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BOUNDS FOR SOLUTIONS TO RETARDED NONLINEAR DOUBLE INTEGRAL INEQUALITIES

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Abstract. We present bounds for the solution to three types retarded nonlinear integral inequalities in two variables. By doing this, we generalizing the results presented in [3, 12]. To illustrate our results, we present some applications.

1. Introduction

In the study of the qualitative behavior for solutions to nonlinear differential and integral equations, some specific types of inequalities are needed. The Gronwall inequality [5] and the nonlinear version by Bihari [1] are fundamental tools in the study of existence, uniqueness, boundedness, stability of solutions of differential, integral, and integro-differential equations. For this reason, several generalizations of the Gronwall inequality have been obtained, see [2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. Retarded integral inequalities have played an extensive role in the study of partial differential and integral equations.

In Section 2 of this article, based on the assumptions (A1)–(A3) below, we derive explicit bounds for the solutions to three types inequalities of retarded nonlinear integral equations in two variables. In Section 3, the bounds are applied for proving the global boundedness of solutions to the initial boundary-value problems. We stud the following three inequalities:

$$
\begin{aligned}
\varphi(u(t,s)) \le a(t,s) + b(t,s)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)} \phi(u(x,y))\Big[f_i(x,y)(w(u(x,y))\\
+ \int_{\alpha_i(t_0)}^{x}\int_{\beta_i(s_0)}^{y} g_i(m,n)w(u(m,n))\,dn\,dm) + h_i(x,y)\Big]\,dy\,dx
\end{aligned}
\tag{1.1}
$$

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$$
\begin{aligned}
\varphi(u(t,s)) \le a(t,s) + b(t,s)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)} &\phi(u(x,y))\Big[f_i(x,y)\phi_1(u(x,y)) \\
&\times (w(u(x,y)) + \int_{\alpha_i(t_0)}^{x}\int_{\beta_i(s_0)}^{y} g_i(m,n)w(u(m,n))\,dn\,dm) \\
&+ h_i(x,y)\phi_2(\log(u(x,y)))\Big]\,dy\,dx
\end{aligned} \tag{1.2}
$$

$$
\begin{aligned}
\varphi(u(t,s)) \le a(t,s) + b(t,s)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)} &\phi(u(x,y))\Big[f_i(x,y)(w(u(x,y)) \\
&+ \int_{\alpha_i(t_0)}^{x}\int_{\beta_i(s_0)}^{y} g_i(m,n)w(u(m,n))\,dn\,dm) \\
&+ h_i(x,y)L(x,y,w(u(x,y)))\Big]\,dy\,dx
\end{aligned} \tag{1.3}
$$

2. Main Results

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}_+ : [0,\infty)$; let t_0, t_1, s_0, s_1 be real numbers such that $I := [t_0, t_1)$; $J := [s_0, s_1)$. Denote by $C^i(M,N)$, the class of all i-times continuously differentiable functions defined on the set M to the set N, $1 \le i \le n$ and $C^0(M,N) = C(M,N)$. The first order partial derivatives of a function $z(x,y)$ defined on $\mathbb{R}^2$ with respect to x and y are denoted by $D_1 z(x,y) (= z_x(x,y))$ and $D_2 z(x,y) (= z_y(x,y))$ respectively. To prove our main results, we first list the following assumptions:

- (A1) $a, b : I \times J \to (0,\infty)$ are nondecreasing in each variable;
- (A2) $\varphi, w \in C(\mathbb{R}_+, \mathbb{R}_+)$, where φ and w are strictly increasing and nondecreasing functions respectively with $\varphi(0) = 0$; $\varphi(t) \to \infty$ as $t \to \infty$ and $w > 0$ on $(0,\infty)$;
- (A3) let $\alpha_i \in C^1(I, I)$ and $\beta_i \in C^1(J, J)$ be non-decreasing with $\alpha_i(t) \le t$ on I and $\beta_i(s) \le s$ on J;
- (A4) let $u, f_i, g_i, h_i \in C(I \times J, \mathbb{R}_+)$, $1 \le i \le n$ and $\phi \in C(\mathbb{R}_+, \mathbb{R}_+)$ a non-decreasing function such that $\phi(r) > 0$ for $r > 0$;
- (A5) let $\phi_1, \phi_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing functions with $\phi_1(r) > 0$ and $\phi_2(r) > 0$ for $r > 0$.

Theorem 2.1. *Assume conditions* (A1)–(A4) *and relation* (1.1) *hold. Then*

$$
u(t,s) \le \varphi^{-1}(G^{-1}(\Psi^{-1}(\Psi(c(t,s)) + b(t,s)D(t,s)))), \tag{2.1}
$$

for all $(t,s) \in [t_0, T_3) \times [s_0, S_3)$ provided that $\varphi^{-1}, G^{-1}, \Psi^{-1}$ are the respective inverses of φ, G, Ψ, and $(T_3, S_3) \in I \times J$ is arbitrarily chosen on the boundary of the planar region: $\mathfrak{R} := \{(t,s) \in I \times J\}$, provided that the following three relations hold:

$$
\begin{gathered}
\Psi(c(t,s)) + b(t,s)D(t,s) \in \mathrm{Dom}(\Psi^{-1}), \\
\Psi^{-1}(\Psi(c(t,s)) + b(t,s)D(t,s)) \in \mathrm{Dom}(G^{-1}), \\
G^{-1}(\Psi^{-1}(\Psi(c(t,s)) + b(t,s)D(t,s))) \in \mathrm{Dom}(\varphi^{-1}),
\end{gathered} \tag{2.2}
$$

where

$$
G(r) := \int_{r_0}^{r} \frac{dp}{\phi(\varphi^{-1}(p))}, \quad r \ge r_0 \ge 0,
$$

$$\Psi(z) := \int_{z_0}^{z} \frac{dl}{w(\varphi^{-1}(G^{-1}(l)))}, \quad z \geq z_0 \geq 0,$$

$$D(t,s) := \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \int_{\beta_i(s_0)}^{\beta_i(s)} f_i(z,y)[1 + \int_{\alpha_i(t_0)}^{z} \int_{\beta_i(s_0)}^{y} g_i(m,n)\, dn\, dm]\, dy\, dz.$$

$$c(t,s) := G(a(t,s)) + b(t,s) \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \int_{\beta_i(s_0)}^{\beta_i(s)} h_i(u,y)\, dy\, du.$$

Proof. By Assumption (A2) and inequality (1.1), we have

$$\begin{aligned} \varphi(u(t,s)) \leq a(T,s) + b(T,s) \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \int_{\beta_i(s_0)}^{\beta_i(s)} \phi(u(x,y)) \Big[f_i(x,y)(w(u(x,y)) \\ + \int_{\alpha_i(t_0)}^{x} \int_{\beta_i(s_0)}^{y} g_i(m,n) w(u(m,n))\, dn\, dm) + h_i(x,y) \Big]\, dy\, dx \end{aligned} \tag{2.3}$$

for all $(t,s) \in [t_0,T] \times J$, $T \leq T_3$. Denote the right hand side of (2.3) by $\eta(t,s)$, then obviously $\eta(t,s)$ is positive and non-decreasing function in each variable such that $\eta(t_0,s) = a(T,s)$. Then, (2.3) is equivalent to

$$u(t,s) \leq \varphi^{-1}(\eta(t,s)). \tag{2.4}$$

$$\begin{aligned} &\eta_t(t,s) \\ &= b(T,s) \sum_{i=1}^{n} \alpha_i'(t) \int_{\beta_i(s_0)}^{\beta_i(s)} \phi(u(\alpha_i(t),y)) \Big[f_i(\alpha_i(t),y)(w(u(\alpha_i(t),y)) \\ &\quad + \int_{\alpha_i(t_0)}^{\alpha_i(t)} \int_{\beta_i(s_0)}^{y} g_i(m,n) w(u(m,n))\, dn\, dm) + h_i(\alpha_i(t),y) \Big] dy \\ &\leq b(T,s) \sum_{i=1}^{n} \alpha_i'(t) \int_{\beta_i(s_0)}^{\beta_i(s)} \phi(\varphi^{-1}(\eta(\alpha_i(t),y))) \Big[f_i(\alpha_i(t),y)(w(\varphi^{-1}(\eta(\alpha_i(t),y))) \\ &\quad + \int_{\alpha_i(t_0)}^{\alpha_i(t)} \int_{\beta_i(s_0)}^{y} g_i(m,n) w(\varphi^{-1}(\eta(m,n)))\, dn\, dm) + h_i(\alpha_i(t),y) \Big] dy, \end{aligned}$$

which implies

$$\begin{aligned} \eta_t(t,s) \leq \phi(\varphi^{-1}(\eta(t,s))) b(T,s) \sum_{i=1}^{n} \alpha_i'(t) \int_{\beta_i(s_0)}^{\beta_i(s)} \Big[f_i(\alpha_i(t),y)(w(\varphi^{-1}(\eta(\alpha_i(t),y))) \\ + \int_{\alpha_i(t_0)}^{\alpha_i(t)} \int_{\beta_i(s_0)}^{y} g_i(m,n) w(\varphi^{-1}(\eta(m,n)))\, dn\, dm) + h_i(\alpha_i(t),y) \Big] dy. \end{aligned} \tag{2.5}$$

Then, (2.5) is equivalent to

$$\begin{aligned} &\frac{\eta_t(t,s)}{\phi(\varphi^{-1}(\eta(t,s)))} \\ &\leq b(T,s) \sum_{i=1}^{n} \alpha_i'(t) \int_{\beta_i(s_0)}^{\beta_i(s)} \Big[f_i(\alpha_i(t),y)(w(\varphi^{-1}(\eta(\alpha_i(t),y))) \\ &\quad + \int_{\alpha_i(t_0)}^{\alpha_i(t)} \int_{\beta_i(s_0)}^{y} g_i(m,n) w(\varphi^{-1}(\eta(m,n))) dn\, dm) + h_i(\alpha_i(t),y) \Big] dy, \end{aligned}$$

for all $(t,s) \in [t_0,T]\times J$. Replace t by v then integrating from t_0 to t with respect to v and making change of variable on right hand side of the above inequality and using the definition of G, we have

$$\begin{aligned}
G(\eta(t,s)) &\le G(\eta(t_0,s)) + b(T,s)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)} h_i(u,y)\,dy\,du \\
&\quad + b(T,s)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)} f_i(u,y)(w(\varphi^{-1}(\eta(u,y))) \\
&\quad + \int_{\alpha_i(t_0)}^{u}\int_{\beta_i(s_0)}^{y} g_i(m,n)w(\varphi^{-1}(\eta(m,n)))\,dn\,dm)\,dy\,du \\
&\le G(a(T,s)) + b(T,s)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(T)}\int_{\beta_i(s_0)}^{\beta_i(s)} h_i(u,y)\,dy\,du \\
&\quad + b(T,s)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)} f_i(u,y)(w(\varphi^{-1}(\eta(u,y))) \\
&\quad + \int_{\alpha_i(t_0)}^{u}\int_{\beta_i(s_0)}^{y} g_i(m,n)w(\varphi^{-1}(\eta(m,n)))\,dn\,dm)\,dy\,du \\
&\le c(T,s) + b(T,s)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)} f_i(u,y)(w(\varphi^{-1}(\eta(u,y))) \\
&\quad + \int_{\alpha_i(t_0)}^{u}\int_{\beta_i(s_0)}^{y} g_i(m,n)w(\varphi^{-1}(\eta(m,n)))\,dn\,dm)\,dy\,du.
\end{aligned} \tag{2.6}$$

Denote the right hand side of (2.6) by $\Gamma(t,s)$, then obviously $\Gamma(t,s)$ is positive and non-decreasing function in each variable such that $\Gamma(t_0,s) = c(T,s)$. Then, (2.6) is equivalent to

$$\eta(t,s) \le G^{-1}(\Gamma(t,s)). \tag{2.7}$$

By the fact that $\alpha_i(t) \le t$ and $\beta_i(s) \le s$ for $(t,s) \in I\times J$, $1 \le i \le n$, and monotonicity of Γ, w and φ^{-1}, we have

$$\begin{aligned}
\Gamma_t(t,s) &= b(T,s)\sum_{i=1}^{n}\alpha_i'(t)\int_{\beta_i(s_0)}^{\beta_i(s)} f_i(\alpha_i(t),y)(w(\varphi^{-1}(\eta(\alpha_i(t),y))) \\
&\quad + \int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{y} g_i(m,n)w(\varphi^{-1}(\eta(m,n)))\,dn\,dm)dy \\
&\le b(T,s)\sum_{i=1}^{n}\alpha_i'(t)\int_{\beta_i(s_0)}^{\beta_i(s)} f_i(\alpha_i(t),y)(w(\varphi^{-1}(\eta(t,y))) \\
&\quad + \int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{y} g_i(m,n)w(\varphi^{-1}(\eta(t,y)))\,dn\,dm)dy \\
&\le b(T,s)w(\varphi^{-1}(G^{-1}(\Gamma(t,s))))\sum_{i=1}^{n}\alpha_i'(t)\int_{\beta_i(s_0)}^{\beta_i(s)} f_i(\alpha_i(t),y) \\
&\quad \times\Big(1 + \int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{y} g_i(m,n)\Big)\,dn\,dm)dy.
\end{aligned} \tag{2.8}$$

Then, (2.8) is written as

$$\begin{aligned}&\frac{\Gamma_t(t,s)}{w(\varphi^{-1}(G^{-1}(\Gamma(t,s))))}\\&\le b(T,s)\sum_{i=1}^{n}\alpha_i'(t)\int_{\beta_i(s_0)}^{\beta_i(s)}f_i(\alpha_i(t),y)\Big(1+\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{y}g_i(m,n)\,dn\,dm\Big)\,dy\end{aligned}$$

Replace t by q then integrating from t_0 to t with respect to q and making change of variable on right hand side of the above inequality and using the definition of Ψ, we obtain

$$\Psi(\Gamma(t,s))\le\Psi(c(T,s))+b(T,s)D(t,s). \tag{2.9}$$

A combination of (2.4), (2.7) and (2.9) yield the desire result (2.1). □

Theorem 2.2. *Assume conditions* (A1)–(A5) *and relation* (1.2) *hold. Then*

- *if* $\phi_1(r)\ge\phi_2(\log(r))$, *we have*
$$u(t,s)\le\varphi^{-1}(G^{-1}(H_1^{-1}(J_1^{-1}(J_1(\widetilde{c}(T,s))+b(T,s)D(t,s))))), \tag{2.10}$$
for $(t,s)\in[t_0,T_1)\times[s_0,S_1)$,
- *if* $\phi_1(r)<\phi_2(\log(r))$, *we have*
$$u(t,s)\le\varphi^{-1}(G^{-1}(H_2^{-1}(J_2^{-1}(J_2(\widetilde{c}(T,s))+b(T,s)D(t,s))))), \tag{2.11}$$

for all $(t,s)\in[t_0,T_2)\times[s_0,S_2)$, *provided that* $\varphi^{-1},G^{-1},H_j^{-1}$ *and* J_j^{-1} *are the respective inverses of* φ,G,H_j *and* H_j; *let* $(T_j,S_j)\in I\times J$ *be arbitrarily chosen on the boundary of the planar region* $\mathfrak{R}_j:=\{(t,s)\in I\times J\}$, $j\in\{1,2\}$, *provided that the following four relations are satisfied*

$$\begin{gathered}J_j(\widetilde{c}(T,s))+b(T,s)D(t,s)\in\operatorname{Dom}(J_j^{-1}),\\ J_j^{-1}(J_j(\widetilde{c}(T,s))+b(T,s)D(t,s))\in\operatorname{Dom}(H_j^{-1}),\\ H_j^{-1}(J_j^{-1}(J_j(\widetilde{c}(T,s))+b(T,s)D(t,s)))\in\operatorname{Dom}(G^{-1}),\\ G^{-1}(H_j^{-1}(J_j^{-1}(J_j(\widetilde{c}(T,s))+b(T,s)D(t,s))))\in\operatorname{Dom}(\varphi^{-1}),\end{gathered}\tag{2.12}$$

where

$$\begin{gathered}\widetilde{c}(t,s)=H_j(G(a(T,s)))+b(T,s)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)}h_i(u,y)\,dy\,du,\\ H_j(r)=\int_{r_0}^{r}\frac{ds}{\phi_j(\varphi^{-1}(G^{-1}(s)))},\\ J_j(r)=\int_{r_0}^{r}\frac{ds}{w(\varphi^{-1}(G^{-1}(H_j^{-1}(s))))},\quad r\ge r_0\ge0.\end{gathered}$$

Proof. By condition (A2) and inequality (1.2), we have

$$\begin{aligned}\varphi(u(t,s))\le a(T,s)+b(T,s)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)}\phi(u(x,y))\Big[f_i(x,y)\\ \times\phi_1(u(x,y))(w(u(x,y))+\int_{\alpha_i(t_0)}^{x}\int_{\beta_i(s_0)}^{y}g_i(m,n)w(u(m,n))\,dn\,dm)\\ +h_i(x,y)\phi_2\big(\log(u(x,y))\big)\Big]\,dy\,dx\end{aligned}\tag{2.13}$$

for all $(t,s) \in [t_0,T] \times J$, $T \le T_1$. Denote the right hand side of (2.13) by $\Theta(t,s)$, then obviously $\Theta(t,s)$ is positive and non-decreasing function in each variable such that $\Theta(t_0,s) = a(T,s)$. Then (2.13) is equivalent to

$$u(t,s) \le \varphi^{-1}(\Theta(t,s)). \tag{2.14}$$

By the fact that $\alpha_i(t) \le t$ and $\beta_i(s) \le s$ for $(t,s) \in I \times J$, $1 \le i \le n$, and monotonicity of ϕ, φ^{-1}, Θ, we have

$$\begin{aligned}
\Theta_t(t,s) &= b(T,s)\sum_{i=1}^{n}\alpha_i'(t)\int_{\beta_i(s_0)}^{\beta_i(s)}\phi(u(\alpha_i(t),y))\Big[f_i(\alpha_i(t),y)\phi_1(u(\alpha_i(t),y))\\
&\quad\times (w(u(\alpha_i(t),y)) + \int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{y} g_i(m,n)w(u(m,n))\,dn\,dm)\\
&\quad + h_i(\alpha_i(t),y)\phi_2(\log(u(\alpha_i(t),y)))\Big]dy\\
&\le b(T,s)\phi(\varphi^{-1}(\Theta(t,s)))\sum_{i=1}^{n}\alpha_i'(t)\int_{\beta_i(s_0)}^{\beta_i(s)}\Big[f_i(\alpha_i(t),y)\phi_1(\varphi^{-1}(\Theta(\alpha_i(t),y)))\\
&\quad\times (w(\varphi^{-1}(\Theta(\alpha_i(t),y))) + \int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{y} g_i(m,n)w(u(m,n))\,dn\,dm)\\
&\quad + h_i(\alpha_i(t),y)\phi_2(\log(\varphi^{-1}(\Theta(\alpha_i(t),y))))\Big]dy,
\end{aligned} \tag{2.15}$$

for all $(t,s) \in [t_0,T] \times J$. From (2.15), we have

$$\begin{aligned}
&\frac{\Theta_t(t,s)}{\phi(\varphi^{-1}(\Theta(t,s)))}\\
&\le b(T,s)\sum_{i=1}^{n}\alpha_i'(t)\int_{\beta_i(s_0)}^{\beta_i(s)}\Big[f_i(\alpha_i(t),y)\phi_1(\varphi^{-1}(\Theta(\alpha_i(t),y)))(w(\varphi^{-1}(\Theta(\alpha_i(t),y)))\\
&\quad + \int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{y} g_i(m,n)w(u(m,n))\,dn\,dm)\\
&\quad + h_i(\alpha_i(t),y)\phi_2(\log(\varphi^{-1}(\Theta(\alpha_i(t),y))))\Big]dy
\end{aligned}$$

Replacing t by v then integrating from t_0 to t with respect to v and making change of variable on right hand side of the above inequality to obtain

$$\begin{aligned}
G(\Theta(t,s)) &\le G(a(T,s)) + b(T,s)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)}\Big[f_i(u,y)\phi_1(\varphi^{-1}(\Theta(u,y)))\\
&\quad\times (w(\varphi^{-1}(\Theta(u,y))) + \int_{\alpha_i(t_0)}^{u}\int_{\beta_i(s_0)}^{y} g_i(m,n)w(u(m,n))\,dn\,dm)\\
&\quad + h_i(u,y)\phi_2(\log(\varphi^{-1}(\Theta(u,y))))\Big]\,dy\,du
\end{aligned} \tag{2.16}$$

When $\phi_1(u) \geq \phi_2(\log(u))$, by (2.16), we have

$$
\begin{aligned}
G(\Theta(t,s)) \leq G(a(T,s)) &+ b(T,s)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)}\phi_1(\varphi^{-1}(\Theta(u,y))) \\
&\times \Big[f_i(u,y)(w(\varphi^{-1}(\Theta(u,y))) \\
&+ \int_{\alpha_i(t_0)}^{u}\int_{\beta_i(s_0)}^{y} g_i(m,n)w(\varphi^{-1}(\Theta(u,y)))\,dn\,dm) + h_i(u,y)]\,dy\,du.
\end{aligned}
\tag{2.17}
$$

Denote the right hand side of (2.17) by $\Lambda(t,s)$, then obviously $\Lambda(t,s)$ is positive and non-decreasing function in each variable such that $\Lambda(t_0,s) = G(a(T,s))$. Then (2.17) is equivalent to

$$
\Theta(t,s) \leq G^{-1}(\Lambda(t,s)),
\tag{2.18}
$$

$$
\begin{aligned}
&\Lambda_t(t,s) \\
&= b(T,s)\sum_{i=1}^{n}\alpha_i'(t)\int_{\beta_i(s_0)}^{\beta_i(s)}\phi_1(\varphi^{-1}(\Theta(\alpha_i(t),y)))\Big[f_i(\alpha_i(t),y)w(\varphi^{-1}(\Theta(\alpha_i(t),y))) \\
&\quad\times\Big(1+\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{y} g_i(m,n)\,dn\,dm\Big) + h_i(\alpha_i(t),y)\Big]dy \\
&\leq b(T,s)\phi_1(\varphi^{-1}(G^{-1}(\Lambda(t,s))))\sum_{i=1}^{n}\alpha_i'(t)\int_{\beta_i(s_0)}^{\beta_i(s)}\Big[f_i(\alpha_i(t),y) \\
&\quad\times w(\varphi^{-1}(G^{-1}(\Lambda(\alpha_i(t),y))))\Big(1+\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{y} g_i(m,n)\,dn\,dm\Big) \\
&\quad + h_i(\alpha_i(t),y)\Big]dy.
\end{aligned}
\tag{2.19}
$$

From (2.19), we have

$$
\begin{aligned}
&\frac{\Lambda_t(t,s)}{\phi_1(\varphi^{-1}(G^{-1}(\Lambda(t,s))))} \\
&\leq b(T,s)\sum_{i=1}^{n}\alpha_i'(t)\int_{\beta_i(s_0)}^{\beta_i(s)}\Big[f_i(\alpha_i(t),y)w(\varphi^{-1}(G^{-1}(\Lambda(\alpha_i(t),y)))) \\
&\quad\times\Big(1+\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{y} g_i(m,n)\,dn\,dm\Big) + h_i(\alpha_i(t),y)\Big]dy,
\end{aligned}
$$

Replacing t by v then integrating from t_0 to t with respect to v and making change of variable on right hand side of the above inequality and using the definition of H_1, we obtain

$$
\begin{aligned}
&H_1(\Lambda(t,s)) \\
&\leq H_1(G(a(T,s))) + b(T,s)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)}\Big[f_i(u,y)w(\varphi^{-1}(G^{-1}(\Lambda(u,y)))) \\
&\quad\times\Big(1+\int_{\alpha_i(t_0)}^{u}\int_{\beta_i(s_0)}^{y} g_i(m,n)\,dn\,dm\Big) + h_i(u,y)\Big]\,dy\,du
\end{aligned}
$$

$$
\begin{aligned}
&\le \widetilde{c}(T,s) + b(T,s)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)} f_i(u,y)w(\varphi^{-1}(G^{-1}(\Lambda(u,y))))\\
&\quad\times\Big(1+\int_{\alpha_i(t_0)}^{u}\int_{\beta_i(s_0)}^{y} g_i(m,n)\,dn\,dm\Big)\,dy\,du. \qquad (2.20)
\end{aligned}
$$

Denote the right hand side of (2.20), such that $\widetilde{\Theta}(t_0,s) = H_1(G(a(T,s)))$. Then (2.20) is equivalent to

$$
\Lambda(t,s) \le H_1^{-1}(\widetilde{\Theta}(t,s)). \qquad (2.21)
$$

By the fact that $\alpha_i(t)\le t, \beta_i(s)\le s$ for $(t,s)\in I\times J$, $1\le i\le n$, and monotonicity of w, φ^{-1} and (2.21), we have

$$
\begin{aligned}
\widetilde{\Theta}_t(t,s) &= b(T,s)\sum_{i=1}^{n}\alpha_i'(t)\int_{\beta_i(s_0)}^{\beta_i(s)} f_i(\alpha_i(t),y)w(\varphi^{-1}(G^{-1}(\Lambda(\alpha_i(t),y))))\\
&\quad\times\Big(1+\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{y} g_i(m,n)\,dn\,dm\Big)dy\\
&\le b(T,s)w(\varphi^{-1}(G^{-1}(H_1^{-1}(\widetilde{\Theta}(t,s)))))\sum_{i=1}^{n}\alpha_i'(t)\int_{\beta_i(s_0)}^{\beta_i(s)} f_i(\alpha_i(t),y)\\
&\quad\times(1+\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{y} g_i(m,n)\,dn\,dm)dy.
\end{aligned} \qquad (2.22)
$$

From (2.22), we have

$$
\begin{aligned}
&\frac{\widetilde{\Theta}_t(t,s)}{w(\varphi^{-1}(G^{-1}(H_1^{-1}(\widetilde{\Theta}(t,s)))))}\\
&\le b(T,s)\sum_{i=1}^{n}\alpha_i'(t)\int_{\beta_i(s_0)}^{\beta_i(s)} f_i(\alpha_i(t),y)\Big(1+\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{y} g_i(m,n)\,dn\,dm\Big)dy.
\end{aligned}
$$

Replacing t by v then integrating from t_0 to t with respect to v and making change of variable on right hand side of the above inequality and using the definition of J_1, we obtain

$$
J_1(\widetilde{\Theta}(t,s)) \le J_1(\widetilde{c}(T,s)) + b(T,s)D(t,s) \qquad (2.23)
$$

As $T\le T_1$ is arbitrary, a combination of (2.14), (2.18), (2.21) and (2.23) yield

$$
u(t,s) \le \varphi^{-1}(G^{-1}(H_1^{-1}(J_1^{-1}(J_1(\widetilde{c}(T,s)) + b(T,s)D(t,s))))).
$$

When $\phi_1(u)\le\phi_2(\log(u))$, by (2.16), we have

$$
\begin{aligned}
&G(\Theta(t,s))\\
&\le G(a(T,s)) + b(T,s)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)}\Big[f_i(u,y)\phi_2(\log(\varphi^{-1}(\Theta(u,y))))\\
&\quad\times(w(\varphi^{-1}(\Theta(u,y))) + \int_{\alpha_i(t_0)}^{u}\int_{\beta_i(s_0)}^{y} g_i(m,n)w(u(m,n))\,dn\,dm)\\
&\quad + h_i(u,y)\phi_2(\log(\varphi^{-1}(\Theta(u,y))))\Big]\,dy\,du\\
&\le G(a(T,s)) + b(T,s)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)}\Big[f_i(u,y)(w(\varphi^{-1}(\Theta(u,y)))
\end{aligned}
$$

$$+\int_{\alpha_i(t_0)}^{u}\int_{\beta_i(s_0)}^{y} g_i(m,n)w(u(m,n))\,dn\,dm) + h_i(u,y)\Big]\phi_2(\varphi^{-1}(\Theta(u,y)))\,dy\,du$$

Similarly to the above process from (2.17) to (2.23), for $T \le T_2$, and as T is arbitrary, we have

$$u(t,s) \le \varphi^{-1}(G^{-1}(H_2^{-1}(J_2^{-1}(J_2(\widetilde{c}(T,s)) + b(T,s)D(t,s))))).$$

□

Theorem 2.3. *Suppose that* (A1)–(A5) *hold and that* $L, M \in C(\mathbb{R}_+^3, \mathbb{R}_+)$ *are such that*

$$0 \le L(t,s,u) - L(t,s,v) \le M(t,s,v)(u-v),$$

for $u > v$. *If* $u(t,s)$ *is a nonnegative and continuous function on* $I \times J$ *satisfying* (1.3), *then we have*

$$\begin{aligned}
u(t,s) \le \varphi^{-1}(G^{-1}(\Psi^{-1}(\Psi(G(a(t,s))\\
&+ b(t,s)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)} h_i(u,y)L(u,y,0)\,dy\,du)\\
&+ b(t,s)\Big\{D(t,s) + \sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)} M(u,y,0)\,dy\,du\Big\}))),
\end{aligned} \tag{2.24}$$

for all $(t,s) \in [t_0, T_4) \times [s_0, S_4)$ *provided that* $\varphi^{-1}, G^{-1}, \Psi^{-1}$ *are the respective inverses of* φ, G, Ψ, *and* $(T_4, S_4) \in I \times J$ *is arbitrarily chosen on the boundary of the planar region,* $\mathfrak{R}_4 := \{(t,s) \in I \times J\}$, *provided that the following three relations are satisfied:*

$$\begin{aligned}
\widetilde{\Delta}(t,s) := \Big[\Psi(G(a(t,s)) + b(t,s)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)} h_i(u,y)L(u,y,0)\,dy\,du)\\
+ b(t,s)\{D(t,s) + \sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)} M(u,y,0)\,dy\,du\}\Big] \in \mathrm{Dom}(\Psi^{-1})
\end{aligned} \tag{2.25}$$

$$\Psi^{-1}(\widetilde{\Delta}(t,s)) \in \mathrm{Dom}(G^{-1}), \quad G^{-1}(\Psi^{-1}(\widetilde{\Delta}(t,s))) \in \mathrm{Dom}(\varphi^{-1}) \tag{2.26}$$

Proof. From assumption (A1) and the inequality (1.3), we have

$$\begin{aligned}
\varphi(u(t,s)) \le a(T,s) + b(T,s)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)} \phi(u(x,y))\Big[f_i(x,y)(w(u(x,y))\\
&+ \int_{\alpha_i(t_0)}^{x}\int_{\beta_i(s_0)}^{y} g_i(m,n)w(u(m,n))\,dn\,dm)\\
&+ h_i(x,y)L(x,y,w(u(x,y)))\Big]\,dy\,dx,
\end{aligned} \tag{2.27}$$

for all $(t,s) \in [t_0, T] \times J$, $T \le T_4$. Denote the right hand side of (2.27) by $\mathfrak{P}(t,s)$, then obviously $\mathfrak{P}(t,s)$ is positive and non-decreasing function in each variable, $\mathfrak{P}(t_0,s) = a(T,s)$. Then, (2.27) is equivalent to

$$u(t,s) \le \varphi^{-1}(\mathfrak{P}(t,s)). \tag{2.28}$$

By the fact that $\alpha_i(t) \le t$ and $\beta_i(s) \le s$ for $(t,s) \in I \times J$, $1 \le i \le n$, and monotonicity of $\mathfrak{P}$, φ^{-1}, ϕ, we have

$$\begin{aligned}
\mathfrak{P}_t(t,s) &= b(T,s)\sum_{i=1}^{n}\int_{\beta_i(s_0)}^{\beta_i(s)} \phi(u(\alpha_i(t),y))\Big[f_i(\alpha_i(t),y)(w(u(\alpha_i(t),y)) \\
&\quad + \int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{y} g_i(m,n)w(u(m,n))\,dn\,dm) \\
&\quad + h_i(\alpha_i(t),y)L(\alpha_i(t),y,w(u(\alpha_i(t),y)))\Big]dy\alpha_i'(t) \\
&\le b(T,s)\phi(\varphi^{-1}(\mathfrak{P}(t,s)))\sum_{i=1}^{n}\int_{\beta_i(s_0)}^{\beta_i(s)}\Big[f_i(\alpha_i(t),y)(w(\varphi^{-1}(\mathfrak{P}(\alpha_i(t),y))) \\
&\quad + \int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{y} g_i(m,n)w(\varphi^{-1}(\mathfrak{P}(m,n)))\,dn\,dm) + h_i(\alpha_i(t),y) \\
&\quad \times L(\alpha_i(t),y,w(\varphi^{-1}(\mathfrak{P}(\alpha_i(t),y))))\Big]\,dy\,\alpha_i'(t).
\end{aligned} \tag{2.29}$$

From (2.29), we have

$$\begin{aligned}
\frac{\mathfrak{P}_t(t,s)}{\phi(\varphi^{-1}(\mathfrak{P}(t,s)))} &\le b(T,s)\sum_{i=1}^{n}\int_{\beta_i(s_0)}^{\beta_i(s)}\Big[f_i(\alpha_i(t),y)(w(\varphi^{-1}(\mathfrak{P}(\alpha_i(t),y))) \\
&\quad + \int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{y} g_i(m,n)w(\varphi^{-1}(\mathfrak{P}(m,n)))\,dn\,dm) \\
&\quad + h_i(\alpha_i(t),y)L(\alpha_i(t),y,w(\varphi^{-1}(\mathfrak{P}(\alpha_i(t),y))))\Big]\,dy\,\alpha_i'(t),
\end{aligned}$$

for all $(t,s) \in [t_0,T] \times J$, $T \le T_4$. Replace t by v then integrating from t_0 to t with respect to v and making change of variable on right hand side of the above inequality and using the definition of G, we have

$$\begin{aligned}
G(\mathfrak{P}(t,s)) &\le G(a(T,s)) + b(T,s)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)}\Big[f_i(u,y)(w(\varphi^{-1}(\mathfrak{P}(u,y))) \\
&\quad + \int_{\alpha_i(t_0)}^{u}\int_{\beta_i(s_0)}^{y} g_i(m,n)w(\varphi^{-1}(\mathfrak{P}(m,n)))\,dn\,dm) + h_i(u,y) \\
&\quad \times \{L(u,y,0) + M(u,y,0)w(\varphi^{-1}(\mathfrak{P}(u,y)))\}\Big]\,dy\,du \\
&\le G(a(T,s)) + b(T,s)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(T)}\int_{\beta_i(s_0)}^{\beta_i(s)} h_i(u,y)L(u,y,0)\,dy\,du \\
&\quad + b(T,s)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)}\Big[f_i(u,y)(1 + \int_{\alpha_i(t_0)}^{u}\int_{\beta_i(s_0)}^{y} g_i(m,n)\,dn\,dm) \\
&\quad + M(u,y,0)\Big]w(\varphi^{-1}(\mathfrak{P}(u,y)))\,dy\,du
\end{aligned} \tag{2.30}$$

Denote the right hand side of (2.30) by $\mathfrak{Q}(t,s)$, then obviously $\mathfrak{Q}(t,s)$ is a positive and nondecreasing function in each variable such that $\mathfrak{Q}(t_0,s) = G(a(T,s))$. Then, (2.30) is equivalent to

$$\mathfrak{P}(t,s) \le G^{-1}(\mathfrak{Q}(t,s)), \tag{2.31}$$

$$\mathfrak{Q}(t_0,s) = G(a(T,s)) + b(T,s)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(T)}\int_{\beta_i(s_0)}^{\beta_i(s)} h_i(u,y)L(u,y,0)\,dy\,du,$$

$$\begin{aligned}
\mathfrak{Q}_t(t,s) &= b(T,s)\sum_{i=1}^{n}\int_{\beta_i(s_0)}^{\beta_i(s)}\Big[f_i(\alpha_i(t),y)(1+\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{y} g_i(m,n)dndm)\\
&\quad + M(\alpha_i(t),y,0)\Big]w(\varphi^{-1}(\mathfrak{P}(\alpha_i(t),y)))dy\alpha_i'(t)\\
&\le b(T,s)w(\varphi^{-1}(G^{-1}(\mathfrak{P}(t,s))))\sum_{i=1}^{n}\int_{\beta_i(s_0)}^{\beta_i(s)}\Big[f_i(\alpha_i(t),y)\\
&\quad \times\Big(1+\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{y} g_i(m,n)dndm\Big) + M(\alpha_i(t),y,0)\Big]dy\alpha_i'(t).
\end{aligned} \tag{2.32}$$

Then, (2.32) is written as

$$\begin{aligned}
&\frac{\mathfrak{Q}_t(t,s)}{w(\varphi^{-1}(G^{-1}(\mathfrak{P}(t,s))))}\\
&\le b(T,s)\sum_{i=1}^{n}\int_{\beta_i(s_0)}^{\beta_i(s)}\Big[f_i(\alpha_i(t),y)\Big(1+\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{y} g_i(m,n)dn\,dm\Big)\\
&\quad + M(\alpha_i(t),y,0)\Big]dy\alpha_i'(t)
\end{aligned} \tag{2.33}$$

Replacing t by v then integrating from t_0 to t with respect to v and making change of variable on right hand side of (2.33) and using the definition of Ψ, we obtain

$$\begin{aligned}
\Psi(\mathfrak{Q}(t,s)) &\le \Psi(G(a(T,s)) + b(T,s)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(T)}\int_{\beta_i(s_0)}^{\beta_i(s)} h_i(u,y)L(u,y,0)\,dy\,du)\\
&\quad + b(T,s)\Big\{D(t,s) + \sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)} M(u,y,0)\,dy\,du\Big\}.
\end{aligned} \tag{2.34}$$

A combination of (2.28), (2.31) and (2.34) yield inequality (2.24). $\square$

Corollary 2.4. *Suppose* (A2)–(A4) *are satisfied. If $p > q > 0$ and $c \ge 0$ are constants such that:*

$$\begin{aligned}
u^p(t,s) &\le c + p\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)} u^q(t,s)\Big[f_i(x,y)(w(u(x,y))\\
&\quad + \int_{\alpha_i(t_0)}^{x}\int_{\beta_i(s_0)}^{y} g_i(m,n)w(u(m,n))\,dn\,dm) + h_i(x,y)\Big]\,dy\,dx,
\end{aligned} \tag{2.35}$$

then

$$u(t,s) \le \sqrt[p-q]{\Psi_*^{-1}(\Psi_*(m_0(t,s)) + (p-q)D(t,s))}, \tag{2.36}$$

$$m_0(t,s) = c^{\frac{p-q}{q}} + (p-q)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)} h_i(u,y)\,dy\,du,$$

$$\Psi_*(r) := \int_{r_0}^{r}\frac{du}{w(\sqrt[p-q]{u})}, \quad r \ge r_0 > 0,$$

Proof. Denote the right hand side of (2.35) by $\Xi(t,s)$, then obviously $\Xi(t,s)$ is positive and non-decreasing function in each variable such that $\Xi(t_0,s)=c$. Then, (2.35) is equivalent to

$$u(t,s) \le \sqrt[p]{\Xi(t,s)}, \tag{2.37}$$

$$\begin{aligned}
&\Xi_t(t,s)\\
&= p\sum_{i=1}^{n}\alpha_i'(t)\int_{\beta_i(s_0)}^{\beta_i(s)}\Big[f_i(\alpha_i(t),y)u^q(\alpha_i(t),y)(w(u(\alpha_i(t),y))\\
&\quad+\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{y}g_i(m,n)w(u(m,n))\,dn\,dm)+h_i(\alpha_i(t),y)u^q(\alpha_i(t),y)\Big]dy\\
&\le p\sum_{i=1}^{n}\alpha_i'(t)\int_{\beta_i(s_0)}^{\beta_i(s)}\{\Xi(\alpha_i(t),y)\}^{q/p}\Big[f_i(\alpha_i(t),y)(w(\sqrt[p]{\Xi(\alpha_i(t),y)})\\
&\quad+\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{y}g_i(m,n)w(\sqrt[p]{\Xi(m,n)})\,dn\,dm)+h_i(\alpha_i(t),y)\Big]dy,
\end{aligned}\tag{2.38}$$

for $(t,s)\in[t_0,T]\times J$. Then, (2.38) is equivalent to

$$\begin{aligned}
\frac{\Xi_t(t,s)}{\{\Xi(t,s)\}^{q/p}} &\le p\sum_{i=1}^{n}\alpha_i'(t)\int_{\beta_i(s_0)}^{\beta_i(s)}\Big[f_i(\alpha_i(t),y)(w(\sqrt[p]{\Xi(\alpha_i(t),y)})\\
&\quad+\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{y}g_i(m,n)w(\sqrt[p]{\Xi(m,n)})\,dn\,dm)+h_i(\alpha_i(t),y)\Big]dy\,.
\end{aligned}$$

Replacing t by v then integrating from t_0 to t with respect to v, making change of variable on right hand side of the above inequality and by using that $m_0(t,s)$ is non-decreasing in each variable, for $t\le T$, we have

$$\begin{aligned}
{}^{(p-q)/p} &\le c^{(p-q)/q}+(p-q)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)}\Big[f_i(u,y)(w(\sqrt[p]{\Xi(u,y)})\\
&\quad+\int_{\alpha_i(t_0)}^{u}\int_{\beta_i(s_0)}^{y}g_i(m,n)w(\sqrt[p]{\Xi(m,n)})\,dn\,dm)+h_i(u,y)\Big]\,dy\,du\\
&\le m_0(T,s)+(p-q)\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{\beta_i(s)}f_i(u,y)(w(\sqrt[p]{\Xi(u,y)})\\
&\quad+\int_{\alpha_i(t_0)}^{u}\int_{\beta_i(s_0)}^{y}g_i(m,n)w(\sqrt[p]{\Xi(m,n)})\,dn\,dm)\,dy\,du
\end{aligned}\tag{2.39}$$

Denote the right hand side of (2.39) by $\gamma(t,s)$, then obviously $\gamma(t,s)$ is positive and non-decreasing function in each variable such that $\gamma(t_0,s)=m_0(T,s)$. Then, (2.39) is equivalent to

$$\Xi(t,s)\le[\gamma(t,s)]^{\frac{p}{p-q}}, \tag{2.40}$$

$$
\begin{aligned}
\gamma_t(t,s) &= (p-q)\sum_{i=1}^{n}\alpha_i'(t)\int_{\beta_i(s_0)}^{\beta_i(s)} f_i(\alpha_i(t),y)(w(\sqrt[p]{\Xi(\alpha_i(t),y)}) \\
&\quad + \int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{y} g_i(m,n)w(\sqrt[p]{\Xi(m,n)})\,dn\,dm)\,dy \\
&\le (p-q)\sum_{i=1}^{n}\alpha_i'(t)\int_{\beta_i(s_0)}^{\beta_i(s)} f_i(\alpha_i(t),y)(w(\sqrt[p-q]{\gamma(\alpha_i(t),y)}) \\
&\quad + \int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{y} g_i(m,n)w(\sqrt[p-q]{\gamma(m,n)})\,dn\,dm)\,dy.
\end{aligned}
\tag{2.41}
$$

Then, (2.41) is written as

$$
\begin{aligned}
&\frac{\gamma_t(t,s)}{w(\sqrt[p-q]{\gamma(t,s)})} \\
&\le (p-q)\sum_{i=1}^{n}\alpha_i'(t)\int_{\beta_i(s_0)}^{\beta_i(s)} f_i(\alpha_i(t),y)\Big(1+\int_{\alpha_i(t_0)}^{\alpha_i(t)}\int_{\beta_i(s_0)}^{y} g_i(m,n)\,dn\,dm\Big)dy.
\end{aligned}
$$

Setting t by l then integrating from t_0 to t with respect to l, making change of variable on right hand side of the above inequality and using $\gamma(t_0,s) = m_0(T,s)$, and the definition of Ψ_*, we have

$$
\Psi_*(\gamma(t,s)) \le \Psi_*(\gamma(t,s)) + (p-q)D(t,s). \tag{2.42}
$$

A combination of (2.37), (2.40), and (2.42) yield the desire result (2.36). □

Remark 2.5.

- For $a(t,s)\equiv c$, $b(t,s)\equiv 1$, $\phi(x)=x$, $g_i\equiv 0\equiv h_i$, $1\le i\le n$. Then Theorem 2.1 reduces to [3, Theorem 2.2].
- For $a(t,s)\equiv c$, $b(t,s)\equiv 1$, $\phi(x)=x$, $g_i\equiv 0$, $1\le i\le n$. Then Theorem 2.1 reduces to [3, Theorem 2.3].
- For $q=1$, $g_i\equiv 0$, $1\le i\le n$, corollary 2.4 reduces to [3, Corollary 2.4].
- For $g_i\equiv 0$, $1\le i\le n$, Theorem 2.1 reduces to [12, Theorem 1].
- For $g_i\equiv 0$, $1\le i\le n$, and $w\equiv 1$, theorem 2.2 reduces to [12, Theorem 2].

3. Applications

In this section, we apply the inequalities established above to achieve the boundedness of partial integro-differential equations, with several retarded arguments, of the form

$$
\begin{aligned}
&\frac{\partial}{\partial s}(z^{p-1}(t,s)z_t(t,s)) \\
&= F\Big[t,s,z(t-l_1(t),s-k_1(s)),\dots,z(t-l_n(t),s-k_n(s)), \\
&\int_{t_0}^{t}\int_{s_0}^{s} Q(t,s,\sigma,\tau,z(t-l_1(t),s-k_1(s)),\dots,z(t-l_n(t),s-k_n(s)))d\sigma d\tau\Big],
\end{aligned}
\tag{3.1}
$$

and

$$\begin{aligned} &D_2(D_1\varphi(z(t,s))) \\ &= F\Big[t, s, z(t-l_1(t), s-k_1(s)), \dots, z(t-l_n(t), s-k_n(s)), \\ &\quad \int_{t_0}^{t}\int_{s_0}^{s} Q(t, s, \sigma, \tau, z(t-l_1(t), s-k_1(s)), \dots, z(t-l_n(t), s-k_n(s)))d\sigma\, d\tau\Big], \end{aligned} \tag{3.2}$$

with the given initial boundary conditions

$$z(t, s_0) = a_1(t), \quad z(t_0, s) = a_2(s), \quad a_1(t_0) = a_2(s_0) = 0, \tag{3.3}$$

where $F \in C(I \times J \times \mathbb{R}^n, \mathbb{R})$, $Q \in C((I \times J) \times (I \times J) \times \mathbb{R}^n, \mathbb{R})$, $a_1 \in C^1(I, \mathbb{R})$, $a_2 \in C^1(J, \mathbb{R})$ and $l_i \in C^1(I, \mathbb{R})$, $k_i \in C^1(J, \mathbb{R})$ are nonincreasing and such that $t - l_i(t) \geq 0$, $t - l_i(t) \in C^1(I, I)$, $s - k_i(s) \geq 0$, $s - k_i(s) \in C^1(J, J)$, $l_i'(t) < 1$, $k_i'(s) < 1$ and $l_i(t_0) = k_i(s_0) = 0$, $1 \leq i \leq n$, for $(t, s) \in I \times J$; let $\varphi \in C^1(\mathbb{R}, \mathbb{R})$ be an increasing function such that $\varphi(|u|) \leq |\varphi(u)|$; let $\varphi(e(t, s)) = \varphi(a_1(t)) + \varphi(a_2(s))$ and

$$M_i = \max_{t\in I} \frac{1}{1 - l_i'(t)}, \quad N_i = \max_{s\in J} \frac{1}{1 - k_i'(s)}, \quad 1 \leq i \leq n. \tag{3.4}$$

The following theorem deals with a boundedness on the solution of (3.2).

Theorem 3.1. *Assume that $F : I \times J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function for which there exist continuous nonnegative functions $f_i(t, s), g_i(t, s)$ and $h_i(t, s)$, $1 \leq i \leq n$, for $(t, s) \in I \times J$ such that:*

$$\begin{gathered} |F(t, s, u_1, \dots, u_n, j)| \leq b(t, s) \sum_{i=1}^{n} \phi(|u_i|)\big[f_i(t, s)w(|u_i|) + |j| + h_i(t, s)\big]. \\ |Q(t, s, v_1, v_2, u_1, u_2, \dots, u_n)| \leq g_i(t, s)w(|u_i|). \end{gathered} \tag{3.5}$$

If $z(t, s)$ is a solution of (3.2) with conditions (3.3), then

$$\begin{aligned} |z(t, s)| \leq \varphi^{-1}\Big(G^{-1}\Big(\Psi^{-1}\Big(\Psi(\overline{c}(t, s)) + b(t, s) \sum_{i=1}^{n} \int_{\phi_i(t_0)}^{\phi_i(t)} \int_{\psi_i(s_0)}^{\psi_i(s)} \overline{f}_i(\delta, \eta) \\ \times \Big(1 + \int_{\phi_i(t_0)}^{\delta} \int_{\psi_i(s_0)}^{\psi_i(s)} \overline{g}_i(\delta_1, \eta_1)d\eta_1 d\delta_1\Big) d\eta\, d\delta\Big)\Big)\Big), \end{aligned} \tag{3.6}$$

where,

$$\begin{gathered} \overline{f}_i(u, v) = M_i N_i f_i(u + l_i(m), v + k_i(p)), \quad \overline{g}_i(u, v) = M_i N_i g_i(u + l_i(\sigma), v + k_i(\tau)), \\ \overline{c}(t, s) = G(\varphi(e(t, s))) + b(t, s) \sum_{i=1}^{n} \int_{\phi_i(t_0)}^{\phi_i(t)} \int_{\psi_i(s_0)}^{\psi_i(s)} \overline{h}_i(u, v)dv\, du, \\ \overline{h}_i(u, v) = M_i N_i h_i(u + l_i(m), v + k_i(p)) \end{gathered}$$

Proof. It is easy to see that the solution $z(t,s)$ of the problem (3.2) with (3.3) satisfies the equivalent integral equation

$$
\begin{aligned}
&\varphi(z(t,s)) \\
&= \varphi(e(t,s)) + \int_{t_0}^{t}\int_{s_0}^{s} F\Big[u, v, z(u-l_1(u), v-k_1(v)), \dots, z(u-l_n(u), v-k_n(v)), \\
&\int_{t_0}^{u}\int_{s_0}^{v} Q(u,v,\sigma,\tau, z(u-l_1(u), v-k_1(v)), \dots, \\
&z(u-l_n(u), v-k_n(v)))d\sigma\, d\tau\Big] dv\, du
\end{aligned} \tag{3.7}
$$

By modulus properties and condition (3.5), equation (3.7) has the form

$$
\begin{aligned}
&|\varphi(z(t,s))| \\
&\le |\varphi(e(t,s))| + b(t,s)\int_{t_0}^{t}\int_{s_0}^{s} \Big|F\Big[u, v, z(u-l_1(u), v-k_1(v)), \dots, \\
&\quad z(u-l_n(u), v-k_n(v)), \int_{t_0}^{u}\int_{s_0}^{v} Q(u,v,\sigma,\tau, z(u-l_1(u), v-k_1(v)), \dots, \\
&\quad z(u-l_n(u), v-k_n(v)))d\sigma\, d\tau\Big]\Big| dv\, du \\
&\le |\varphi(e(t,s))| + b(t,s)\int_{t_0}^{t}\int_{s_0}^{s} \sum_{i=1}^{n} [\phi(|z(m-l_i(m), p-k_i(p))|)(f_i(m,p) \\
&\quad \times (w(|z(m-l_i(m), p-k_i(p))|) \\
&\quad + \int_{t_0}^{m}\int_{s_0}^{p} g_i(\sigma,\tau) w(|z(\sigma-l_i(\sigma), \tau-k_i(\tau))|)d\tau\, d\sigma) \\
&\quad + h_i(m,p)|\phi(z(m-l_i(m), p-k_i(p)))|]dp\, dm \\
&\le |\varphi(e(t,s))| + b(t,s)\sum_{i=1}^{n} M_i N_i \int_{\phi_i(t_0)}^{\phi_i(t)}\int_{\psi_i(s_0)}^{\psi_i(s)} \Big[\phi(|z(\phi_i(m), \psi_i(p))|) \\
&\times (f_i(\phi_i(m)+l_i(m), \psi_i(p)+k_i(p))(w(|z(\phi_i(m), \psi_i(p))|) \\
&\quad + \int_{\phi_i(t_0)}^{\phi_i(m)}\int_{\psi_i(s_0)}^{\psi_i(p)} M_i N_i g_i(\phi_i(\sigma)+l_i(\sigma), \psi_i(\tau)+k_i(\tau)) \\
&\quad \times w(|z(\phi_i(\sigma), \psi_i(\tau))|)d\psi_i(\tau)\, d\phi_i(\sigma)) + h_i(\phi_i(m)+l_i(m), \psi_i(p) \\
&\quad + k_i(p))\phi(|z(\phi_i(m), \psi_i(p))|))\Big] d\psi_i(p)\, d\phi_i(m)
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\varphi(|z(t,s)|) \\
&\le |\varphi(e(t,s))| + b(t,s)\sum_{i=1}^{n}\int_{\phi_i(t_0)}^{\phi_i(t)}\int_{\psi_i(s_0)}^{\psi_i(s)} \Big[\phi(|z(\delta,\eta)|)(\overline{f}_i(\delta,\eta)(w(|z(\delta,\eta)|) \\
&\quad + \int_{\phi_i(t_0)}^{\delta}\int_{\psi_i(s_0)}^{\eta} \overline{g}_i(\delta_1,\eta_1) w(|z(\delta_1,\eta_1)|)d\eta_1\, d\delta_1) + \overline{h}_i(\delta,\eta)\phi(|z(\delta,\eta)|))\Big] d\eta\, d\delta
\end{aligned} \tag{3.8}
$$

Now an immediate application of inequality (2.1) to (3.8) yields the desired result (3.6). $\square$

Theorem 3.2. *Assume that $F : I \times J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function for which there exist continuous nonnegative functions $f_i(t,s), g_i(t,s)$ and $h_i(t,s)$, $1 \le i \le n$, for $(t,s) \in I \times J$ such that:*

$$\begin{gathered} |F(t,s,u_1,\dots,u_n,j)| \le \sum_{i=1}^{n} |u_i|^q [f_i(t,s) w(|u_i|) + |j| + h_i(t,s)], \\ |Q(t,s,v_1,v_2,u_1,u_2,\dots,u_n)| \le g_i(t,s) w(|u_i|), \\ |a_1^p(t) + a_2^p(s)| \le c \end{gathered} \tag{3.9}$$

If $z(t,s)$ is a solution of (3.1) *with the condition* (3.3)*, then*

$$\begin{aligned} u(t,s) \le [\Psi_*^{-1}(\Psi_*(\widetilde{m}_0(t,s)) + (p-q) \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \int_{\beta_i(s_0)}^{\beta_i(s)} \widetilde{f}_i(z,y) \\ \times \Big(1 + \int_{\alpha_i(t_0)}^{z} \int_{\beta_i(s_0)}^{y} \widetilde{g}_i(m,n)\, dn\, dm\Big)\, dy\, dz)]^{1/(p-q)}, \end{aligned} \tag{3.10}$$

where

$$\widetilde{f}_i(\mathfrak{u},\mathfrak{v}) = M_i N_i f_i(\mathfrak{u} + l_i(u), \mathfrak{v} + k_i(v)), \quad \widetilde{g}_i(\mathfrak{u},\mathfrak{v}) = M_i N_i g_i(\mathfrak{u} + l_i(\sigma), \mathfrak{v} + k_i(\tau)),$$
$$\widetilde{h}_i(\mathfrak{u},\mathfrak{v}) = M_i N_i h_i(\mathfrak{u} + l_i(u), \mathfrak{v} + k_i(v)),$$
$$\widetilde{m}_0(t,s) = c^{(p-q)/p} + (p-q) \sum_{i=1}^{n} \int_{\phi_i(t_0)}^{\phi_i(t)} \int_{\psi_i(s_0)}^{\psi_i(s)} \widetilde{h}_i(u,y)\, dy\, du$$

Proof. It is easy to see that the solution $z(t,s)$ of (3.1) with (3.3) satisfies the equivalent integral equation

$$\begin{aligned} {}^p = a_1^p(t) + a_2^p(s) + p \int_{t_0}^{t} \int_{s_0}^{s} F[u,v,z(u-l_1(u), v-k_1(v)),\dots, \\ z(u-l_n(u), v-k_n(v)), \int_{t_0}^{u} \int_{s_0}^{v} Q(u,v,\sigma,\tau, z(u-l_1(u), v-k_1(v)),\dots, \\ z(u-l_n(u), v-k_n(v))) d\sigma d\tau] dv\, du \end{aligned} \tag{3.11}$$

By modulus properties and condition (3.9), equation (3.11) has the form

$$\begin{aligned} &|z^p(t,s)| \\ &\le c + p \int_{t_0}^{t} \int_{s_0}^{s} \Big| F\Big[u,v,z(u-l_1(u), v-k_1(v)),\dots, z(u-l_n(u), v-k_n(v)), \\ &\quad \int_{t_0}^{u} \int_{s_0}^{v} Q(u,v,\sigma,\tau,z(u-l_1(u), v-k_1(v)),\dots, \\ &\quad z(u-l_n(u), v-k_n(v))) d\sigma d\tau\Big]\Big|\, dv\, du \\ &\le c + p \int_{t_0}^{t} \int_{s_0}^{s} \sum_{i=1}^{n} \Big[|z(u-l_i(u), v-k_i(v))|^q . f_i(u,v) (w(|z(u-l_i(u), v-k_i(v))|) \\ &\quad + \int_{t_0}^{u} \int_{s_0}^{v} g_i(\sigma,\tau) w(|z(\sigma - l_i(\sigma), \tau - k_i(\tau))|) d\tau\, d\sigma) \\ &\quad + h_i(u,v) |z(u-l_i(u), v-k_i(v))|^q \Big]\, dv\, du \end{aligned}$$

$$
\begin{aligned}
&\le c + p\sum_{i=1}^{n} M_i N_i \int_{\phi_i(t_0)}^{\phi_i(t)} \int_{\psi_i(s_0)}^{\psi_i(s)} \Big[|z(\phi_i(u),\psi_i(v))|^q f_i(\phi_i(u)+l_i(u),\psi_i(v)+k_i(v)) \\
&\quad \times (w(|z(\phi_i(u),\psi_i(v))|) + \int_{\phi_i(t_0)}^{\phi_i(u)} \int_{\psi_i(s_0)}^{\psi_i(v)} M_i N_i g_i(\phi_i(\sigma)+l_i(\sigma),\psi_i(\tau) \\
&\quad + k_i(\tau)) w(|z(\phi_i(\sigma),\psi_i(\tau))|) d\psi_i(\tau) d\phi_i(\sigma)) + h_i(\phi_i(u)+l_i(u),\psi_i(v) \\
&\quad + k_i(v)) |\phi(z(\phi_i(u),\psi_i(v)))|\Big] d\psi_i(v) d\phi_i(u) \\
&\le c + p\sum_{i=1}^{n} \int_{\phi_i(t_0)}^{\phi_i(t)} \int_{\psi_i(s_0)}^{\psi_i(s)} [|z(\delta,\eta)|^q . \widetilde{f}_i(\delta,\eta)(w(|z(\delta,\eta)|) \\
&\quad + \int_{\phi_i(t_0)}^{\delta} \int_{\psi_i(s_0)}^{\eta} \widetilde{g}_i(\delta_1,\eta_1) w(|z(\delta_1,\eta_1)|) d\eta_1 d\delta_1) + \widetilde{h}_i(\delta,\eta)|z(\delta,\eta)|^q] d\eta\, d\delta
\end{aligned}
$$

Now an immediate application of inequality (2.36) to above inequality yields the desired result (3.10). □

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