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PERIODIC AND SUBHARMONIC SOLUTIONS FOR FOURTH-ORDER p-LAPLACIAN DIFFERENCE EQUATIONS

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Abstract. Using critical point theory, we obtain criteria for the existence and multiplicity of periodic and subharmonic solutions to fourth-order p-Laplacian difference equations. The proof is based on the Linking Theorem in combination with variational technique. Recent results in the literature are generalized and improved.

1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of all natural numbers, integers and real numbers respectively. For a, $b \in \mathbb{Z}$, define $\mathbb{Z}(a) = \{a, a+1, \dots\}$, $\mathbb{Z}(a, b) = \{a, a+1, \dots, b\}$ when $a < b$. The symbol * denotes the transpose of a vector.

In this paper, we consider the forward and backward difference equation

$$\Delta^2\big(\gamma_{n-2}\varphi_p(\Delta^2 u_{n-2})\big) = f(n, u_{n+1}, u_n, u_{n-1}), \quad n \in \mathbb{Z}, \tag{1.1}$$

where Δ is the forward difference operator $\Delta u_n = u_{n+1} - u_n$, $\Delta^2 u_n = \Delta(\Delta u_n)$, γ_n is real valued for each $n \in \mathbb{Z}$, $\varphi_p(s)$ is the p-Laplacian operator $\varphi_p(s) = |s|^{p-2}s(1 < p < \infty)$, $f \in C(\mathbb{Z} \times \mathbb{R}^3, \mathbb{R})$, γ_n and $f(n, v_1, v_2, v_3)$ are T-periodic in n for a given positive integer T.

We may think of (1.1) as a discrete analogue of the fourth-order functional differential equation

$$\frac{d^2}{dt^2}\big[\gamma(t)\varphi_p\big(\frac{d^2 u(t)}{dt^2}\big)\big] = f(t, u(t+1), u(t), u(t-1)), \quad t \in \mathbb{R}. \tag{1.2}$$

This equation includes the equation

$$u^{(4)}(t) = f(t, u(t)), \quad t \in \mathbb{R}, \tag{1.3}$$

which is used to model deformations of elastic beams [8, 31]. Equations similar in structure to (1.2) arise in the study of the existence of solitary waves of lattice differential equations, see Smets and Willem [35].

The theory of nonlinear difference equations has been widely used to study discrete models appearing in many fields such as computer science, economics, neural networks, ecology, cybernetics, etc. For the general background of difference equations, one can refer to monographs [1, 19]. Since the last decade, there has been

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much progress on the qualitative properties of difference equations, which included results on stability and attractivity [12, 23, 26, 42] and results on oscillation and other topics, see [1, 2, 3, 4, 5, 6, 18, 19, 20, 21, 22, 24, 25, 35, 36, 37, 38, 39, 40, 41].

The motivation of this paper is as follows. It is well known that critical point theory is a powerful tool that deals with the problems of differential equations [8, 11, 15, 27, 31]. Starting in 2003, critical point theory has been employed to establish sufficient conditions on the existence of periodic solutions of difference equations. Particularly, Guo and Yu [14, 15, 16] and Shi et al. [32] studied the existence of periodic solutions of second-order nonlinear difference equations by using the critical point theory. Compared to first-order or second-order difference equations, the study of higher-order equations, and in particular, fourth-order equations, has received considerably less attention (see, for example, [1, 9, 10, 13, 24, 29, 30, 34, 37] and the references contained therein). Yan, Liu [37] in 1997 and Thandapani, Arockiasamy [34] in 2001 studied the fourth-order difference equation

$$\Delta^2\big(\gamma_n\Delta^2 u_n\big) + f(n,u_n) = 0, \quad n\in\mathbb{Z}, \tag{1.4}$$

and obtained criteria for the oscillation and nonoscillation of solutions for equation (1.4). In 2005, Cai, Yu and Guo [7] have obtained some criteria for the existence of periodic solutions of the fourth-order difference equation

$$\Delta^2\left(\gamma_{n-2}\Delta^2 u_{n-2}\right) + f(n,u_n) = 0, \quad n\in\mathbb{Z}. \tag{1.5}$$

In 1995, Peterson and Ridenhour considered the disconjugacy of equation (1.5) when $\gamma_n\equiv 1$ and $f(n,u_n)=q_nu_n$ (see [29]). However, to the best of our knowledge, the results on periodic solutions of fourth-order p-Laplacian difference equations are very scarce in the literature. We found that [7] is the only paper which deals with the problem of periodic solutions to fourth-order difference equation (1.5). Furthermore, since (1.1) contains both advance and retardation, there are very few manuscripts dealing with this subject. The main purpose of this paper is to give some sufficient conditions for the existence and multiplicity of periodic and subharmonic solutions to fourth-order p-Laplacian difference equations. The main approach used in our paper is the variational technique and the Linking Theorem. In particular, our results not only generalize the results in the literature [7], but also improve them. In fact, one can see the Remarks 1.4 and 1.9 for details. The motivation for the present work stems from the recent papers in [9, 14, 16].

Let

$$\underline{\gamma} = \min_{n\in\mathbb{Z}(1,T)}\{\gamma_n\}, \quad \overline{\gamma} = \max_{n\in\mathbb{Z}(1,T)}\{\gamma_n\}.$$

Our main results read as follows.

Theorem 1.1. *Assume that the following hypotheses are satisfied:*

- (F0) $\gamma_n>0$ *for all* $n\in\mathbb{Z}$*;*
- (F1) *there exists a functional* $F(n,v_1,v_2)\in C^1(\mathbb{Z}\times\mathbb{R}^2,\mathbb{R})$ *with* $F(n,v_1,v_2)\geq 0$ *and it satisfies*

$$F(n+T,v_1,v_2) = F(n,v_1,v_2),$$
$$\frac{\partial F(n-1,v_2,v_3)}{\partial v_2} + \frac{\partial F(n,v_1,v_2)}{\partial v_2} = f(n,v_1,v_2,v_3);$$

- (F2) *there exist constants* $\delta_1>0$, $\alpha\in\big(0,\frac{\underline{\gamma}}{2^{p/2}p}(c_1/c_2)^p\lambda_{\min}^p\big)$ *such that*

$$F(n,v_1,v_2)\leq \alpha\Big(\sqrt{v_1^2+v_2^2}\Big)^p, \quad \textit{for } n\in\mathbb{Z} \textit{ and } v_1^2+v_2^2\leq\delta_1^2;$$

(F3) *there exist constants* $\rho_1 > 0$, $\zeta > 0$, $\beta \in \left(\frac{\bar{\gamma}}{2^{p/2}p}(c_2/c_1)^p \lambda_{\max}^p, +\infty\right)$ *such that*

$$F(n, v_1, v_2) \geq \beta\Big(\sqrt{v_1^2 + v_2^2}\Big)^p - \zeta, \quad \text{for } n \in \mathbb{Z} \text{ and } v_1^2 + v_2^2 \geq \rho_1^2,$$

where c_1, c_2 *are constants which can be referred to* (2.4)*, and* $\lambda_{\min}, \lambda_{\max}$ *are constants which can be referred to* (2.7)*.*

Then for any given positive integer $m > 0$*, Equation* (1.1) *has at least three* mT*-periodic solutions.*

Remark 1.2. By (F3) it is easy to see that there exists a constant $\zeta' > 0$ such that

(F3')

$$F(n, v_1, v_2) \geq \beta\Big(\sqrt{v_1^2 + v_2^2}\Big)^p - \zeta', \quad \forall (n, v_1, v_2) \in \mathbb{Z} \times \mathbb{R}^2.$$

As a matter of fact, let $\zeta_1 = \max\big\{|F(n, v_1, v_2) - \beta\big(\sqrt{v_1^2 + v_2^2}\big)^p + \zeta| : n \in \mathbb{Z}, v_1^2 + v_2^2 \leq \rho_1^2\big\}$, $\zeta' = \zeta + \zeta_1$, we can easily get the desired result.

Corollary 1.3. *Assume that* (F0–(F3) *are satisfied. Then for any given positive integer* $m > 0$*,* (1.1) *has at least two nontrivial* mT*-periodic solutions.*

Remark 1.4. The statement in in the above corollary is the same as [7, Theorem 1.1]

Theorem 1.5. *Assume that* (F0), (F1) *and the following conditions are satisfied:*

(F4) $\lim_{\rho \to 0} \frac{F(n, v_1, v_2)}{\rho^p} = 0$, $\rho = \sqrt{v_1^2 + v_2^2}$ *for all* $(n, v_1, v_2) \in \mathbb{Z} \times \mathbb{R}^2$*;*

(F5) *there exist constants* $\theta > p$ *and* $a_1 > 0$, $a_2 > 0$ *such that*

$$F(n, v_1, v_2) \geq a_1\big(\sqrt{v_1^2 + v_2^2}\big)^\theta - a_2, \quad \forall (n, v_1, v_2) \in \mathbb{Z} \times \mathbb{R}^2.$$

Then for any given positive integer $m > 0$*, Equation* (1.1) *has at least three* mT*-periodic solutions.*

Corollary 1.6. *Assume that* (F0), (F1), (F4), (F5) *are satisfied. Then for any given positive integer* $m > 0$*, Equation* (1.1) *has at least two nontrivial* mT*-periodic solutions.*

If $f(n, u_{n+1}, u_n, u_{n-1}) = q_n g(u_n)$, then (1.1) reduces to the fourth-order nonlinear equation

$$\Delta^2\left(\gamma_{n-2}\varphi_p(\Delta^2 u_{n-2})\right) = q_n g(u_n), \ n \in \mathbb{Z}, \tag{1.6}$$

where $g \in C(\mathbb{R}, \mathbb{R}), q_{n+T} = q_n > 0$, for all $n \in \mathbb{Z}$. Then, we have the following results.

Theorem 1.7. *Assume that* (F0) *and the following hypotheses are satisfied:*

(G1) *there exists a functional* $G(v) \in C^1(\mathbb{R}, \mathbb{R})$ *with* $G(v) \geq 0$ *and it satisfies*

$$\frac{dG(v)}{dv} = g(v);$$

(G2) *there exist constants* $\delta_2 > 0$, $\alpha \in \left(0, \frac{\gamma}{p}(c_1/c_2)^p \lambda_{\min}^p\right)$ *such that* $G(v) \leq \alpha|v|^p$*, for* $|v| \leq \delta_2$*;*

(G3) *there exist constants* $\rho_2 > 0$, $\zeta > 0$, $\beta \in \big(\frac{\bar{\gamma}}{p}(c_2/c_1)^p \lambda_{\max}^p, +\infty\big)$ *such that*

$$G(v) \geq \beta |v|^p - \zeta, \quad \text{for } |v| \geq \rho_2,$$

where c_1, c_2 *are constants which can be referred to* (2.4)*, and* $\lambda_{\min}, \lambda_{\max}$ *are constants which can be referred to* (2.7)*.*

Then for any given positive integer $m > 0$*, Equation* (1.6) *has at least three* mT*-periodic solutions.*

Corollary 1.8. *Assume that* (F0), (G1)–(G3) *are satisfied. Then for any given positive integer* $m > 0$*, Equation* (1.6) *has at least two nontrivial* mT*-periodic solutions.*

Remark 1.9. The statement of above corollary is the same as [7, Theorem 1.2].

The rest of the paper is organized as follows. First, in Section 2 we shall establish the variational framework associated with (1.1) and transfer the problem of the existence of periodic solutions of (1.1) into that of the existence of critical points of the corresponding functional. Some related fundamental results will also be recalled. Then, in Section 3, we shall complete the proof of the results by using the critical point method. Finally, in Section 4, we shall give an example to illustrate the main result.

2. Variational structure and some lemmas

In order to apply the critical point theory, we shall establish the corresponding variational framework for (1.1) and give some basic notation and useful lemmas. For the basic knowledge of variational methods, the reader is referred to [17, 27, 28, 31].

Let S be the set of sequences $u = (\dots, u_{-n}, \dots, u_{-1}, u_0, u_1, \dots, u_n, \dots) = \{u_n\}_{n=-\infty}^{+\infty}$, that is

$$S = \{\{u_n\} : u_n \in \mathbb{R}, n \in \mathbb{Z}\}.$$

For any $u, v \in S$, $a, b \in \mathbb{R}$, $au + bv$ is defined by

$$au + bv = \{au_n + bv_n\}_{n=-\infty}^{+\infty}.$$

Then S is a vector space. For any given positive integers m and T, E_{mT} is defined as a subspace of S by

$$E_{mT} = \{u \in S : u_{n+mT} = u_n, \ \forall n \in \mathbb{Z}\}.$$

Clearly, E_{mT} is isomorphic to $\mathbb{R}^{mT}$. E_{mT} can be equipped with the inner product

$$\langle u, v\rangle = \sum_{j=1}^{mT} u_j v_j, \ \forall u, v \in E_{mT}, \tag{2.1}$$

by which we introduce the norm

$$\|u\| = \Big(\sum_{j=1}^{mT} u_j^2\Big)^{1/2}, \quad \forall u \in E_{mT}. \tag{2.2}$$

It is obvious that E_{mT} with the inner product (2.1) is a finite dimensional Hilbert space and linearly homeomorphic to $\mathbb{R}^{mT}$.

On the other hand, we define the norm $\|\cdot\|_r$ on E_{mT} as follows:

$$\|u\|_r = \Big(\sum_{j=1}^{mT} |u_j|^r\Big)^{1/r}, \tag{2.3}$$

for all $u \in E_{mT}$ and $r > 1$.

Since $\|u\|_r$ and $\|u\|_2$ are equivalent, there exist constants c_1, c_2 such that $c_2 \geq c_1 > 0$, and

$$c_1\|u\|_2 \leq \|u\|_r \leq c_2\|u\|_2, \quad \forall u \in E_{mT}. \tag{2.4}$$

Clearly, $\|u\| = \|u\|_2$. For all $u \in E_{mT}$, define the functional J on E_{mT} as follows:

$$J(u) = \sum_{n=1}^{mT} \big[\frac{1}{p}\gamma_{n-1}|\Delta^2 u_{n-1}|^p - F(n, u_{n+1}, u_n)\big], \tag{2.5}$$

where

$$\frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = f(n, v_1, v_2, v_3).$$

Clearly, $J \in C^1(E_{mT}, \mathbb{R})$ and for any $u = \{u_n\}_{n\in\mathbb{Z}} \in E_{mT}$, by using $u_0 = u_{mT}$, $u_1 = u_{mT+1}$, we can compute the partial derivative as

$$\frac{\partial J}{\partial u_n} = \Delta^2\left(\gamma_{n-2}\varphi_p(\Delta^2 u_{n-2})\right) - f(n, u_{n+1}, u_n, u_{n-1}).$$

Thus, u is a critical point of J on E_{mT} if and only if

$$\Delta^2\left(\gamma_{n-2}\varphi_p(\Delta^2 u_{n-2})\right) = f(n, u_{n+1}, u_n, u_{n-1}), \quad \forall n \in \mathbb{Z}(1, mT).$$

Due to the periodicity of $u = \{u_n\}_{n\in\mathbb{Z}} \in E_{mT}$ and $f(n, v_1, v_2, v_3)$ in the first variable n, we reduce the existence of periodic solutions of (1.1) to the existence of critical points of J on E_{mT}. That is, the functional J is just the variational framework of (1.1).

Let P be the $mT \times mT$ matrix defined by

$$P = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ -1 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}.$$

By matrix theory, we see that the eigenvalues of P are

$$\lambda_k = 2\big(1 - \cos(\frac{2k}{mT}\pi)\big), \quad k = 0, 1, 2, \dots, mT-1. \tag{2.6}$$

Thus, $\lambda_0 = 0$, $\lambda_1 > 0$, $\lambda_2 > 0, \dots, \lambda_{mT-1} > 0$. Therefore,

$$\begin{aligned} \lambda_{\min} &= \min\{\lambda_1, \lambda_2, \dots, \lambda_{mT-1}\} = 2\Big(1 - \cos(\frac{2}{mT}\pi)\Big), \\ \lambda_{\max} &= \max\{\lambda_1, \lambda_2, \dots, \lambda_{mT-1}\} \\ &= \begin{cases} 4, & \text{if } mT \text{ is even}, \\ 2\left(1 + \cos(\frac{1}{mT}\pi)\right), & \text{if } mT \text{ is odd}. \end{cases} \end{aligned} \tag{2.7}$$

Let

$$W = \ker P = \{u \in E_{mT} | Pu = 0 \in \mathbb{R}^{mT}\}.$$

Then

$$W = \{u \in E_{mT} | u = \{c\},\ c \in \mathbb{R}\}.$$

Let V be the direct orthogonal complement of E_{mT} to W; i.e.; $E_{mT} = V \oplus W$. For convenience, we identify $u \in E_{mT}$ with $u = (u_1, u_2, \dots, u_{mT})^*$.

Let E be a real Banach space, $J \in C^1(E, \mathbb{R})$; i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to satisfy the Palais-Smale condition ((PS) condition for short) if any sequence $\{u^{(k)}\} \subset E$ for which $\{J(u^{(k)})\}$ is bounded and $J'(u^{(k)}) \to 0 (k \to \infty)$ possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 2.1 (Linking Theorem [31]). *Let E be a real Banach space, $E = E_1 \oplus E_2$, where E_1 is finite dimensional. Suppose that $J \in C^1(E, \mathbb{R})$ satisfies the (PS) condition and*
(J_1) there exist constants $a > 0$ and $\rho > 0$ such that $J|_{\partial B_\rho \cap E_2} \geq a$;
(J_2) there exists an $e \in \partial B_1 \cap E_2$ and a constant $R_0 \geq \rho$ such that $J|_{\partial Q} \leq 0$, where $Q = (\bar{B}_{R_0} \cap E_1) \oplus \{re | 0 < r < R_0\}$.
Then J possesses a critical value $c \geq a$, where

$$c = \inf_{h \in \Gamma} \sup_{u \in Q} J(h(u)),$$

and $\Gamma = \{h \in C(\bar{Q}, E) \mid h|_{\partial Q} = id\}$, where id denotes the identity operator.

Lemma 2.2. *Assume that* (F0), (F1), (F3) *are satisfied. Then the functional J is bounded from above in E_{mT}.*

Proof. By (F3') and (2.4), for any $u \in E_{mT}$,

$$\begin{aligned}
J(u) &= \sum_{n=1}^{mT} \Big[\frac{1}{p}\gamma_{n-1}|\Delta^2 u_{n-1}|^p - F(n, u_{n+1}, u_n)\Big] \\
&\leq \frac{\bar{\gamma}}{p} c_2^p \Big[\sum_{n=1}^{mT} (\Delta u_n - \Delta u_{n-1})^2\Big]^{p/2} - \sum_{n=1}^{mT} F(n, u_{n+1}, u_n) \\
&\leq \frac{\bar{\gamma}}{p} c_2^p (x^* P x)^{p/2} - \sum_{n=1}^{mT} \Big[\beta\big(\sqrt{u_{n+1}^2 + u_n^2}\big)^p - \zeta'\Big] \\
&\leq \frac{\bar{\gamma}}{p} c_2^p \lambda_{\max}^{p/2} \|x\|_2^p - \beta\Big[\Big(\sum_{n=1}^{mT}\big(\sqrt{u_{n+1}^2 + u_n^2}\big)^p\Big)^{1/p}\Big]^p + mT\zeta' \\
&\leq \frac{\bar{\gamma}}{p} c_2^p \lambda_{\max}^{p/2} \|x\|_2^p - \beta c_1^p \Big[\sum_{n=1}^{mT}\big(u_{n+1}^2 + u_n^2\big)\Big]^{p/2} + mT\zeta' \\
&= \frac{\bar{\gamma}}{p} c_2^p \lambda_{\max}^{p/2} \|x\|_2^p - \beta c_1^p \big(2\|u\|_2^2\big)^{p/2} + mT\zeta' \\
&\leq \frac{\bar{\gamma}}{p} c_2^p \lambda_{\max}^{p/2} \|x\|_2^p - 2^{p/2}\beta c_1^p \|u\|_2^p + mT\zeta',
\end{aligned}$$

where $x = (\Delta u_1, \Delta u_2, \dots, \Delta u_{mT})^*$. Since

$$\|x\|_2^p = \Big[\sum_{n=1}^{mT} (u_{n+1} - u_n, u_{n+1} - u_n)\Big]^{p/2} = (u^* P u)^{p/2} \leq \lambda_{\max}^{p/2} \|u\|_2^p,$$

we have

$$J(u) \leq \frac{\bar{\gamma}}{p} c_2^p \lambda_{\max}^p \|u\|_2^p - 2^{p/2}\beta c_1^p \|u\|_2^p + mT\zeta' \leq mT\zeta'.$$

The proof is complete. $\square$

Remark 2.3. The case $mT = 1$ is trivial. For the case $mT = 2$, P has a different form, namely,

$$P = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

However, in this special case, the argument need not to be changed and we omit it.

Lemma 2.4. *Assume that* (F0), (F1), (F3) *are satisfied. Then the functional* J *satisfies the (PS) condition.*

Proof. Let $\{J\big(u^{(k)}\big)\}$ be a bounded sequence from the lower bound; i.e., there exists a positive constant M_1 such that

$$-M_1 \le J\left(u^{(k)}\right), \ \forall k \in \mathbf{N}.$$

By the proof of Lemma 2.2, it is easy to see that

$$-M_1 \le J\big(u^{(k)}\big) \le \big(\frac{\bar{\gamma}}{p} c_2^p \lambda_{\max}^p - 2^{p/2}\beta c_1^p\big)\|u^{(k)}\|_2^p + mT\zeta', \quad \forall k \in \mathbf{N}.$$

Therefore,

$$\big(2^{p/2}\beta c_1^p - \frac{\bar{\gamma}}{p} c_2^p \lambda_{\max}^p\big)\|u^{(k)}\|_2^p \le M_1 + mT\zeta'.$$

Since $\beta > \frac{\bar{\gamma}}{2^{p/2}p}(c_2/c_1)^p\lambda_{\max}^p$, it is not difficult to know that $\{u^{(k)}\}$ is a bounded sequence in E_{mT}. As a consequence, $\{u^{(k)}\}$ possesses a convergence subsequence in E_{mT}. Thus the (PS) condition is verified. □

3. Proof of main results

In this Section, we shall prove our main results by using the critical point method.

Proof of Theorem 1.1. Assumptions (F1) and (F2) imply that $F(n, 0) = 0$ and $f(n, 0) = 0$ for $n \in \mathbb{Z}$. Then $u = 0$ is a trivial mT-periodic solution of (1.1).

By Lemma 2.4, J is bounded from above on E_{mT}. We define $c_0 = \sup_{u\in E_{mT}} J(u)$. The proof of Lemma 2.4 implies $\lim_{\|u\|_2\to+\infty} J(u) = -\infty$. This means that $-J(u)$ is coercive. By the continuity of $J(u)$, there exists $\bar{u} \in E_{mT}$ such that $J(\bar{u}) = c_0$. Clearly, $\bar{u}$ is a critical point of J.

We claim that $c_0 > 0$. Indeed, by (F2), for any $u \in V$, $\|u\|_2 \le \delta_1$, we have

$$\begin{aligned}
J(u) &= \sum_{n=1}^{mT}\Big[\frac{1}{p}\gamma_{n-1}|\Delta^2 u_{n-1}|^p - F(n, u_{n+1}, u_n)\Big] \\
&\ge \frac{1}{p}\underline{\gamma} c_1^p \Big[\sum_{n=1}^{mT}(\Delta u_n - \Delta u_{n-1})^2\Big]^{p/2} - \sum_{n=1}^{mT} F(n, u_{n+1}, u_n) \\
&\ge \frac{1}{p}\underline{\gamma} c_1^p (x^* P x)^{p/2} - \sum_{n=1}^{mT} F(n, u_{n+1}, u_n) \\
&\ge \frac{1}{p}\underline{\gamma} c_1^p \lambda_{\min}^{p/2}\|x\|_2^p - \alpha\sum_{n=1}^{mT}\big(\sqrt{u_{n+1}^2 + u_n^2}\big)^p \\
&= \frac{1}{p}\underline{\gamma} c_1^p \lambda_{\min}^{p/2}\|x\|_2^p - \alpha\Big[\Big(\sum_{n=1}^{mT}\big(\sqrt{u_{n+1}^2 + u_n^2}\big)^p\Big)^{1/p}\Big]^p
\end{aligned}$$

$$\geq \frac{1}{p}\underline{\gamma} c_1^p \lambda_{\min}^{p/2} \|x\|_2^p - \alpha c_2^p \Big[\sum_{n=1}^{mT} \left(u_{n+1}^2 + u_n^2 \right) \Big]^{p/2},$$

where $x = (\Delta u_1, \Delta u_2, \dots, \Delta u_{mT})^*$. Since

$$\|x\|_2^p = \Big[\sum_{n=1}^{mT} (u_{n+1} - u_n, u_{n+1} - u_n) \Big]^{p/2} = (u^* P u)^{p/2} \geq \lambda_{\min}^{p/2} \|u\|_2^p,$$

we have

$$J(u) \geq \frac{1}{p}\underline{\gamma} c_1^p \lambda_{\min}^p \|u\|_2^p - \alpha c_2^p \left(2\|u\|_2^2 \right)^{p/2} = \left(\frac{1}{p}\underline{\gamma} c_1^p \lambda_{\min}^p - 2^{p/2} c_2^p \alpha \right) \|u\|_2^p.$$

Take $\sigma = \left(\frac{1}{p}\underline{\gamma} c_1^p \lambda_{\min}^p - 2^{p/2} c_2^p \alpha \right) \delta_1^p$. Then

$$J(u) \geq \sigma, \quad \forall u \in V \cap \partial B_{\delta_1}.$$

Therefore, $c_0 = \sup_{u \in E_{mT}} J(u) \geq \sigma > 0$. At the same time, we have also proved that there exist constants $\sigma > 0$ and $\delta_1 > 0$ such that $J|_{\partial B_{\delta_1} \cap V} \geq \sigma$. That is to say, J satisfies the condition (J_1) of the Linking Theorem.

Noting that $\sum_{n=1}^{mT} \gamma_{n-1} \left| \Delta^2 u_{n-1} \right|^p = 0$, for all $u \in W$, we have

$$J(u) = \frac{1}{p} \sum_{n=1}^{mT} \gamma_{n-1} |\Delta^2 u_{n-1}|^p - \sum_{n=1}^{mT} F(n, u_{n+1}, u_n) = - \sum_{n=1}^{mT} F(n, u_{n+1}, u_n) \leq 0.$$

Thus, the critical point $\bar{u}$ of J corresponding to the critical value c_0 is a nontrivial mT-periodic solution of (1.1).

To obtain another nontrivial mT-periodic solution of (1.1) different from $\bar{u}$, we need to use the conclusion of Lemma 2.1. We have known that J satisfies the (PS) condition on E_{mT}. In the following, we shall verify the condition (J_2).

Take $e \in \partial B_1 \cap V$, for any $z \in W$ and $r \in \mathbb{R}$, let $u = re + z$. Then

$$\begin{aligned}
J(u) &= \sum_{n=1}^{mT} \big[\frac{1}{p} \gamma_n |\Delta^2 u_n|^p - F(n, u_{n+1}, u_n) \big] \\
&\leq \sum_{n=1}^{mT} \big[\frac{\bar{\gamma}}{p} r^p \left| \Delta^2 e_n \right|^p - F(n, re_{n+1} + z_{n+1}, re_n + z_n) \big] \\
&\leq \frac{\bar{\gamma}}{p} r^p c_2^p \Big[\sum_{n=1}^{mT} \left(\Delta e_n - \Delta e_{n-1} \right)^2 \Big]^{p/2} - \sum_{n=1}^{mT} F(n, re_{n+1} + z_{n+1}, re_n + z_n) \\
&\leq \frac{\bar{\gamma}}{p} r^p c_2^p (y^* P y)^{p/2} - \sum_{n=1}^{mT} \Big\{ \beta \left(\sqrt{(re_{n+1} + z_{n+1})^2 + (re_n + z_n)^2} \right)^p - \zeta' \Big\} \\
&\leq \frac{\bar{\gamma}}{p} r^p c_2^p (y^* P y)^{p/2} - \beta c_1^p \Big\{ \sum_{n=1}^{mT} \left[(re_{n+1} + z_{n+1})^2 + (re_n + z_n)^2 \right] \Big\}^{p/2} + mT\zeta' \\
&\leq \frac{\bar{\gamma}}{p} r^p c_2^p \lambda_{\max}^{p/2} \|y\|_2^p - \beta c_1^p \Big[2 \sum_{n=1}^{mT} \left(re_n + z_n \right)^2 \Big]^{p/2} + mT\zeta' \\
&= \frac{\bar{\gamma}}{p} r^p c_2^p \lambda_{\max}^{p/2} \|y\|_2^p - \beta c_1^p r^p 2^{p/2} - \beta c_1^p 2^{p/2} \|z\|_2^p + mT\zeta',
\end{aligned}$$

where $y=(\Delta e_1,\Delta e_2,\dots,\Delta e_{mT})^*$. Since

$$\|y\|_2^p=\Big[\sum_{n=1}^{mT}(e_{n+1}-e_n,e_{n+1}-e_n)\Big]^{p/2}=(e^*Pe)^{p/2}\le\lambda_{\max}^{p/2},$$

we have

$$J(u)\le\Big(\frac{\bar\gamma}{p}c_2^p\lambda_{\max}^p-\beta c_1^p2^{p/2}\Big)r^p-\beta c_1^p2^{p/2}\|z\|_2^p+mT\zeta'\le-\beta c_1^p2^{p/2}\|z\|_2^p+mT\zeta'.$$

Thus, there exists a positive constant $R_1>\delta_1$ such that for any $u\in\partial Q$, $J(u)\le 0$, where $Q=(\bar B_{R_1}\cap W)\oplus\{re:0<r<R_1\}$. By the Linking Theorem, J possesses a critical value $c\ge\sigma>0$, where

$$c=\inf_{h\in\Gamma}\sup_{u\in Q}J(h(u)),$$

and $\Gamma=\{h\in C(\bar Q,E_{mT})\mid h|_{\partial Q}=id\}$.

Let $\tilde u\in E_{mT}$ be a critical point associated to the critical value c of J, i.e., $J(\tilde u)=c$. If $\tilde u\ne\bar u$, then the conclusion of Theorem 1.1 holds. Otherwise, $\tilde u=\bar u$. Then $c_0=J(\bar u)=J(\tilde u)=c$; that is, $\sup_{u\in E_{mT}}J(u)=\inf_{h\in\Gamma}\sup_{u\in Q}J(h(u))$. Choosing $h=id$, we have $\sup_{u\in Q}J(u)=c_0$. Since the choice of $e\in\partial B_1\cap V$ is arbitrary, we can take $-e\in\partial B_1\cap V$. Similarly, there exists a positive number $R_2>\delta_1$, for any $u\in\partial Q_1$, $J(u)\le 0$, where $Q_1=(\bar B_{R_2}\cap W)\oplus\{-re|0<r<R_2\}$.

Again, by the Linking Theorem, J possesses a critical value $c'\ge\sigma>0$, where

$$c'=\inf_{h\in\Gamma_1}\sup_{u\in Q_1}J(h(u)),$$

and $\Gamma_1=\{h\in C(\bar Q_1,E_{mT})\mid h|_{\partial Q_1}=id\}$.

If $c'\ne c_0$, then the proof is finished. If $c'=c_0$, then $\sup_{u\in Q_1}J(u)=c_0$. Due to the fact $J|_{\partial Q}\le 0$ and $J|_{\partial Q_1}\le 0$, J attains its maximum at some points in the interior of sets Q and Q_1. However, $Q\cap Q_1\subset W$ and $J(u)\le 0$ for any $u\in W$. Therefore, there must be a point $u'\in E_{mT}$, $u'\ne\tilde u$ and $J(u')=c'=c_0$. The proof is complete. □

Similarly to above argument, we can also prove Theorems 1.5 and 1.7, so their proofs are omitted. Due to Theorems 1.1, 1.5 and 1.7, the conclusion of Corollaries 1.3, 1.6 and 1.8 are obviously true.

4. Example

As an application of Theorem 1.1, we give an example to illustrate our main result.

Example 4.1. Assume that for all $n\in\mathbb{Z}$,

$$\begin{aligned}\Delta^2\left(\gamma_{n-2}\varphi_p(\Delta^2u_{n-2})\right)=\mu u_n\Big[&\Big(3+\sin^2(\pi n/T)\Big)(u_{n+1}^2+u_n^2)^{\frac{\mu}{2}-1}\\&+\Big(3+\sin^2\left(\pi(n-1)/T\right)\Big)(u_n^2+u_{n-1}^2)^{\frac{\mu}{2}-1}\Big],\end{aligned}\tag{4.1}$$

where γ_n is real valued for each $n\in\mathbb{Z}$ and $\gamma_{n+T}=\gamma_n>0$, $1<p<+\infty$, $\mu>p$, T is a given positive integer. We have

$$\begin{aligned}f(n,v_1,v_2,v_3)=\mu v_2\Big[&\Big(3+\sin^2(\pi n/T)\Big)(v_1^2+v_2^2)^{\frac{\mu}{2}-1}\\&+\Big(3+\sin^2\left(\pi(n-1)/T\right)\Big)(v_2^2+v_3^2)^{\frac{\mu}{2}-1}\Big]\end{aligned}$$

and
$$F(n,v_1,v_2)=[3+\sin^2(\pi n/T)](v_1^2+v_2^2)^{\frac{\mu}{2}}.$$
Then
$$\begin{aligned}&\frac{\partial F(n-1,v_2,v_3)}{\partial v_2}+\frac{\partial F(n,v_1,v_2)}{\partial v_2}\\&=\mu v_2\Big[\Big(3+\sin^2(\pi n/T)\Big)(v_1^2+v_2^2)^{\frac{\mu}{2}-1}+\Big(3+\sin^2\big(\pi(n-1)/T\big)\Big)(v_2^2+v_3^2)^{\frac{\mu}{2}-1}\Big].\end{aligned}$$
It is easy to verify all the assumptions of Theorem 1.1 are satisfied. Consequently, for any given positive integer $m>0$, (4.1) has at least three mT-periodic solutions.

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