$$

Consider (4.4) on $(t_k, t_{k+1}]$, and take t_k as the initial time. Then we establish a comparison system

$$\begin{gathered}\dot{W}_k(t) = -(1+\sigma)W_k(t) + W_k(t-r), \quad t \in (t_k, t_{k+1}]\\ W_k(t_k^+) = \tilde{d}_k^2 V(t_k)\\ W_{kt_k} = \tilde{d}_k^2 V^2(t - t_k).\end{gathered}$$

By Lemma 3.5, $V(t) \le W_k(t)$ for every $t \in (t_k, t_{k+1}]$. And $W_k(t)$ is continuous on $[t_k - r, t_{k+1}]$, because $t_k - t_{k-1} > r$. From Halanay inequality,

$$\begin{aligned} W_k(t) &\le \Big\{\tilde{d}_k^2 \cdot \prod_{i=0}^{k-1} \tilde{d}_i^2 G e^{-u(t_k - t_0 - kr)}\Big\} e^{-u(t-t_k)} \\ &= \prod_{i=1}^{k} \tilde{d}_i^2 G e^{-u(t-t_0-kr)}, \quad t \in (t_k, t_{k+1}]. \end{aligned}$$

Hence,

$$V(t) \le \prod_{i=1}^{k} \tilde{d}_i^2 G e^{-u(t-t_0-kr)}, \quad t \in (t_k, t_{k+1}].$$

By mathematical induction,

$$V(t) \le \prod_{i=1}^{k} \tilde{d}_i^2 G e^{-u(t-t_0-kr)} < \lambda^2 \prod_{i=1}^{k} \tilde{d}_i^2 e^{-u(t_k-t_0-kr)} \le A^2, \quad t \in (t_k, t_{k+1}],\ k \in \mathbb{N}.$$

Namely, $|x(t)| < A$, $t \in [t_0, +\infty)$. The proof is complete. $\square$

Remark 4.6. In Theorem 4.3, it requests that $a_{i_k}(t) \le 0$ if $t \in (t_{k-1}, t_k]$; However, Theorems 4.7 and 4.9 establish criterions, where $a_{i_k}(t)$ do not have to be negative.

Theorem 4.7. *Consider (λ, A) with $0 < \lambda < A$. Assume that there exist constants $t_0 \in \mathbb{R}^+$ and $\delta > 0$ such that:*

$$\begin{gathered} \Big(1 + \int_{t_0}^{t_0+r/2} |b_{i_1}(u)| du\Big) e^{\int_{t_0}^{t} a_{i_1}(s) ds} \le \frac{A}{\lambda}, \quad t \in (t_0, t_0 + r/2]; \\ \Big[1 + \int_{t_0-r}^{t_0} |b_{i_1}(s+r)| ds\Big]^2 + \int_{-r}^{0}\int_{t_0+s}^{t_0} b_{i_1}^2(z+r)\, dz\, ds \le \frac{A^2}{6\lambda^2}; \\ -\frac{1}{2r} \le g_{i_k}(t) \le -r b_{i_k}^2(t+r), \quad t \in (t_{k-1}, t_k]; \\ d_k \le \frac{\delta}{A h_k(t_k)}, \quad t_k - t_{k-1} \ge 2r, \quad k \in \mathbb{N}_+; \end{gathered}$$

for any $k \in \mathbb{N}_+$,

$$\begin{gathered} \max\{d_k \cdot h_k(t_k), h_k(t_k - r)\}\Big(1 + \int_{t_k}^{t_k+\frac{r}{2}} |b_{i_{k+1}}(u)| du\Big) e^{\int_{t_k}^{t} a_{i_{k+1}}(s) ds} \le 1, \\ t \in (t_k, t_k + \frac{r}{2}]; \end{gathered}$$

$$\Big[\delta + A \int_{t_k-r}^{t_k} |b_{i_{k+1}}(s+r)| h_k(s) ds\Big]^2 + A^2 \int_{-r}^{0}\int_{t_k+s}^{t_k} b_{i_{k+1}}^2(z+r) h_k^2(z)\, dz\, ds \le \frac{A^2}{6};$$

where $g_i(t) = a_i(t) + b_i(t+r)$, $h_k(t) = e^{\frac{1}{2}\int_{t_{k-1}}^{t-0.5r}[a_{i_k}(s) + b_{i_k}(s+r)] ds}$, $i \in \Lambda$, $k \in \mathbb{N}_+$. Then system (4.4) is λ-A-practically stable.

Proof. For any $\varphi \in C([-r, 0], \mathbb{R})$, $\|\varphi\| < \lambda$, let $x(t)$ be the solution of (4.4). For each $k \in \mathbb{N}_+$, we define a function:

$$V_k(u) = \Big[x(u) + \int_{u-r}^{u} b_{i_k}(s+r) x(s) ds\Big]^2 + \int_{-r}^{0}\int_{u+s}^{u} b_{i_k}^2(z+r) x^2(z)\, dz\, ds,$$

$u \in (t_{k-1}, t_k]$. (I) when $t \in (t_0, t_1]$, by Lemma 3.6,

$$\begin{aligned}|x(t)| &\le \|\varphi\|(1 + \int_{t_0}^{t_0+r/2} |b_{i_1}(u)|du)e^{\int_{t_0}^t a_{i_1}(s)ds} \\ &< \lambda(1 + \int_{t_0}^{t_0+r/2} |b_{i_1}(u)|du)e^{\int_{t_0}^t a_{i_1}(s)ds} \\ &\le A, \quad t \in (t_0, t_0 + r/2];\end{aligned}$$

$$\begin{aligned}|x(t)| &\le \sqrt{6V(t_0)}e^{\frac{1}{2}\int_{t_0}^{t-r/2} a_{i_1}(s)+b_{i_1}(s+r)ds} \\ &= \Big\{6\Big[x(t_0) + \int_{t_0-r}^{t_0} b_{i_1}(s+r)x(s)ds\Big]^2 \\ &\quad + 6\int_{-r}^0\int_{t_0+s}^{t_0} b_{i_1}^2(z+r)x^2(z)\,dz\,ds\Big\}^{1/2} h_1(t) \\ &< \Big\{6[\lambda + \lambda\int_{t_0-r}^{t_0} |b_{i_1}(s+r)|ds]^2 + 6\int_{-r}^0\int_{t_0+s}^{t_0} b_{i_1}^2(z+r)\lambda^2\,dz\,ds\Big\}^{1/2} h_1(t) \\ &\le Ah_1(t), \quad t \in (t_0 + r/2, t_1].\end{aligned}$$

So, $|x(t)| < A$, if $t \in (t_0, t_1]$.

(II) Assume that $|x(t)| < A$, if $t \in (t_j, t_j + \frac{r}{2}]$, and $|x(t)| < Ah_{j+1}(t)$, if $t \in [t_j + \frac{r}{2}, t_{j+1}]$, where $j = 0, 1, \dots, k-1$. Then we need to prove that

$$|x(t)| < A,\ t \in (t_k, t_k + \frac{r}{2}]; \quad |x(t)| < Ah_{k+1}(t),\ t \in [t_k + \frac{r}{2}, t_{k+1}]. \tag{4.7}$$

From Corollary 3.7,

$$\begin{aligned}|x(t)| &\le \max\Big\{d_k x(t_k), \sup_{s\in[t_k-r,t_k]}\{x(s)\}\Big\}\Big(1 + \int_{t_k}^{t_k+r/2} |b_{i_{k+1}}(u)|du\Big)e^{\int_{t_k}^t a_{k+1}(s)ds} \\ &< A\max\{d_k h_k(t_k), h_k(t_k - r)\}\Big(1 + \int_{t_k}^{t_k+r/2} |b_{i_{k+1}}(u)|du\Big)e^{\int_{t_k}^t a_{i_{k+1}}(s)ds} \\ &\le A, \quad t \in (t_k, t_k + r/2];\end{aligned}$$

$$\begin{aligned}|x(t)|^2 &\le 6V_{k+1}(t_k^+)e^{\int_{t_k}^{t-r/2} a_{i_{k+1}}(s)+b_{i_{k+1}}(s+r)ds} \\ &= 6\Big\{\Big[x(t_k^+) + \int_{t_k-r}^{t_k} b_{i_{k+1}}(s+r)x(s)ds\Big]^2 \\ &\quad + \int_{-r}^0\int_{t_k+s}^{t_k} b_{i_{k+1}}^2(z+r)x^2(z)\,dz\,ds\Big\}h_{k+1}^2(t) \\ &< 6\Big\{\Big[\delta + \int_{t_k-r}^{t_k} |b_{i_{k+1}}(s+r)|Ah_k(s)ds\Big]^2 \\ &\quad + \int_{-r}^0\int_{t_k+s}^{t_k} b_{i_{k+1}}^2(z+r)A^2h_k^2(z)\,dz\,ds\Big\}h_{k+1}^2(t) \\ &\le A^2h_{k+1}^2(t), \quad t \in (t_k + r/2, t_{k+1}].\end{aligned}$$

Hence, (4.7) holds. By mathematical induction, $|x(t)| < A,\ t \ge t_0$. Namely, system (4.4) is λ-A-practically stable. $\square$

Corollary 4.8. *Consider (λ, A) with $0<\lambda<A$. Assume that $a_i(t)=a_i$, $b_i(t)=b_i$, where $a_i, b_i \in \mathbb{R}, i\in\Lambda$. If there is a $\delta>0$ and the following assumptions are satisfied:*

$$(1+\frac{1}{2}|b_{i_1}|r)e^{a_{i_1}u} \le \frac{A}{\lambda}, \quad u\in(0,\frac{1}{2}]; \quad \frac{3}{2}b_{i_1}^2 r^2+2|b_{i_1}|r+1 \le \frac{A^2}{6\lambda^2};$$

$$-\frac{1}{2r} \le c_i \le -rb_i^2, \quad i\in\Lambda; \quad d_k \le \frac{\delta}{Aw_k(t_k)}, \quad \tau_k \ge 2r, \quad k\in\mathbb{N}_+;$$

for any $k\in\mathbb{N}_+$ and $u\in(0,\frac{1}{2}]$, $(1+\frac{1}{2}|b_{i_{k+1}}|r)e^{\frac{1}{2}c_{i_k}(\tau_k-1.5r)+ua_{i_{k+1}}} \le 1$;

$$\Big[\delta + A|b_{i_{k+1}}|\int_{t_k-r}^{t_k} w_k(s)ds\Big]^2 + A^2 b_{i_{k+1}}^2 \int_{t_k-r}^{t_k}(z-t_k+r)w_k^2(z)dz \le \frac{A^2}{6};$$

where $c_i=a_i+b_i, w_k(t)=e^{\frac{1}{2}c_{i_k}(t-t_{k-1}-0.5r)}$, $\tau_k=t_k-t_{k-1}$, $i\in\Lambda$, $k\in\mathbb{N}_+$, then system (4.4) is λ-A-uniformly practically stable.

Theorem 4.9. *Consider (λ, A) with $0<\lambda<A$. Assume that there are constants $t_0\in\mathbb{R}^+$, $\beta>0$ and $\delta>0$ such that*

(i) $|b_{i_k}(t)| \le ru_{i_k}(t)$ *for* $t\in(t_{k-1},t_k]$; $d_k \le \frac{\delta}{A}$;
(ii) $\big(1+r^2u_{i_1}(t_0)\big)g_1(t) \le \frac{A}{\lambda}$ *for* $t\in(t_0,t_1]$;
(iii) $\big[\delta + Ar^2u_{i_{k+1}}(t_k^+)\big]g_{k+1}(t) \le A$ *for* $t\in(t_k,t_{k+1}]$, $k\in\mathbb{N}_+$; *where*

$$u_i(t)=\frac{e^{\int_0^t a_i(s)ds}}{1+r\int_t^{t+\beta} e^{\int_0^u a_i(s)ds}du}, \quad g_k(t)=e^{\int_{t_{k-1}}^t [a_{i_k}(s)+ru_{i_k}(s)]ds}, \quad i\in\Lambda,\ k\in\mathbb{N}_+.$$

Then system (4.4) is λ-A-practically stable.

Proof. For any $\varphi\in C([-r,0],\mathbb{R})$, $\|\varphi\|<\lambda$, let $x(t)$ be the solution of (4.4). Define

$$V_i(t)=|x(t)|+ru_i(t)\int_{t-h}^t |x(s)|ds, \quad i\in\Lambda.$$

For any $k\in\mathbb{N}_+$ and $t\in(t_{k-1},t_k]$, we have

$$\begin{aligned}
\dot u_{i_k}(t) &= \frac{a_{i_k}(t)e^{\int_0^t a_{i_k}(s)ds}}{1+r\int_t^{t+\beta}e^{\int_0^u a_{i_k}(s)ds}du} - \frac{e^{r\int_0^t a_{i_k}(s)ds}}{(1+r\int_t^{t+\beta}e^{\int_0^u a_{i_k}(s)ds}du)^2} \\
&\quad \times \Big(e^{\int_0^{t+\beta} a_{i_k}(s)ds} - e^{\int_0^t a_{i_k}(s)ds}\Big) \\
&= a_{i_k}(t)u_{i_k}(t) - ru_{i_k}^2(t)(e^{\int_t^{t+\beta} a_{i_k}(s)ds}-1) \\
&\le a_{i_k}(t)u_{i_k}(t) + ru_{i_k}^2(t);
\end{aligned}$$

$$\begin{aligned}
&\dot V_{i_k}(t) \\
&\le [a_{i_k}(t)+ru_{i_k}(t)]|x(t)| + r\dot u_{i_k}(t)\int_{t-h}^t |x(s)|ds + (|b_{i_k}(t)|-ru_{i_k}(t))|x(t-r)| \\
&\le [a_{i_k}(t)+ru_{i_k}(t)]\Big(|x(t)|+ru_{i_k}(t)\int_{t-h}^t |x(s)|ds\Big) - [a_{i_k}(t)+ru_{i_k}(t)]r \\
&\quad \times u_{i_k}(t)\int_{t-h}^t |x(s)|ds + r\cdot[a_{i_k}(t)u_{i_k}(t)+ru_{i_k}^2(t)]\int_{t-h}^t |x(s)|ds \\
&\le [a_{i_k}(t)+ru_{i_k}(t)]V(t).
\end{aligned}$$

So, $|x(t)| \le V_{i_k}(t) \le V_{i_k}(t_k^+)e^{\int_{t_k}^t [a_{i_k}(s)+ru_{i_k}(s)]ds}$, $t \in (t_{k-1}, t_k]$.

(I) If $t \in (t_0, t_1]$, then

$$\begin{aligned} |x(t)| &\le V(t_0^+)e^{\int_{t_0}^t a_{i_1}(s)+ru_{i_1}(s)ds} \\ &= \Big(|x(t_0)| + ru_{i_1}(t_0)\int_{t_0-r}^t |x(s)|ds\Big)g_1(t) \\ &< \Big(\lambda + ru_{i_1}(t_0)\int_{t_0-r}^t \lambda ds\Big)g_1(t) \le A. \end{aligned}$$

(II) Assume that $|x(t)| < A$, if $t \in (t_i - 1, t_i]$, $i = 1, 2, \dots, k$. If $t \in (t_k, t_{k+1}]$, then

$$\begin{aligned} |x(t)| &\le V(t_k^+)e^{\int_{t_k}^t a_{i_{k+1}}(s)+ru_{i_{k+1}}(s)ds} \\ &= \Big(|x(t_k^+)| + ru_{i_{k+1}}(t_k)\int_{t_k-r}^t |x(s)|ds\Big)g_{k+1}(t) \\ &< \Big(A\frac{\delta}{A} + ru_{i_{k+1}}(t_k)\int_{t_k-r}^t Ads\Big)g_{k+1}(t) \le A. \end{aligned}$$

By mathematical induction, $|x(t)| < A$, $t \ge t_0$. The proof is complete. $\square$

Corollary 4.10. *Assume that $a_{i_k}(t) \equiv 0$, $k \in \mathbb{N}_+$. And suppose there exist constants $t_0 \ge 0$, and $\beta, \delta > 0$, such that*

$$|b_{i_k}(t)| \le \frac{h}{1+\beta r}; \quad d_k \le \frac{\delta}{A};$$

$$\big(1 + \frac{r^2}{1+\beta r}\big)e^{\frac{r(t_1-t_0)}{1+\beta r}} \le \frac{A}{\lambda}; \quad \big(\delta + \frac{Ar^2}{1+\beta r}\big)e^{\frac{r(t_{k+1}-t_k)}{1+\beta r}} \le A, \quad k \in \mathbb{N}_+.$$

Then system (4.4) is λ-A-practically stable.

Lastly, we consider the n-dimensional system with constant coefficients. That is, $f_{i_k}(t, x(t), x(t-r)) = A_{i_k}x(t) + B_{i_k}x(t-r)$ in (2.4).

$$\begin{gathered} \dot{x}(t) = A_{i_k}x(t) + B_{i_k}x(t-r), \quad t \in (t_{k-1}, t_k] \\ x(t_k^+) = d_k x(t_k), \quad k \in \mathbb{N}_+ \\ x_{t_0} = \varphi, \end{gathered} \tag{4.8}$$

where $x \in \mathbb{R}^n$, $d_k \in \mathbb{R}_+$, $\varphi \in C([-r, 0], \mathbb{R})$, A_i and B_i are $n \times n$ matrices, $k \in \mathbb{N}_+$, $i \in \Lambda$.

Theorem 4.11. *Consider (λ, A) with $0 < \lambda < A$. Assume that there is a $\eta > 0$, such that the linear matrix inequality with respect to symmetric matrices $\{P_i > 0\}_{i=1}^m$*

$$\begin{pmatrix} A_i^T P_i + P_i A_i + (1+\eta)P_i & P_i B_i \\ B_i^T P_i & -P_i \end{pmatrix} < 0, \quad i \in \Lambda \tag{4.9}$$

is solvable. And suppose that there exist constants $\delta > 1$ and $\beta > 0$ such that

$$\beta < \frac{\ln\delta}{\sigma} + 1 + \eta - \delta e^{\beta r}, \quad d_k^2 \ge \frac{1}{\delta},$$

$$\delta^{k+1}\chi^k \prod_{i=0}^k d_i^2 \lambda_{\max}(P_{i_1})e^{-\beta(t_k-t_0)} \le \lambda_{\min}(P_{i_{k+1}})\frac{A^2}{\lambda^2}, \ k \in \mathbb{N},$$

where $d_0 = 1$, $\sigma = \sup\{t_{n+1} - t_n : n \in \mathbb{N}\} < +\infty$, $\chi = \max\big\{\frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_j)} : i, j \in \Lambda, i \neq j\big\}$. *Then system* (4.8) *is* λ-A-*uniformly practically stable.*

Proof. By (4.9), we have

$$A_i^T P_i + P_i A_i + P_i B_i P_i^{-1} B_i^T P_i + (1+\eta) P_i < 0, \quad i \in \Lambda.$$

For any $\varphi \in C([-r, 0], \mathbb{R})$, $\|\varphi\| < \lambda$ and $t_0 \geq 0$, let $x(t)$ be the solution of (4.8). Establish a function $V_i = x(t)^T P_i x(t)$ with respect to the i-th subsystem of (4.8), and define $\overline{V}_i(t) = \sup_{-r \leq \theta \leq 0} V_i(t+\theta)$, $i \in \Lambda$. It follows that

$$\begin{aligned}
\dot{V}_i(t) &= 2x^T(t) P_i \dot{x}(t) \\
&= 2x^T(t) P_i (A_i x(t) + B_i x(t-r)) \\
&\leq x^T(t)(A_i^T P_i + P_i A_i + P_i B_i P_i^{-1} B_i^T P_i) x(t) + x^T(t-r) P_i x(t-r) \\
&\leq -(1+\eta) V_i(t) + \overline{V}_i(t),\ t \geq t_0.
\end{aligned}$$

At the switching time,

$$V_{i_{k+1}}(t_k^+) = x^T(t_k^+) P_{i_{k+1}} x(t_k^+) = d_k^2 x^T(t_k) P_{i_{k+1}} x(t_k) = d_k^2 V_{i_{k+1}}(t_k), \quad k \in \mathbb{N}_+.$$

If $t \in [t_0 - r, t_0]$, it is easy to verify that $V_{i_1}(t) \leq \lambda_{\max}(P_{i_1}) |x(t)|^2$. Hence,

$$\overline{V}_{i_1}(t_0) \leq \lambda_{\max}(P_{i_1}) \|\varphi\|^2 < \lambda^2 \cdot \lambda_{\max}(P_{i_1}).$$

(I) If $t \in (t_0, t_1]$, we claim that

$$V_{i_1}(t) \leq \delta d_0 \overline{V}_{i_1}(t_0) e^{-\beta(t-t_0)} := \alpha_0 e^{-\beta(t-t_0)}. \tag{4.10}$$

If not, define

$$g_0(t) = \begin{cases} V_{i_1}(t) e^{\beta(t-t_0)}, & t \in (t_0, t_1] \\ V_{i_1}(t), & t \in [t_0 - r, t_0]. \end{cases}$$

Then, there is a $\tilde{t}_0 \in (t_0, t_1]$, such that $g_0(\tilde{t}_0) > \alpha_0$. Define

$$\begin{gathered}
t_0^* = \inf\{t \in (t_0, t_1] : g_0(t) > \alpha_0\}; \\
t_{0*} = \sup\{t \in [t_0, t_0^*] : g_0(t) \leq \overline{V}_{i_1}(t_0)\}.
\end{gathered}$$

Then $t_0 \leq t_{0*} < t_0^* < t_1$, and $\delta g_0(t) \geq \alpha_0 \geq g_0(s)$ for any $s \in [t_0 - r, t_0^*]$, $t \in [t_{0*}, t_0^*]$. From Lemma 3.4, $\delta > \frac{g_0(t_0^*)}{g_0(t_{0*})} = \delta$. This contradiction proves (4.10).

(II) Assume that $V_{i_{j+1}}(t) \leq \alpha_j e^{-\beta(t-t_0)}$, where $t \in (t_j, t_{j+1}]$, $j = 0, 1, \dots, k-1$, and

$$\alpha_j = \delta^{j+1} \chi^j \prod_{i=0}^{j} d_i^2 \cdot \overline{V}_{i_1}(t_0).$$

Below, we are to prove $V_{i_{k+1}}(t) \leq \alpha_k e^{-\beta(t-t_0)}$ for every $t \in (t_k, t_{k+1}]$. Note that $V_{i_{k+1}}(t) \leq \chi V_{i_{j+1}}(t) \leq \chi \alpha_j e^{-\beta(t-t_0)}$, if $t \in (t_j, t_{j+1}]$, $j = 0, 1, \dots, k-1$.

$$V_{i_{k+1}}(t_k^+) = d_k^2 V_{i_{k+1}}(t_k) \leq \chi d_k^2 V_{i_k}(t_k) \leq \chi d_k^2 \alpha_{k-1} e^{-\beta(t_k - t_0)}.$$

Define

$$g_k(t) = \begin{cases} V_{i_{k+1}}(t) e^{\beta(t-t_0)}, & t \in (t_0, t_{k+1}] \\ V_{i_{k+1}}(t), & t \in [t_0 - r, t_0]. \end{cases}$$

Because $\alpha_i \le \alpha_j$ for any $i \le j$, we have $g_k(t) \le \delta(d_k^2\chi\alpha_{k-1}) := \alpha_k$, if $t \in [t_k - r, t_k]$. We claim that $g_k(t) \le \alpha_k$ for each $t \in (t_k, t_{k+1}]$. Otherwise, there is a $\tilde{t}_k \in (t_k, t_{k+1}]$, such that $g_k(\tilde{t}_k) > \alpha_k$. Define

$$
\begin{gathered}
t_k^* = \inf\{t \in (t_k, \tilde{t}_k] : g_k(t) > \alpha_k\};\\
E_k = \{t \in (t_k, t_k^*] : g_k(t) \le \alpha_{k-1} d_k^2 \chi\};\\
t_{k*} = \begin{cases} t_k, & \text{if } E_k = \emptyset\\ \sup E_k, & \text{if } E_k \neq \emptyset. \end{cases}
\end{gathered}
$$

Hence, $t_k \le t_{k*} < t_k^* \le t_{k+1}$, and $\delta g_k(t) \ge \alpha_k \ge g_k(s)$, for any $s \in [t_k - r, t_k^*]$, $t \in (t_{k*}, t_k^*]$. By Lemma 3.4, $\delta > \frac{g_k(t_k^*)}{g_k(t_{k*}^+)} = \delta$. This contradiction proves $g_k(t) \le \alpha_k$, for each $t \in (t_k, t_{k+1}]$. That is, $V_{i_{k+1}}(t) \le \alpha_k e^{-\beta(t-t_0)}$, if $t \in (t_k, t_{k+1}]$. By mathematical induction,

$$
V_{i_{k+1}} \le \alpha_k e^{-\beta(t-t_0)}, \quad t \in (t_k, t_{k+1}],\ k \in \mathbb{N}.
$$

It is easy to verify that $\lambda_{\min}(P_{i_{k+1}})|x(t)|^2 \le V_{i_{k+1}}(t)$. Furthermore,

$$
\begin{aligned}
|x(t)| &\le \sqrt{\frac{1}{\lambda_{\min}(P_{i_{k+1}})} V_{i_{k+1}}(t)}\\
&\le \Big[\frac{1}{\lambda_{\min}(P_{i_{k+1}})} \delta^{k+1}\chi^k \prod_{i=0}^{k} d_i^2 e^{-\beta(t_k-t_0)} \overline{V}_{i_1}(t_0)\Big]^{1/2}\\
&< \Big[\frac{\delta^{k+1}\chi^k}{\lambda_{\min}(P_{i_{k+1}})} \prod_{i=0}^{k} d_i^2 e^{-\beta(t_k-t_0)} \lambda_{\max}(P_{i_1})\lambda^2\Big]^{1/2}\\
&\le A, \quad t \in (t_k, t_{k+1}],\ k \in \mathbb{N}.
\end{aligned}
$$

So, system (4.8) is λ-A-uniformly practically stable. □

Corollary 4.12. *Consider (λ, A) with $0 < \lambda < A$. Suppose that there exist constants $\eta > 0$, $\delta > 1$, and $\beta > 0$, such that the linear matrix inequality with respect to symmetric matrices $\{P_i > 0\}_{i=1}^m$*

$$
\begin{pmatrix} A_i^T P_i + P_i A_i + (1+\eta)P_i & P_i B_i\\ B_i^T P_i & -P_i \end{pmatrix} < 0, \quad i \in \Lambda, \tag{4.11}
$$

has a common solution $P_i = P$, $i \in \Lambda$; and the following assumptions are satisfied:

$$
\begin{gathered}
\beta < \frac{\ln\delta}{\sigma} + 1 + \eta - \delta e^{\beta r}, \quad d_k^2 \ge \frac{1}{\delta},\\
\delta^{n+1} \prod_{k=0}^{n} d_k^2 \lambda_{\max}(P) e^{-\beta(t_n - t_0)} \le \lambda_{\min}(P)\frac{A^2}{\lambda^2}, \quad n \in \mathbb{N},
\end{gathered}
$$

where $d_0 = 1$, $\sigma = \sup\{t_{n+1} - t_n : n \in \mathbb{N}\}$. Then system(4.8) is λ-A-uniformly practically stable.

Remark 4.13. According to [2], the system is exponentially stable if (4.9) is true. Then, Theorem 4.11 and Corollary 4.10 tells us that the switched system with delay can keep practically stable under some impulse perturbation, if the subsystem are of some good quality, such as exponential stability. That is to say, the restriction on impulses is loose in this case.

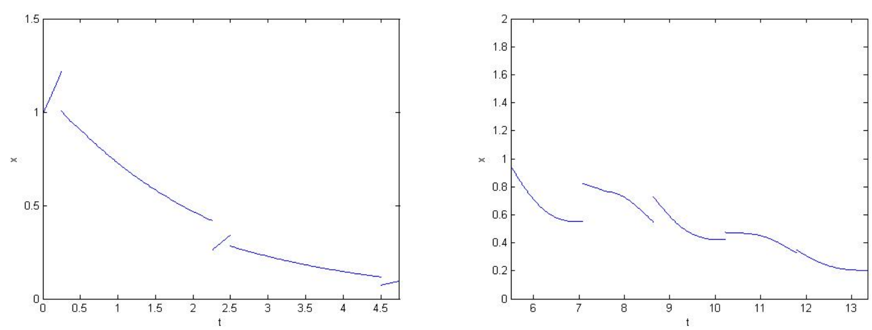
5. Examples

In this section, several examples are given to illustrate our theorems.

Example 5.1. Consider system (4.1), where

$$\dot{x}(t) = a_1 x(t) + b_1 x(t-r) = -x(t) + \frac{1}{2}x(t-0.25);$$
$$\dot{x}(t) = a_2 x(t) + b_2 x(t-r) = \frac{3}{8}x(t) + \frac{1}{2}x(t-0.25);$$
$$S = \{(0.25,2),(2,1),(0.25,2),\dots\};$$
$$d_k = \begin{cases} \frac{\sqrt{5}e^{0.1}}{4} & \text{if } k \text{ is even} \\ \frac{\sqrt{6}e^{0.0125}}{3} & \text{if } k \text{ is odd.} \end{cases}$$

Given $\lambda = 1$, $A = 1.8$, we set $p_1 = -1.75$, $p_2 = 1$, $\delta_1 = 1.5$, $\delta_2 = 3.2$, $\sigma_1 = 2$, $\sigma_2 = 0.25$, $\beta = 0.1$. According to Theorem 4.1, it is easy to verify that the system is λ-A-uniformly practically stable (see Figure 1(a)).



(a) (b)

Figure 1. (a) $t_0 = 0$, $\varphi = 0.99$; (b) $\varphi = 0.95$

Example 5.2. Consider system (4.4), where

$$\dot{x}(t) = a_1(t)x(t) + b_1(t)x(t-r) = (-0.6 + \frac{\sin^4(t)}{8})x(t) + \frac{1}{2}\sin^2(t)x(t-\frac{\pi}{6}),$$
$$\dot{x}(t) = a_2(t)x(t) + b_2(t)x(t-r) = (-0.6 + \frac{\cos^4(t)}{8})x(t) + \frac{1}{2}\cos^2(t)x(t-\frac{\pi}{6});$$
$$S = \{(\frac{\pi}{2},1),(\frac{\pi}{2},2),(\frac{\pi}{2},1),(\frac{\pi}{2},2),\dots\}; d_1 = 1.5, \quad d_k = \frac{k^2}{k^2-1}, \quad k \ge 2.$$

Given $\lambda = 1$, $A = 3$, we set $t_0 = \frac{7\pi}{4}$, $\sigma = 0.075$. According to Theorem 4.3, it is easy to verify that the system is λ-A-practically stable (see Figure 1(b)).

Example 5.3. Consider system (4.4), where

$$\dot{x}(t) = a_1(t)x(t) + b_1(t)x(t-r) = -\sin(\pi t)x(t) + \frac{1}{4}e^{\frac{\cos(\pi t)-1}{\pi}}x(t-\frac{1}{2}),$$
$$\dot{x}(t) = a_2(t)x(t) + b_2(t)x(t-r) = -\sin(2\pi t)x(t) + \frac{1}{4}e^{\frac{\cos(2\pi t)-1}{2\pi}}x(t-\frac{1}{2});$$

$$S = \{(2,1),(2,2),(2,1),(2,2),\dots\}; \quad d_k = \frac{9}{20}, \quad k \in \mathbb{N}_+.$$

Given $\lambda = 1$, $A = 2$, we set $t_0 = 0$, $\delta = 0.9$, $\beta = 2$. Then

$$\begin{aligned}
u_1(t) &= \frac{e^{\int_0^t -\sin(\pi s)ds}}{1+\frac{1}{2}\int_t^{t+2} e^{\int_0^u -\sin(\pi s)ds}du} \\
&= \frac{e^{[\cos(\pi t)-1]/\pi}}{1+\frac{1}{2}\int_t^{t+2} e^{[\cos(\pi u)-1]/\pi}du} \\
&\geq \frac{e^{[\cos(\pi t)-1]/\pi}}{1+\frac{1}{2}(e^0+e^{\frac{-1}{\pi}})} \\
&\geq \frac{1}{2}e^{\frac{\cos(\pi t)-1}{\pi}} = \frac{1}{r}b_1(t).
\end{aligned}$$

$$\begin{aligned}
u_1(t) &= \frac{e^{[\cos(\pi t)-1]/\pi}}{1+\frac{1}{2}\int_t^{t+2} e^{[\cos(\pi u)-1]/\pi}du} \\
&\leq \frac{e^{[\cos(\pi t)-1]/\pi}}{1+\frac{1}{2}(e^{\frac{-2}{\pi}}+e^{\frac{-1}{\pi}})} \\
&\leq 0.6142e^{\frac{\cos(\pi t)-1}{\pi}} \leq \frac{5}{8}e^{\frac{\cos(\pi t)-1}{\pi}}.
\end{aligned}$$

$$\begin{aligned}
u_2(t) &= \frac{e^{\int_0^t -\sin(2\pi s)ds}}{1+\frac{1}{2}\int_t^{t+2} e^{\int_0^u -\sin(2\pi s)ds}du} \\
&= \frac{e^{[\cos(2\pi t)-1]/(2\pi)}}{1+\frac{1}{2}\int_t^{t+2} e^{[\cos(2\pi u)-1]/(2\pi)}du} \\
&\geq \frac{e^{[\cos(2\pi t)-1]/(2\pi)}}{1+\frac{1}{2}(e^0+e^{\frac{-1}{2\pi}})} \\
&\geq \frac{1}{2}e^{\frac{\cos(2\pi t)-1}{2\pi}} = \frac{1}{r}b_2(t).
\end{aligned}$$

$$\begin{aligned}
u_2(t) &= \frac{e^{[\cos(2\pi t)-1]/(2\pi)}}{1+\frac{1}{2}\int_t^{t+2} e^{[\cos(2\pi u)-1]/(2\pi)}du} \\
&\leq \frac{e^{[\cos(2\pi t)-1]/(2\pi)}}{1+\frac{1}{2}(e^{\frac{-2}{2\pi}}+e^{\frac{-1}{2\pi}})} \\
&\leq \frac{100}{179}e^{\frac{2\cos(\pi t)-1}{2\pi}}.
\end{aligned}$$

Consequently, if $t \in (t_0, t_1]$,

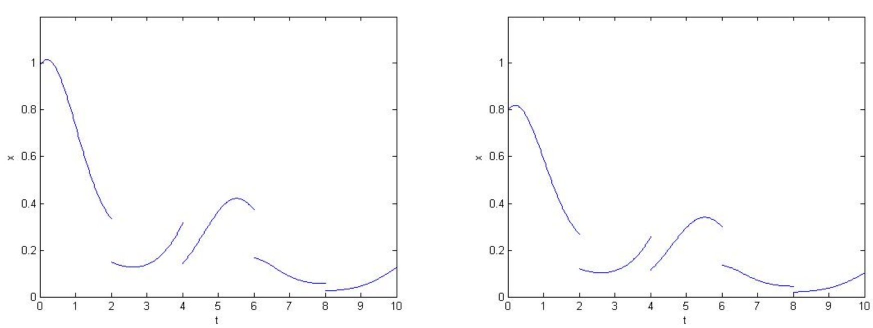
$$\begin{aligned}
\Big(1+r^2u_{i_1}(t_0)\Big)e^{\int_{t_0}^t a_{i_1}(s)+ru_{i_1}(s)ds} &\leq [1+\frac{1}{4}\times\frac{5}{8}]e^{\int_0^2 \frac{1}{2}\times\frac{5}{8}e^{\frac{\cos(\pi s)-1}{\pi}}ds} \\
&\leq \frac{37}{32}e^{\frac{5}{16}(e^0+e^{\frac{-1}{\pi}})} \\
&\leq 2 = \frac{A}{\lambda};
\end{aligned}$$

if $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}_+$,

$$\begin{aligned}
&\Big[\delta + Ar^2 u_1(t_k^+)\Big] e^{\int_{t_k}^{t} [a_1(s) + r u_1(s)] ds} \\
&\le [0.9 + 2 \times \frac{1}{4} \times 0.6142] e^{\int_{t_k}^{t_k+2} \frac{1}{2} \times \frac{5}{8} e^{\frac{\cos(\pi s) - 1}{\pi}} ds} \\
&\le (0.9 + 0.3071) e^{\frac{5}{16}(e^0 \times \frac{2}{3} + e^{\frac{-1}{2\pi}} \times \frac{1}{3} + e^{\frac{-1}{\pi}} \times \frac{1}{3} + e^{\frac{-3}{2\pi}} \times \frac{2}{3})} \\
&\le 2 = A.
\end{aligned}$$

$$\begin{aligned}
\Big[\delta + Ar^2 u_2(t_k^+)\Big] e^{\int_{t_k}^{t} [a_2(s) + r u_2(s)] ds} &\le [0.9 + 2 \times \frac{1}{4} \times \frac{100}{179}] e^{\int_{t_k}^{t_k+2} \frac{1}{2} \times \frac{100}{179} e^{\frac{\cos(2\pi s) - 1}{2\pi}} ds} \\
&\le (0.9 + \frac{50}{179}) e^{\frac{50}{179}(e^0 + e^{\frac{-1}{2\pi}})} \\
&\le 2 = A.
\end{aligned}$$

And $d_k = 9/20 = \delta/A$ yields $d_k \le \delta/A$. According to Theorem 4.9, it is easy to verify that the system is λ-A-practically stable (see Figure 2).



(a) (b)

FIGURE 2. (a) $\varphi = 0.99$; (b) $\varphi = 0.8$

Example 5.4. Consider system (4.8), where

$$A_1 = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -4 & 0 \\ 0 & -3 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

$$S = \{(2,1), (2,2), (2,1), (2,2), \dots\}; \quad d_1 = \frac{20 e^{0.3}}{11}, \quad d_k = \frac{e^{0.3}}{\sqrt{1.1}}, \ k \ge 2.$$

$r = 1$. Given $\lambda = 1, A = 2$, we set $\eta = 0.9$, $\delta = 1.1$, $\sigma = 2$, $\beta = 0.3$, $P_1 = P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. According to Theorem 4.11, it is easy to verify that the system is λ-A-practically stable.

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