Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 255, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

DISTRIBUTION OF THE PRÜFER ANGLE IN p-LAPLACIAN EIGENVALUE PROBLEMS

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Abstract. The Prüfer angle is an effective tool for studying Sturm-Liouville problems and p-Laplacian eigenvalue problems. In this article, we show that for the p-Laplacian eigenvalue problem, when x is irrational in $(0,1)$, a sequence of modified Prüfer angles (after modulo π_p) is equidistributed in $(0,\pi_p)$. As a function of x, ψ_n is also asymptotic to the uniform distribution on $(0,\pi_p)$.

1. Introduction

It is well known that when a real number x is irrational, the sequence $\{x_n = \langle nx\rangle\}$ is dense in $(0,1)$. Here for any $t \in \mathbb{R}$, the fractional part of t is denoted by $\langle t\rangle := t - [t]$. It is equivalent to saying that $\{\xi_n = \sin(n\pi x)\}$ is dense in $[-1,1]$. Furthermore, the above sequence $\{x_n\}$ is equidistributed in $(0,1)$ in the sense below ([10, p.105]).

Definition. A sequence $\{x_n\} \subset (0,1)$ is said to be equidistributed in $(0,1)$ if for any subinterval $(a,b) \subset (0,1)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{(a,b)}(x_n) = b - a.$$

The above property is a basic one in ergodic theory. It tells us that the sequence spreads evenly in the interval $(0,1)$. In fact, this equidistribution theorem is equivalent to the property that for any $f \in L^1(0,1)$,

$$\int_0^1 f(x)\,dx = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n),$$

which in term is equivalent to saying that the transformation $T(\theta) = \langle \theta + x\rangle$ is ergodic [10, 7].

Consider the Sturm-Liouville problem

$$-u'' + q(x)u = \lambda u \tag{1.1}$$

2000 *Mathematics Subject Classification.* 34B24, 37A30.
Key words and phrases. p-Laplacian eigenvalue problem; Prüfer angle; equidistribution; uniform distribution.

Submitted November 11, 2014. Published December 4, 2014.

subject to boundary conditions

$$\begin{aligned} u(0)\cos\alpha + u'(0)\sin\alpha = 0 \\ u(1)\cos\beta + u'(1)\sin\beta = 0 \end{aligned} \tag{1.2}$$

where $\alpha, \beta \in [0, \pi)$, and $q \in L^1(0,1)$. We call α, β the boundary phases.

The Prüfer substitution

$$u = r(x)\sin\theta(x), \quad u' = r(x)\cos\theta(x) \tag{1.3}$$

is a useful method to study the Sturm-Liouville problem, such as the existence of countably many simple eigenvalues, oscillations of the nth eigenfunction, the asymptotics of the eigenvalues and eigenfunctions [6]. In [1], Atkinson showed that the Sturm-Liouville properties are also valid when the coefficient function q is L^1. His method is also this Prüfer substitution in spirit. Furthermore, Binding and Volkmer [4] (see also [5]) showed that one can use the Prüfer substitution method to show the distribution of periodic and anti-periodic eigenvalues for periodic Sturm-Liouville problems. (Traditionally the Hill discriminant function is used to prove this distribution.) Thus the Prüfer angle is an effective tool for the Sturm-Liouville theory. It would be interesting to explore further properties of this Prüfer angle. In this paper, we shall study the equidistribution property.

In recent years, the Prüfer angle has been used to show that another class of degenerate boundary value problems, the p-Laplacian eigenvalue problem, observes the Sturm-Liouville properties, as shown by Binding and Drabek [3] (see also [2]). Let (λ_n, y_n) be the nth eigenpair of the boundary value problem

$$\begin{gathered} -(|y'|^{p-2}y')' = (p-1)(\lambda - q(x))|y|^{p-2}y, \\ y(0)S_p'(\alpha) + y'(0)S_p(\alpha) = 0, \\ y(1)S_p'(\beta) + y'(1)S_p(\beta) = 0. \end{gathered} \tag{1.4}$$

Here, S_p is called the generalized sine function and defined as the solution of the initial value problem

$$\begin{gathered} (|S_p'(x)|^{p-2}S_p'(x))' + (p-1)|S_p(x)|^{p-2}S_p(x) = 0, \\ S_p(0) = S_p'(0) - 1 = 0. \end{gathered}$$

It is known that the function S_p is $2\pi_p$-periodic on $\mathbb{R}$, where

$$\pi_p \equiv 2\int_0^1 (1-t^p)^{-1/p}dt,$$

and for all $x \in \mathbb{R}$, the following identity holds:

$$|S_p(x)|^p + |S_p'(x)|^p = 1.$$

Note that π_p is strictly decreasing in p [2]. When $p = 2$, we have $\pi_2 = \pi$ and $S_p(x) = \sin x$. Moreover, for $q = 0$ and $p = 2$, the Dirichlet eigenvalues and eigenfunctions are $\lambda_n = (n\pi)^2$ and $y_n = \sin(n\pi x)$.

For $a > 0$, let us define the fractional part of $\langle t\rangle_a$ ($t \pmod a$) by

$$\langle t\rangle_a := t - a\cdot[t/a]\,.$$

When $a = 1$, we denote this fractional part simply by $\langle t\rangle$. As discussed above, when x is irrational, the sequence $\{\langle n\pi x\rangle_\pi\}$ is equidistributed in $(0,\pi)$. We shall see that a sequence of modified Prüfer angles $\{\langle\psi_n(x)\rangle_{\pi_p}\}$ also observe this ergodic behavior. Consider the modified Prüfer substitution

$$y(x) = R(x)S_p(\psi(x)), \quad y'(x) = \lambda^{1/p}R(x)S_p'(\psi(x)). \tag{1.5}$$

We call $\psi(x)$ the modified Prüfer angle at x of the problem (1.4). It becomes $\psi_n(x)$ when associated with the nth eigenpair (λ_n, y_n). We note that in literature $\psi(x)$ can also help to give estimates for the eigenvalues and nodal points. See [8].

Theorem 1.1. *Fix any irrational number $x \in (0,1)$. For any boundary phases α and β, the sequence $\{\langle\psi_n(x)\rangle_{\pi_p}\}$ is equidistributed in $(0,\pi_p)$.*

We remark that $\psi_n(x)$ can be viewed as the phase of the eigenfunction y_n at x, analogous to the argument of the function $\sin(n\pi x)$. Moreover $\psi_n(x)$ demonstrates another property of uniform distribution, just like $\langle n\pi x\rangle_\pi$.

Theorem 1.2. *For $q \in L^1(0,1)$, the distribution of the modifed Prüfer angle ψ_n defined in (1.5) is asymptotic to the uniform distribution on $(0,\pi_p)$. That is, for all $t \in (0,\pi_p)$,*

$$P_n(t) := \mu\{x \in (0,1) : \langle\psi_n(x)\rangle_{\pi_p} < t\} \to \frac{t}{\pi_p} \quad \text{as } n \to \infty\,.$$

Here μ denotes the Lebesgue measure on $\mathbb{R}$.

The above two theorems are the main results of this paper. To prove them, we need to use the following lemma. Define $CT_p(x) \equiv S_p'(x)/S_p(x)$ and let $CT_p^{-1}(x)$ be the inverse function of $CT_p(x)$ taking value in $(0,\pi_p)$.

Lemma 1.3. *The modified Prüfer angle $\psi_n(x)$, defined in (1.5) for the p-Laplacian eigenvalue problem (1.4), has the asymptotic formula*

$$\psi_n(x) = \lambda_n^{1/p}x + \psi_n(0) + O\Big(\frac{1}{\lambda_n^{1-1/p}}\Big), \tag{1.6}$$

where

$$\psi_n(0) = \begin{cases} 0, & \text{if } \alpha = 0, \\ CT_p^{-1}\big(-\frac{CT_p(\alpha)}{\lambda_n^{1/p}}\big), & \text{if } \alpha > 0. \end{cases}$$

Proof. Since $\frac{y'(x)}{\lambda^{1/p}y(x)} = \frac{S_p'(\psi(x))}{S_p(\psi(x))}$, differentiating both sides with respect to x, we have

$$\psi'(x) = \lambda^{1/p} - \frac{q(x)}{\lambda^{1-1/p}}|S_p(\psi(x))|^p = \lambda^{1/p} + O\Big(\frac{1}{\lambda^{1-1/p}}\Big). \tag{1.7}$$

Integrating (1.7) with respect to the nth eigenfunction from 0 to x and we have

$$\psi_n(x) = \lambda_n^{1/p}x + \psi_n(0) + O\Big(\frac{1}{\lambda_n^{1-1/p}}\Big). \tag{1.8}$$

This completes the proof. □

Remark. If the eigenvalues $\lambda_n \to \infty$, then when $\alpha > 0$,

$$\lim_{n\to\infty} \psi_n(0) = \frac{\pi_p}{2}. \tag{1.9}$$

In section 2, we shall prove Theorem 1.2. The proof of Theorem 1.1, using Weyl's criterion, will be given in section 3. In section 4, we shall see that the classical Prüfer angle $\theta_n(x)$ after modulo π_p is not equidistributed in $(0,\pi_p)$. Nor is the sequence asymptotic to the uniform distribution.

The question that whether the classical Prüfer angle is dense in $(0, \pi_p)$ or not is still open. The problem seems to be related to continued fractions with bounded and unbounded elements. It would be interesting to study this question.

As discussed above, the eigenvalues and eigenfunctions of the Sturm-Liouville operators H_q behaves like H_0, the case when the potential function $q = 0$. Say, with Dirichlet boundary conditions has the asymptotics $y_n \sim A\sin(n\pi x)$ and the nodal points $x_k^{(n)} \sim \frac{k}{n}$. For these asymptotic results, the use of another modified Prüfer angle $\phi_n = \psi_n/\sqrt{\lambda_n}$ so that

$$\phi_n' = 1 - \frac{q}{\lambda_n}\sin^2(\sqrt{\lambda_n}\phi_n(x)),$$

gives the simplest proof. The situation with the p-Laplacian operator is analogous. This paper establishes another analogy of equidistribution between $\langle\psi_n(x)\rangle_\pi$, and $\langle n\pi x\rangle_\pi$ which is associated with $q = 0$. It supports the fact that $\psi_n/\sqrt{\lambda_n}$ was a better choice.

2. Proof of Theorem 1.2

Lemma 2.1. *For any $t \in (0, \pi_p)$, $a > 0$, $b \in \mathbb{R}$, we have*

(a) $\mu\{x \in (0, \pi_p) : \langle x + b\rangle_{\pi_p} < t\} = t$.

(b) $\mu\{x \in (0, \pi_p) : \langle ax\rangle_{\pi_p} < t\} = \frac{t[a]}{a} + \min\left\{\frac{t}{a}, \pi_p\left(1 - \frac{[a]}{a}\right)\right\}$.

Proof. (a) Without loss of generality, we assume that $b \in (0, \pi_p)$. The statement is trivial when $t \geq b$. If $t < b$, it is easy to see that the measure is still t.

(b) First, it is clearly that if $k\pi_p \leq ax < k\pi_p + t$ for $k \in \mathbb{N} \cup \{0\}$, then $\langle ax\rangle_{\pi_p} < t$. This means that for $k = 0, \dots, [a] - 1$,

$$\frac{k\pi_p}{a} \leq x < \frac{t}{a} + \frac{k\pi_p}{a}.$$

When $k = [a]$, the contribution is either $\frac{t}{a}$ or $\pi_p(1 - \frac{[a]}{a})$. We conclude that (b) is also valid. $\square$

Corollary 2.2. *Let $t \in (0, \pi_p)$ and $n \in \mathbb{N}$. As $n \to \infty$,*

$$\begin{aligned}\mu\{x \in (0, \pi_p) : \langle nx + o(1)\rangle_{\pi_p} < t\} &= \mu\{x \in (0, \pi_p) : \langle nx\rangle_{\pi_p} < t\} + o(1)\\ &= t + o(1).\end{aligned}$$

Proof. From Lemma 2.1(b), $\mu\{x \in (0, \pi_p) : \langle nx\rangle_{\pi_p} < t\} = t$. It is clear that if $k\pi_p \leq nx + o(1) < k\pi_p + t$ for $k \in \mathbb{N} \cup \{0\}$, then $\langle nx + o(1)\rangle_{\pi_p} < t$. This means that for $k = 0, \dots, n - 1$,

$$\frac{k\pi_p}{n} + o(\frac{1}{n}) \leq x < \frac{k\pi_p}{n} + \frac{t}{n} + o(\frac{1}{n}).$$

The case $k = n$ only contributes $o(\frac{1}{n})$. We conclude that the formula is valid. $\square$

We also need an eigenvalue asymptotic result proved in [9].

Lemma 2.3. *The eigenvalues λ_n in the p-Laplacian eigenvalue problem* (1.4) *has the following asymptotic formula*

$$\begin{aligned}\lambda_n^{1/p} &= n_{\alpha\beta}\pi_p + \frac{\widetilde{CT_p}(\beta)^{(p-1)} - \widetilde{CT_p}(\alpha)^{(p-1)}}{(n_{\alpha\beta}\pi_p)^{p-1}} + \frac{1}{p(n_{\alpha\beta}\pi_p)^{p-1}}\int_0^1 q + o(\frac{1}{n^{p-1}}) \\ &= n_{\alpha\beta}\pi_p + o(1).\end{aligned} \tag{2.1}$$

where

$$n_{\alpha\beta} = \begin{cases} n, & \text{if } \alpha=\beta=0 \\ n-\frac{1}{2}, & \text{if } \alpha=0<\beta \text{ or } \alpha>0=\beta \\ n-1, & \text{if } \alpha,;\beta>0\,, \end{cases}$$

and, for any $\gamma \in [0,\pi_p)$,

$$\widetilde{CT_p}(\gamma)^{(p-1)} = \begin{cases} 0 & \text{if } \gamma=0 \\ |CT_p(\gamma)|^{p-2}CT_p(\gamma) & \text{if } \gamma>0\,. \end{cases}$$

Proof of Theorem 1.2. From (1.8) and (2.1),

$$\psi_n(x) = \lambda_n^{1/p}x + \psi_n(0) + o(1) = n_{\alpha\beta}\pi_p x + \psi_n(0) + o(1).$$

Hence by Lemma 2.1,

$$\begin{aligned}P_n(t) &:= \mu\{x\in(0,1) : \langle\psi_n(x)\rangle_{\pi_p} < t\} \\ &= \mu\{x\in(0,1) : \langle n_{\alpha\beta}\pi_p x + \psi_n(0) + o(1)\rangle_{\pi_p} < t\} \\ &= \frac{1}{\pi_p}\mu\{x\in(0,\pi_p) : \langle n_{\alpha\beta}x\rangle_{\pi_p} < t\} + o(1) \\ &= \frac{t[n_{\alpha\beta}]}{\pi_p n_{\alpha\beta}} + \min\big\{\frac{t}{n_{\alpha\beta}}, \pi_p\big(1-\frac{[n_{\alpha\beta}]}{n_{\alpha\beta}}\big)\big\} + o(1).\end{aligned}$$

By the definition of $n_{\alpha\beta}$, we conclude that $P_n(t) \to \frac{t}{\pi_p}$ as $n\to\infty$. ☐

3. Proof of Theorem 1.1

We shall make use of Weyl criterion, a Fourier analytic equivalent condition for a equidistributed sequence. The criterion was given by Weyl in 1916 and has proved to be very useful. The interested reader may refer to [10, p. 115-123] for a clear and interesting exposition.

Theorem 3.1. *A sequence $\{x_n\}$ is equidistributed in $(0,\pi_p)$ if and only if for any $k\in\mathbb{Z}\setminus\{0\}$,*

$$\lim_{N\to\infty}\sum_{n=1}^N \exp(\frac{2ik\pi x_n}{\pi_p}) = 0$$

Remark. When $x_n = \langle nx\rangle$ in the interval $(0,1)$ with x irrational, then by a scaling, the Weyl criterion is

$$\lim_{N\to\infty}\sum_{n=1}^N \exp(2\pi iknx) = 0. \tag{3.1}$$

It means that along the unit circle on the complex plane, as we move by an argument of $2\pi kx$ each time, the points do not overlap, but are so evenly distributed on the unit circle that their average tends to 0.

Lemma 3.2. *Let $\{b_n\}$ be a sequence in $\mathbb{R}$ such that $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} b_n = b$. Let the sequence $\{a_n\}$ satisfy $a_n = b_n + o(1)$ as $n\to\infty$. Then $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} a_n = b$.*

Proof. Since $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} b_n = b$, we find that, for $\varepsilon > 0$, there exists a $N_1\in\mathbb{N}$ such that for all $N\ge N_1$,

$$\Big|\frac{1}{N}\sum_{n=1}^{N} b_n - b\Big| \le \frac{\varepsilon}{3}.$$

On the other hand, $a_n = b_n + o(1)$ as $n\to\infty$. Given $\varepsilon>0$, there exists a $N_2\in\mathbb{N}$ such that for all $n\ge N_2$, $|a_n - b_n|\le\frac{\varepsilon}{3}$. Now let

$$M = \sum_{n=1}^{N_2-1} |a_n - b_n|.$$

Let $N_0\in\mathbb{N}$ be such that $N_0\ge\max\{N_1, N_2, \frac{3M}{\varepsilon}\}$. Then for all $N\ge N_0$,

$$\begin{aligned}\Big|\frac{1}{N}\sum_{n=1}^{N} a_n - b\Big| &\le \Big|\frac{1}{N}\sum_{n=1}^{N}(a_n - b_n)\Big| + \Big|\frac{1}{N}\sum_{n=1}^{N} b_n - b\Big|\\ &< \frac{M + (N-N_2+1)\cdot\varepsilon/3}{N} + \frac{\varepsilon}{3} < \varepsilon\,.\end{aligned}$$

This completes the proof. $\square$

Proof of Theorem 1.1. By Theorem 3.1, it suffices to show that

$$\lim_{N\to\infty}\sum_{n=1}^{N}\exp\Big(\frac{2ik\pi\psi_n(x)}{\pi_p}\Big) = 0.$$

Fixed $x\in\mathbb{R}$, from (1.8) and (2.1),

$$\psi_n(x) = \lambda_n^{1/p}x + \psi_n(0) + o(1) = n_{\alpha\beta}\pi_p x + \psi_n(0) + o(1).$$

If $\alpha=\beta=0$, then $\psi_n(0)=0$ and $n_{\alpha\beta}=n$. Hence

$$\psi_n(x) = n\pi_p x + o(1).$$

Since $\{\langle n\pi_p x\rangle_{\pi_p}\}$ is equidistributed in $(0,\pi_p)$, by Lemma 3.2, $\{\langle\psi_n(x)\rangle_{\pi_p}\}$ is also equidistributed.

If $\alpha>0=\beta$, then by (1.9), $\psi_n(0)=\frac{\pi_p}{2}+o(1)$, and $n_{\alpha\beta}=n-\frac{1}{2}$. Thus

$$\psi(x) = (n-\frac{1}{2})\pi_p x + \frac{\pi_p}{2} + o(1).$$

So when $x\in(0,1)$ is irrational, by taking any $k\in\mathbb{Z}\setminus\{0\}$,

$$\frac{1}{N}\sum_{n=1}^{N}\exp\Big(2\pi ik(n-\frac{1}{2})x + \pi ik\Big) = e^{\pi ik(1-x)}\cdot\frac{1}{N}\sum_{n=1}^{N}\exp(2\pi iknx),$$

which converges to 0 as $N\to\infty$. By Weyl's criterion, $\{\langle\psi_n(x)\rangle\}_{\pi_p}$ is also equidistributed.

The other cases $\alpha=0<\beta$ and $\alpha,\beta>0$ are similar. $\square$

4. Classical Prüfer angle

The classical Prüfer angle $\theta(x)$ is defined through

$$y = R(x)S_p(\theta(x)), \quad y' = R(x)S_p'(\theta(x)),$$

and the Prüfer angle $\theta_n(x)$ associated with the nth eigenpair satisfies

$$CT_p(\theta_n(x)) = \lambda_n^{\frac{1}{p}}\, CT_p(\psi_n(x)). \tag{4.1}$$

We denote

$$b_n := \langle\theta_n(x)\rangle_{\pi_p} = CT_p^{-1}\Big(\lambda_n^{\frac{1}{p}} CT_p(n_{\alpha\beta}\pi_p x + \psi_n(0) + o(1))\Big),$$

taking value of the inverse function CT_p^{-1} in $(0, \pi_p)$.

Theorem 4.1. *For $x \in (0,1)$, the sequence of classical Prüfer angle $\{\langle\theta_n(x)\rangle_{\pi_p}\}$ is NOT equidistributed in $(0, \pi_p)$. Nor asymptotic to the uniform distribution.*

Proof. Let I be the subinterval $(CT_p^{-1}(1), \pi_p/2) \subset (0, \pi_p/2)$. We shall see that for any $x \in (0,1)$,

$$\lim_{N\to\infty} \sum_{n=1}^{N} \chi_I(b_n) \neq \frac{1}{2} - \frac{CT_p^{-1}(1)}{\pi_p}. \tag{4.2}$$

Observe that

$$\begin{aligned} \chi_I(b_n) = 1 &\Leftrightarrow CT_p^{-1}\left(\lambda_n^{1/p} CT_p(\psi_n(x))\right) \in I = \left(CT_p^{-1}(1), \frac{\pi_p}{2}\right) \\ &\Leftrightarrow \lambda_n^{1/p} CT_p(\psi_n(x)) \in (0,1) \\ &\Leftrightarrow \langle n_{\alpha\beta}\pi_p x + \psi_n(0) + o(1)\rangle_{\pi_p} \in \big(CT_p^{-1}(\lambda_n^{-1/p}), \frac{\pi_p}{2}\big) \end{aligned} \tag{4.3}$$

If $\alpha = \beta = 0$, then $n_{\alpha\beta} = n$ and $\psi_n(0) = 0$. Hence $\chi_I(b_n) = 1$ if and only if

$$\langle\psi_n(x)\rangle_{\pi_p} = \langle n\pi_p x + o(1)\rangle_{\pi_p} \in J_n := \left(CT_p^{-1}\left(\lambda_n^{-1/p}\right), \frac{\pi_p}{2}\right).$$

Since $\lim_{n\to\infty} CT_p^{-1}(\lambda_n^{-1/p}) = \frac{\pi_p}{2}$, the probability of b_n in I tends to 0 as $n \to \infty$. Therefore.

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi_I(b_n) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi_{J_n}(\langle\psi_n(x)\rangle_{\pi_p}) = 0 < \frac{1}{2} - \frac{CT_p^{-1}(1)}{\pi_p},$$

because $|J_n| \to 0$ as $n \to \infty$.

In case $\alpha > 0 = \beta$, by (4.3),

$$\begin{aligned} \chi_I(b_n) = 1 \quad &\Leftrightarrow \langle(n - \tfrac{1}{2})\pi_p x + \tfrac{\pi_p}{2} + o(1)\rangle_{\pi_p} \in \left(CT_p^{-1}\left(\lambda_n^{-1/p}\right), \tfrac{\pi_p}{2}\right) \\ &\Leftrightarrow \langle(n - \tfrac{1}{2})\pi_p x + o(1)\rangle_{\pi_p} \in \left(\tfrac{\pi_p}{2} + CT_p^{-1}(\lambda_n^{-1/p}), \pi_p\right), \end{aligned}$$

by Lemma 2.1(a). Therefore by a similar argument as above,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi_I(b_n) = 0 < \frac{1}{2} - \frac{CT_p^{-1}(1)}{\pi_p}.$$

Therefore, (4.2) is also valid. The other cases are similar.

On the other hand, from (4.1),

$$\langle\theta_n(x)\rangle_{\pi_p} < t \Leftrightarrow CT_p^{-1}(\lambda_n^{\frac{1}{p}} CT_p(\psi_n(x))) < t$$

$$\Leftrightarrow \langle \psi_n(x) \rangle_{\pi_p} < CT_p^{-1}(\lambda_n^{\frac{-1}{p}} CT_p(t)).$$

Now for any $t \in (0, \pi_p)$, $CT_p^{-1}(\lambda_n^{\frac{-1}{p}} CT_p(t)) \to \frac{\pi_p}{2}$. Hence

$$\mu\{x \in (0,1) : \langle \theta_n(x) \rangle_{\pi_p} < t\} = P_n(\frac{\pi_p}{2} + o(1)) \to \frac{1}{2},$$

as $n \to \infty$, by Theorem 1.2. $\square$

Acknowledgements. The authors were partially supported by Ministry of Science and Technology, Taiwan (formerly National Science Council). We would also like to thank Prof. Jhishen Tsay for ther helpful discussions.

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