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SOLUTIONS TO QUASI-LINEAR DIFFERENTIAL EQUATIONS WITH ITERATED DEVIATING ARGUMENTS

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ABSTRACT. We establish sufficient conditions for the existence and uniqueness of solutions to quasi-linear differential equations with iterated deviating arguments, complex Banach space. The results are obtained by using the semigroup theory for parabolic equations and fixed point theorems. The main results are illustrated by an example.

1. Introduction

Let $(X, \|\cdot\|)$ be a complex Banach space. For each t, $0 \le t \le T < \infty$ and $x \in X$, let $A(t,x) : D(A(t,x)) \subset X \to X$ be a linear operator on X. We study the following problem in X:

$$\begin{gathered}\frac{du}{dt} + A(t,u(t))u(t) = f(t,u(t),u(w_1(t,u(t)))), \quad t>0;\\ u(0)=u_0,\end{gathered} \tag{1.1}$$

where $u : \mathbb{R}_+ \to X$, $u_0 \in X$, $w_1(t,u(t)) = h_1(t,u(h_2(t,\dots,u(h_m(t,u(t)))\dots)))$, $f : \mathbb{R}_+ \times X \times X \to X$ and $h_j : \mathbb{R}_+ \times X \to \mathbb{R}_+$, $j = 1,2,3,\dots,m$ are continuous functions. The non-linear functions f and h_j satisfy appropriate conditions in terms of their arguments (see Section 2).

The class of quasi-linear differential equations is one of the most important classes that arise in the study of gas dynamics, continuum mechanics, traffic flow models, nonlinear acoustics, and groundwater flows, to mention only a few of their application in real world problems (see [2, 15, 16]). Thus the theory of quasi-linear differential equations and their generalizations become as one of the most rapid developing areas in applied mathematics. In this article we consider one such generalization. We establish the existence and uniqueness theory for a class quasi-linear differential equation to a class quasi-linear differential equation with iterated deviating arguments. The main results of this article are new and complement to the existing ones that generalize some results of [7, 8, 28]. Different sufficient conditions for the existence and uniqueness of a solution to quasi-linear differential equations can be found in [1, 3, 8, 16, 17, 18, 22, 26]. Further, we refer to [4, 7, 14, 13] for a

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nice introductions to the theory of differential equations with deviating arguments and references cited therein for more details.

The plentiful applications of differential equations with deviating arguments has motivated the rapid development of the theory of differential equations with deviating arguments and their generalization in the recent years (see [7, 8, 9, 10, 11, 19, 27, 28, 29, 30]). Stević [28] has proved some sufficient conditions for the existence of a bounded solution on the whole real line for the system

$$u'(t) = Au(t) + G(t, u(t), u(v_1(t)), u'(g(t))),$$

where $v_1(t) = f_1(t, x(f_2(t, \dots, x(f_k(t, x(t)), \dots)))$, $A = \operatorname{diag}(A_1, A_2)$ is an $n \times n$ real constant matrix with A_1 and A_2 matrices of dimensions $p \times p$ and $q \times q$, respectively, $p+q = n$, and $G : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $f_j : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $j = 1, 2, 3, \dots, k$ and $g : \mathbb{R} \to \mathbb{R}$. The results are obtained when $G(\cdot, x, y, z)$ satisfies Lipschitz condition in x, y, z; $f_j(\cdot, x)$ satisfies Lipschitz condition in x and A satisfies some stability condition [28].

Recently, Kumar et al [19] established sufficient conditions for the existence of piecewise continuous mild solutions in a Banach space X to the problem

$$\begin{gathered} \frac{d}{dt}u(t) + Au(t) = g(t, u(t), u[v_1(t, u(t))]), \quad t \in I = [0, T_0],\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k^-)), \quad k = 1, 2, \dots, m, \\ u(0) = u_0, \end{gathered} \tag{1.2}$$

where

$$v_1(t, u(t)) = f_1(t, u(f_2(t, \dots, u(f_m(t, u(t))) \dots)))$$

and $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators. The functions g and f_i satisfy appropriate conditions. I_k $(k = 1, 2, \dots, m)$ are bounded functions for $0 = t_0 < t_1 <, \dots, < t_m < t_{m+1} = T_0$, and $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, $u(t_k^-)$ and $u(t_k^+)$ represent the left and right hand limits of $u(t)$ at $t = t_k$, respectively. For more details, we refer the reader to [19].

With strong motivation from [19, 27, 28, 29], we establish the sufficient condition for the existence as well as uniqueness of a solution for a quasi-linear functional differential equation with iterated deviating argument in a Banach space.

We organize this article as follows. The preliminaries and assumptions on the functions f, h_j and the operator $A_0 \equiv A(0, u_0)$ are provided in Section 2. The local existence and uniqueness of a solution to Problem (1.1) have established in Section 3. The application of the main results are illustrated by an example at the end.

2. Preliminaries and assumptions

In this section, we recall some basic facts, lemmas and theorems that are useful in the remaining sections. We make assumptions on the functions f, h_j and the operator $A_0 \equiv A(0, u_0)$ that are required for the proof of the main results. The material covered in this section can be found, in more detail, in Friedman [5] and Tanabe [31].

Let $\mathcal{L}(X)$ denote the Banach algebra of all bounded linear operators on X. For $T \in [0, \infty)$, let $\{A(t) : 0 \leq t \leq T\}$ be a family of closed linear operators on the Banach space X such that

(B1) the domain $D(A)$ of $A(t)$ is dense in X and independent of t;

(B2) the resolvent $R(\lambda; A(t))$ exists for all $\operatorname{Re}\lambda \le 0$, for each $t \in [0,T]$, and

$$\|R(\lambda; A(t))\| \le \frac{C}{|\lambda|+1}, \operatorname{Re}\lambda \le 0, \quad t \in [0,T]$$

for some constant $C > 0$ (independent of t and λ);

(B3) there are constants $C > 0$ and $\rho \in (0,1]$ with

$$\|[A(t) - A(\tau)]A^{-1}(s)\| \le C|t-\tau|^{\rho},$$

for $t, s, \tau \in [0,T]$, where C and ρ are independent of t, τ and s.

Note taht assumption (B2) implies that for each $s \in [0,T]$, $-A(s)$ generates a strongly continuous analytic semigroup $\{e^{-tA(s)} : t \ge 0\}$ in $\mathcal{L}(X)$. Also the assumption (B3) implies that there exists a constant $C > 0$ such that

$$\|A(t)A^{-1}(s)\| \le C, \tag{2.1}$$

for all $0 \le s, t \le T$. Hence, for each t, the functional $y \to \|A(t)y\|$ defines an equivalent norm on $D(A) = D(A(0))$ and the mapping $t \to A(t)$ from $[0,T]$ into $\mathcal{L}(X_1, X)$ is uniformly Hölder continuous.

The following theorem will give the existence of a solution to the homogeneous Cauchy problem

$$\frac{du}{dt} + A(t)u = 0; \quad u(t^*) = u_0, \quad t > t^* \ge 0\,. \tag{2.2}$$

Theorem 2.1 ([5, Lemma II. 6.1]). *Let the assumptions* (B1)–(B3) *hold. Then there exists a unique fundamental solution* $\{U(t,s) : 0 \le s \le t \le T\}$ *to* (2.2).

Let $C^{\beta}([t^*,T]; X)$ denote the space of all X-valued functions $h(t)$, that are uniformly Hölder continuous on $[t^*,T]$ with exponent β, where $0 < \beta \le 1$. Then the solution to the problem

$$\frac{du}{dt} + A(t)u = h(t), \quad u(t^*) = u_0,\ t > t^* \ge 0 \tag{2.3}$$

is given by the following theorem.

Theorem 2.2 ([5, Theorem II. 3.1]). *Let the assumptions* (B1)–(B3) *hold. If* $h \in C^{\beta}([t^*,T]; X)$ *and* $u_0 \in X$*, then there exists a unique solution of* (2.3) *and the solution*

$$u(t) = U(t,t^*)u_0 + \int_{t^*}^{t} U(t,s)h(s)ds, \quad t^* \le t \le T$$

is continuously differentiable on $(t^*,T]$.

We need the following assumption to establish the existence of a local solution to (1.1):

(H1) The operator $A_0 = A(0,u_0)$ is a closed linear with domain D_0 dense in X and the resolvent $R(\lambda; A_0)$ exists for all Re $\lambda \le 0$ such that

$$\|(\lambda I - A_0)^{-1}\| \le \frac{C}{1+|\lambda|} \quad \text{for all } \lambda \text{ with } \operatorname{Re}\lambda \le 0 \tag{2.4}$$

for some positive constant $C($ independent of $\lambda)$.

Then the negative fractional powers of the operator A_0 is well defined. For $\alpha > 0$, we define

$$A_0^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-tA_0} t^{\alpha-1} dt.$$

Then $A_0^{-\alpha}$ is a bijective and bounded linear operator on X. We define the positive fractional powers of A_0 by $A_0^\alpha \equiv [A_0^{-\alpha}]^{-1}$. Then A_0^α is closed linear operator and domain $D(A_0^\alpha)$ is dense in X. If $\varsigma > \upsilon$, then $D(A_0^\varsigma) \subset D(A_0^\upsilon)$. For $0 < \alpha \le 1$, we denote $X_\alpha = D(A_0^\alpha)$. Then $(X_\alpha, \|\cdot\|)$ is a Banach space equipped with the graph norm

$$\|x\|_\alpha = \|A_0^\alpha x\|.$$

If $0 < \alpha \le 1$, the embeddings $X_\alpha \hookrightarrow X$ are dense and continuous.

For each $\alpha > 0$, we define $X_{-\alpha} = (X_\alpha)^*$, the dual space of X_α, endowed with the natural norm

$$\|x\|_{-\alpha} = \|A_0^{-\alpha} x\|.$$

Then $(X_{-\alpha}, \|\cdot\|_{-\alpha})$ is a Banach space. The following assumptions are necessary for proving the main result.

Let $R, R_1 > 0$ and $B_\alpha = \{x \in X_\alpha : \|x\|_\alpha < R\}$, $B_{\alpha-1} = \{y \in X_{\alpha-1} : \|y\|_{\alpha-1} < R_1\}$.

(H2) The operator $A(t,x)$ is well defined on D_0 for all $t \in [0,T]$, for some $\alpha \in [0,1)$ and for any $x \in B_\alpha$. Furthermore, $A(t,x)$ satisfies

$$\|[A(t,x) - A(s,y)]A^{-1}(s,y)\| \le C(R)[|t-s|^\theta + \|x-y\|_\alpha^\gamma] \tag{2.5}$$

for some constants $\theta, \gamma \in (0,1]$ and $C(R) > 0$, for any $t, s \in [0,T]$ and $x, y \in B_\alpha$.

(H3) For $\alpha \in (0,1)$, there exist constants $C_f \equiv C_f(t,R,R_1) > 0$ and $0 < \theta, \gamma, \rho \le 1$ such that the non-linear map $f : [0,T] \times B_\alpha \times B_{\alpha-1} \to X$ satisfies

$$\|f(t,x,x_1) - f(s,y,y_1)\| \le C_f(|t-s|^\theta + \|x-y\|_\alpha^\gamma + \|x_1-y_1\|_{\alpha-1}^\rho) \tag{2.6}$$

for every $t, s \in [0,T]$, $x, y \in B_\alpha$, $x_1, y_1 \in B_{\alpha-1}$.

(H4) For some $\alpha \in [0,1)$, there exist constants $C_{h_j} \equiv C_h(t,R) > 0$ and $0 < \theta_i \le 1$, such that $h_j : B_{\alpha-1} \times [0,T] \to [0,T]$ satisfies $h_j(\cdot,0) = 0$,

$$|h_j(t,x) - h_j(s,y)| \le C_{h_j}(|t-s|^{\theta_i} + \|x-y\|_{\alpha-1}^\gamma), \tag{2.7}$$

for all $x, y \in B_\alpha$, for all $s, t \in [0,T]$ and $j = 1, 2, \dots, m$.

(H5) Let $u_0 \in X_\beta$ for some $\beta > \alpha$, and

$$\|u_0\|_\alpha < R. \tag{2.8}$$

(H6) The operator A_0^{-1} is completely continuous operator.

Remark 2.3. We note that the assumptions (H1) and (H6) imply that $A^{-\nu}$ is completely continuous for any $0 < \nu \le 1$. Indeed, we define

$$A_{0,\vartheta}^{-\nu} = \frac{1}{\Gamma(\alpha)} \int_\vartheta^\infty e^{-tA_0} t^{\nu-1} dt.$$

We write $A_{0,\vartheta}^{-\nu} = A_0^{-1}(A_0 A_{0,\vartheta}^{-\nu})$. As $A_0 A_{0,\vartheta}^{-\nu}$ is a bounded operator for each $\vartheta > 0$, so $A_{0,\vartheta}^{-\nu}$ is completely continuous. Also $\|A_{0,\vartheta}^{-\nu} - A_0^{-\nu}\| \to 0$ as $\vartheta \to 0$. Thus $A_0^{-\nu}$ is completely continuous.

Let us state the following Lemmas that will be used in the subsequent sections. Let $C^\beta([t^*,T];X)$ denote the space of all Hölder continuous function from $[t^*,T]$ into X.

Lemma 2.4 ([6, Lemma 1.1])**.** *If $g \in C^\beta([t^*,T];X)$, then $H : C^\beta([t^*,T];X) \to C([t^*,T];X_1)$ by*

$$Hg(t) = \int_{t^*}^t U(t,s)g(s)ds, \quad t^* \le t \le T$$

is a bounded mapping and $\|Hg\|_{C([t^*,T];X_1)} \le C\|g\|_{C^\beta([t^*,T];X)}$, for some $C > 0$.

As a consequence of Lemma 2.4, we obtain

Corollary 2.5. *For $v \in X_1$, define*

$$R(v;g) = U(t,0)v + \int_0^t U(t,s)g(s)ds,\ 0 \le t \le T.$$

Then R is a bounded linear mapping from $X_1 \times C^\beta([0,T];X)$ into $C([0,T];X_1)$.

3. Main results

We establish the existence and uniqueness of a local solution to Problem (1.1). Let I denote the interval $[0,\delta]$ for some positive number δ to be specified later. For $0 \le \alpha \le 1$, let $\mathcal{C}_\alpha = \{u | u : I \to X_\alpha \text{ is continuous}\}$ Then $(\mathcal{C}_\alpha, \|\cdot\|_\infty)$ is a Banach space, where $\|\cdot\|_\infty$ is defined as

$$\|u\|_\infty = \sup_{t\in I} \|u(t)\|_\alpha \text{ for } u \in C(I;X_\alpha).$$

Let

$$Y_\alpha \equiv C_{L_\alpha}(I;X_{\alpha-1}) = \big\{y \in \mathcal{C}_\alpha : \|y(t)-y(s)\|_{\alpha-1} \le L_\alpha|t-s| \text{ for all } t,s \in I\big\}$$

for some positive constant L_α to be specified later. Then Y_α is a Banach space endowed with the supremum norm of $\mathcal{C}_\alpha$.

Definition 3.1. A function $u : I \to X$ is said to be a solution to Problem (1.1) if u satisfies the following:

(i) $u(\cdot) \in C_{L_\alpha}(I;X_{\alpha-1}) \cap C^1((0,\delta);X) \cap C(I;X)$;
(ii) $u(t) \in X_1$ for all $t \in (0,\delta)$;
(iii) $\frac{du}{dt} + A(t,u(t))u(t) = f(t,u(t),u(w_1(u(t),t)))$ for all $t \in (0,\delta)$;
(iv) $u(0) = u_0$.

We choose $R > 0$ small enough such that the assumptions (H2)–(H5) hold. Let $K > 0$ and $0 < \eta < \beta - \alpha$ be fixed constants, where $0 < \alpha < \beta \le 1$. Define

$$\mathcal{W}(\delta,\mathcal{K},\eta) = \big\{x \in \mathcal{C}_\alpha \cap Y_\alpha : x(0) = u_0, \|x(t)-x(s)\|_\alpha \le K|t-s|^\eta \text{ for all } t,s \in I\big\}.$$

Then $\mathcal{W}$ is a non-empty, closed, bounded and convex subset of $\mathcal{C}_\alpha$. We prove the following theorem for the local existence and uniqueness of a solution to Problem (1.1). The proof is based on the ideas of Haloi et al [8] and Sobolevskiĭ [26].

Theorem 3.2. *Let $u_0 \in X_\beta$, where $0 < \alpha < \beta \le 1$. Let the assumptions (H1)–(H6) hold. Then there exist a solution $u(t)$ to Problem* (1.1) *in $I = [0,\delta]$ for some positive number $\delta \equiv \delta(\alpha,u_0)$ such that $u \in \mathcal{W} \cap C^1((0,\delta);X)$.*

Proof. Let $x \in \mathcal{W}$. It follows from (H5) that

$$\|x(t)\|_\alpha < R \quad \text{for } t \in I \tag{3.1}$$

for sufficiently small $\delta > 0$. By assumption (H2), the operator

$$A_x(t) = A(t, x(t))$$

is well defined for each $t \in I$. Also it follows from assumption (H2) and inequality (2.1) that

$$\|[A_x(t) - A_x(s)]A_0^{-1}\| \le C|t-s|^\mu \quad \text{for } \mu = \min\{\theta, \gamma\eta\}, \tag{3.2}$$

for $C > 0$ is a constant independent of δ and $x \in \mathcal{W}$. We note that $A_x(0) = A_0$. Assumption (H1) and inequality (3.2) imply that

$$\|(\lambda I - A_x(t))^{-1}\| \le \frac{C}{1+|\lambda|} \quad \text{for } \lambda \text{ with } \operatorname{Re}\lambda \le 0,\ t \in I, \tag{3.3}$$

sufficiently small $\delta > 0$. Again from assumption (H3), it follows that

$$\|[A_x(t) - A_x(s)]A_x^{-1}(\tau)\| \le C|t-s|^\mu \quad \text{for } t, \tau, s \in I. \tag{3.4}$$

It follows from the assumption (H2), (3.3) and (3.4) that the operator $A_x(t)$ satisfies the assumptions (B1)–(B3). Thus there exists a unique fundamental solution $U_x(t,s)$ for $s, t \in I$, corresponding to the operator $A_x(t)$ (see Theorem 2.1). For $x \in \mathcal{W}$, we put $f_x(t) = f(t, x(t), x(w_1(t, x(t))))$. Then the assumptions (H3) and (H4) imply that f_x is Hölder continuous on I of exponent $\gamma = \min\{\theta, \gamma\eta, \theta_1\rho, \theta_2\eta\gamma\rho, \theta_3\eta^2\gamma^2\rho, \dots, \theta_m\eta^{m-1}\gamma^{m-1}\rho\}$. By Theorem 2.1, there exists a unique solution ψ_x to the problem

$$\begin{gathered} \frac{d\psi_x(t)}{dt} + A_x(t)\psi_x(t) = f_x(t), \quad t \in I; \\ \psi(0) = u_0 \end{gathered} \tag{3.5}$$

and given by

$$\psi_x(t) = U_x(t,0)u_0 + \int_0^t U_x(t,s) f_x(s)ds, \quad t \in I.$$

Now for each $x \in \mathcal{W}$, we define a map Q by

$$Qx(t) = U_x(t,0)u_0 + \int_0^t U_x(t,s) f_x(s)ds \quad \text{for each } t \in I.$$

By Lemma 2.4 and the assumption (H5), the map Q is well defined and $Q : \mathcal{W} \to \mathcal{C}_\alpha$. We show that Q maps from $\mathcal{W}$ into $\mathcal{W}$ for sufficiently small $\delta > 0$. Indeed, if $t_1, t_2 \in I$ with $t_2 > t_1$, then we have

$$\begin{aligned} &\|Qx(t_2) - Qx(t_1)\|_{\alpha-1} \\ &\le \|[U_x(t_2,0) - U_x(t_1,0)]u_0\|_{\alpha-1} \\ &\quad + \Big\| \int_0^{t_2} U_x(t_2,s) f_x(s)ds - \int_0^{t_1} U_x(t_1,s) f_x(s)ds \Big\|_{\alpha-1}. \end{aligned} \tag{3.6}$$

Using the bounded inclusion $X \to X_{\alpha-1}$, we estimate the first term on the right hand side of (3.6) as (cf. [5, Lemma II. 14.1]),

$$\big\|\big(U_x(t_2,0) - U_x(t_1,0)\big)u_0\big\|_{\alpha-1} \le C_1\|u_0\|_\alpha (t_2 - t_1), \tag{3.7}$$

where C_1 is some positive constant. Using [5, Lemma 14.4], we obtain the estimate for the second term on the right hand side of (3.6) as

$$\begin{aligned} &\|\int_0^{t_2} U_x(t_2,s)f_x(s)ds - \int_0^{t_1} U_x(t_1,s)f_x(s)ds\|_{\alpha-1} \\ &\le C_2 M_f(t_2-t_1)(|\log(t_2-t_1)|+1), \end{aligned} \tag{3.8}$$

where $M_f = \sup_{s\in[0,T]} \|f_x(s)\|$ and C_2 is some positive constant.

Thus from (3.7) and (3.8), we obtain

$$\|Qx(t_2) - Qx(t_1)\|_{\alpha-1} \le L_\alpha |t_2 - t_1|, \tag{3.9}$$

where $L_\alpha = \max\{C_1\|u_0\|_\alpha, C_2 M_f(|\log(t_2-t_1)|+1)\}$ that depends on C_1, C_2, M_f, δ. Finally, we show that

$$\|Qx(t+\Delta t) - Qx(t)\|_\alpha \le K_1(\Delta t)^\eta$$

for some constants $K_1 > 0$, $0 < \eta < 1$ and for $t \in [0,\delta]$. For $0 \le \alpha < \beta \le 1$, $0 \le t \le t + \Delta t \le \delta$, we have

$$\begin{aligned} &\|Qx(t+\Delta t) - Qx(t)\|_\alpha \\ &\le \|\Big[U_x(t+\Delta t,0) - U_x(t,0)\Big]u_0\|_\alpha \\ &\quad + \|\int_0^{t+\Delta t} U_x(t+\Delta t,s)f_x(s)ds - \int_0^t U_x(t,s)f_x(s)ds\|_\alpha. \end{aligned} \tag{3.10}$$

Using [5, Lemma II. 14.1] and [5, Lemma II. 14.4], we obtain the following two estimates

$$\|[U_x(t+\Delta t,0) - U_x(t,0)]u_0\|_\alpha \le C(\alpha,u_0)(\Delta t)^{\beta-\alpha}; \tag{3.11}$$

$$\begin{aligned} &\Big\|\int_0^{t+\Delta t} U_x(t+\Delta t,s)f_x(s)ds - \int_0^t U_x(t,s)f_x(s)ds\Big\|_\alpha \\ &\le C(\alpha)M_f(\Delta t)^{1-\alpha}(1+|\log\Delta t|). \end{aligned} \tag{3.12}$$

Using (3.11) and (3.12) in (3.10), we obtain

$$\begin{aligned} &\|Qx(t+\Delta t) - Qx(t)\|_\alpha \\ &\le (\Delta t)^\eta\Big[C(\alpha,u_0)\delta^{\beta-\alpha-\eta} + C(\alpha)M_f\delta^\nu(\Delta t)^{1-\alpha-\eta-\nu}(|\log\Delta t|+1)\Big] \end{aligned}$$

for any $\nu > 0, \nu < 1-\alpha-\eta$. Thus for sufficiently small $\delta > 0$, we have

$$\|Qx(t+\Delta t) - Qx(t)\|_\alpha \le K_1(\Delta t)^\eta \text{ for all } t \in [0,\delta],$$

some positive constant K_1. Thus Q maps $\mathcal{W}$ into $\mathcal{W}$.

We show that Q is continuous in $\mathcal{W}$. Let $t \in [0,\delta]$. Let $x_1, x_2 \in \mathcal{W}$. We put $\phi_1(t) = \psi_{x_1}(t)$ and $\phi_2(t) = \psi_{x_2}(t)$. Then for $j = 1,2$, we have

$$\begin{gathered} \frac{d\phi_j(t)}{dt} + A_{x_j}(t)\phi_j(t) = f_{x_j}(t), \quad t \in (0,\delta]; \\ \phi_j(0) = u_0. \end{gathered} \tag{3.13}$$

Then from (3.13), we have

$$\frac{d(\phi_1-\phi_2)(t)}{dt} + A_{x_1}(t)(x_1-x_2)(t) = [A_{x_2}(t) - A_{x_1}(t)]\phi_2(t) + [f_{x_1}(t) - f_{x_2}(t)]$$

for $t \in (0, \delta]$. We note that $A_0 x_2(t)$ is uniformly Hölder continuous for $\tau \le t \le \delta$ and for $\tau > 0$ which is followed form [5, Lemma II. 14.3] and [5, Lemma II.14.5]. Again Lemma 2.4 implies that

$$\|A_0 \int_0^t U_{x_2}(t,s) f_{x_2}(s) ds\| \le C_3$$

for some positive constant C_3. Thus we have the bound

$$\|A_0 \phi_2(t)\| \le C_4 t^{\beta-1} \tag{3.14}$$

for some positive constant C_4 and $t \in (0, \delta]$. Further, in view of (2.1) and (3.4), the operator $[A_{x_2}(t) - A_{x_1}(t)]A_0^{-1}$ is uniformly Hölder continuous for $\tau \le t \le \delta$ and $\tau > 0$. Hence, $[A_{x_2}(t) - A_{x_1}(t)]\phi_2(t)$ is uniformly Hölder continuous for $\tau \le t \le \delta$ and for $\tau > 0$. By Theorem 2.1, we obtain that for any $\tau \le t \le \delta$ and $\tau > 0$,

$$\begin{aligned} &\phi_1(t) - \phi_2(t) \\ &= U_{x_1}(t,\tau)[\phi_1(\tau) - \phi_2(\tau)] \\ &\quad + \int_\tau^t U_{x_1}(t,s)\Big\{[A_{x_2}(s) - A_{x_1}(s)]\phi_2(s) + [f_{x_1}(s) - f_{x_2}(s)]\Big\} ds. \end{aligned} \tag{3.15}$$

Using the bound (3.14), we take the limit as $\tau \to 0$ in (3.15), and passing to the limit, we obtain

$$\phi_1(t) - \phi_2(t) = \int_0^t U_{x_1}(t,s)\Big\{[A_{x_2}(s) - A_{x_1}(s)]\phi_2(t) + [f_{x_1}(s) - f_{x_2}(s)]\Big\} ds.$$

Using (2.5), (2.6), (2.7) and [5, inequlity II. 14.12], we obtain

$$\begin{aligned} \|Qx_1(t) - Qx_2(t)\|_\alpha &\le CC(R) \int_0^t (t-s)^{-\alpha} \|x_1(s) - x_2(s)\|_\alpha^\gamma s^{\beta-1} ds \\ &\quad + CC_f \int_0^t (t-s)^{-\alpha} \Big\{ \|x_1(s) - x_2(s)\|_\alpha^\gamma \\ &\quad + \|x_1(w_1(x_1(s),s)) - x_2(w_1(x_2(s),s))\|_{\alpha-1}^\rho \Big\} ds, \end{aligned} \tag{3.16}$$

where C is some positive constant. Now using the bounded inclusion $X_\alpha \to X_{\alpha-1}$, inequalities (2.6) and (2.7), we obtain

$$\begin{aligned} &\|x_1(w_1(x_1(s),s)) - x_2(w_1(x_2(s),s))\|_{\alpha-1}^\rho \\ &= \|x_1(h_1(t, x_1(h_2(t, \dots, x_1(h_m(t, x_1(t))) \dots)))) \\ &\quad - x_2(h_1(t, x_2(h_2(t, \dots, x_2(h_m(t, x_2(t))) \dots))))\|_{\alpha-1}^\rho \\ &\le \|x_1(h_1(t, x_1(h_2(t, \dots, x_1(h_m(t, x_1(t))) \dots)))) \\ &\quad - x_1(h_1(t, x_2(h_2(t, \dots, x_2(h_m(t, x_2(t))) \dots))))\|_{\alpha-1}^\rho \\ &\quad + \|x_1(h_1(t, x_2(h_2(t, \dots, x_2(h_m(t, x_2(t))) \dots)))) \\ &\quad - x_2(h_1(t, x_2(h_2(t, \dots, x_2(h_m(t, x_2(t))) \dots))))\|_{\alpha-1}^\rho \\ &\le L_\alpha^\rho |h_1(t, x_1(h_2(t, \dots, x_1(h_m(t, x_1(t))) \dots))) \\ &\quad - h_1(t, x_2(h_2(t, \dots, x_2(h_m(t, x_2(t))) \dots)))|^\rho + \|x_1 - x_2\|_\alpha^\rho \\ &\le L_\alpha^\rho C_{h_1}^\rho \|x_1(h_2(t, \dots, x_1(h_m(t, x_1(t))) \dots)) \\ &\quad - x_2(h_2(t, \dots, x_2(h_m(t, x_2(t))) \dots))\|_{\alpha-1}^{\gamma\rho}] + \|x_1 - x_2\|_\alpha^\rho \end{aligned}$$

$$
\begin{aligned}
&\le L_\alpha^\rho C_{h_1}^\rho \Big[L_\alpha^\rho |h_2(t,\dots,x_1(h_m(t,x_1(t)))\dots) \\
&\quad - h_2(t,\dots,x_2(h_m(t,x_2(t)))\dots)|^{\gamma\rho} + \|x_1-x_2\|_\alpha^{\gamma\rho}\Big] + \|x_1-x_2\|_\alpha^\rho \\
&\quad \dots \\
&\le \Big[1 + L_\alpha^\rho C_{h_1}^\rho + (L_\alpha^\rho)^2 C_{h_1}^\rho C_{h_2}^\rho + \dots + (L_\alpha^\rho)^m C_{h_1}^\rho \dots C_{h_m}^\rho\Big] \|x_1-x_2\|_\alpha^\kappa \\
&= \widetilde{C}\|x_1-x_2\|_\alpha^\kappa,
\end{aligned}
\tag{3.17}
$$

where $\kappa = \min\{\rho, \gamma\rho, \gamma^2\rho, \dots, \gamma^{m-1}\rho\}$ and

$$
\widetilde{C} = 1 + L_\alpha^\rho C_{h_1}^\rho + (L_\alpha^\rho)^2 C_{h_1}^\rho C_{h_2}^\rho + \dots + (L_\alpha^\rho)^m C_{h_1}^\rho \dots C_{h_m}^\rho.
$$

Using (3.17) in (3.16), we obtain

$$
\begin{aligned}
\|Qx_1(t) - Qx_2(t)\|_\alpha &\le CC(R)\int_0^t (t-s)^{-\alpha}\|x_1(s)-x_2(s)\|_\alpha^\gamma s^{\beta-1} ds \\
&\quad + (1+\widetilde{C})CC_f \frac{\delta^{1-\alpha}}{1-\alpha} \sup_{t\in[0,\delta]} \|x_1(t)-x_2(t)\|_\alpha^\mu \\
&\le \widetilde{K}\delta^{\beta-\alpha} \sup_{t\in[0,\delta]} \|x_1(t)-x_2(t)\|_\alpha^\mu,
\end{aligned}
$$

where $\mu = \min\{\gamma,\kappa\}$ and $\widetilde{K} = \max\left\{\frac{CC(R)}{1-\alpha}, \frac{(1+\widetilde{C})CC_f}{1-\alpha}\right\}$. Thus

$$
\sup_{t\in[0,\delta]} \|Qx_1(t) - Qx_2(t)\|_\alpha \le \widetilde{K}\delta^{\beta-\alpha} \sup_{t\in[0,\delta]} \|x_1(t)-x_2(t)\|_\alpha^\mu. \tag{3.18}
$$

This shows that the operator Q is continuous in $\mathcal{W}(\delta,K,\eta)$. Again it follows from inequality (3.1) that the functions $x(t)$ in $\mathcal{W}(\delta,K,\eta)$ is uniformly bounded and is equicontinuous (by the definition of $\mathcal{W}(\delta,K,\eta)$). If we can show that the set $\{\psi_x(t) : x \in W(\delta,K,\eta)\}$ for each $t\in[0,\delta]$, is contained in a compact subset of $\mathcal{C}_\alpha$, then the image of $\mathcal{W}(\delta,K,\eta)$ under Q is contained in a compact subset of Y_α which follows from the Ascoli-Arzela theorem.

For each $t \in [0,\delta]$, we have

$$
\psi_x(t) = A_0^{-\nu} A_0^\nu \psi_x(t), \quad \text{for } 0 < \nu < \beta - \alpha.
$$

As $\{A_0^\nu \psi_x(t) : x \in \mathcal{W}(\delta,K,\eta)\}$ is a bounded set and $A_0^{-\nu}$ is completely continuous, so $\{\psi_x(t) : x \in W(\delta,K,\eta)\}$ for each $t\in[0,\delta]$, is contained in a compact subset of $\mathcal{C}_\alpha$.

Thus by the Schauder fixed point theorem, Q has a fixed point x in $\mathcal{W}(\delta,K,\eta)$; that is,

$$
x(t) = U_x(t,0)u_0 + \int_0^t U_x(t,s)f_x(s)ds \quad \text{for each } t \in I.
$$

It is clear from Theorem 2.2 that $x \in C^1((0,\delta);X)$. Thus x is a solution to problem (1.1) on I. □

The solution to Problem (1.1) is unique with stronger assumptions. We outline the proof of the following theorem that gives the uniqueness of the solution. For more details, we refer to Haloi et al [8].

Theorem 3.3. *Let $u_0 \in X_\beta$, where $0 < \alpha < \beta \le 1$. Let the assumptions* (H1)–(H5) *hold with $\rho = 1$ and $\gamma = 1$. Then there exist a positive number $\delta \equiv \delta(\alpha, u_0)$ and a unique solution $u(t)$ to Problem* (1.1) *in $[0,\delta]$ such that $u \in \mathcal{W} \cap C^1((0,\delta); X)$.*

Proof. We define

$$\mathcal{W}(\delta, \mathcal{K}, \eta) = \Big\{ y \in \mathcal{C}_\alpha \cap Y_\alpha : y(0) = u_0, \|y(t) - y(s)\|_\alpha \le K|t-s|^\eta \text{ for all } t, s \in [0,\delta] \Big\}.$$

For $v \in \mathcal{W}$ and $[0,\delta]$, we set $w_v(t) = Qv(t)$, where $w_v(t)$ is the solution to the problem

$$\begin{gathered} \frac{dw_v(t)}{dt} + A_v(t) w_v(t) = f_v(t), \quad t \in [0,\delta]; \\ w(0) = u_0. \end{gathered} \tag{3.19}$$

That is, $Qv(t)$ is given by

$$Qv(t) = w_v(t) = U_v(t,0)u_0 + \int_0^t U_v(t,s) f_v(s) ds, \quad t \in [0,\delta]. \tag{3.20}$$

We choose $\delta > 0$ such that $\widetilde{K}\delta^{\beta-\alpha} < 1/2$, where

$$\widetilde{K} = \max\Big\{ \frac{CC(R)}{1-\alpha}, \frac{(1+\widetilde{C})CC_f}{1-\alpha} \Big\}$$

for some positive constant C. Then it follows from (3.18) that the map Q defined by (3.20) is contraction on $\mathcal{W}$. Thus by the Banach fixed point theorem Q has unique fixed point in $\mathcal{W}$. $\square$

Remark 3.4. The value of δ in Theorem 3.2 and Theorem 3.3 depends on the constants C in (2.4), R, $\|u_0\|_\beta$ and $R - \|u\|_\alpha$ for $0 < \alpha < \beta \le 1$. So, any solution $u(t)$ on $[0,\delta]$ is global solution to Problem (1.1), it is sufficient to show $[A(t, u(t))]$ satisfies the a priori bound

$$\|[A(t,u(t))]^\beta u(t)\| \le D$$

for any $t \in [0,T]$ and for some positive constant D independent of t.

4. Application

Let $X = L^2(\Omega)$, where Ω is a bounded domain with smooth boundary in $\mathbb{R}^n$. For $T \in [0,\infty)$, we define

$$\Omega_T = \big\{ (t,x,y,z) : x \in \Omega, 0 < t < T, y, z \in X \big\}.$$

We consider the following quasi-linear initial value problem in X [5, 8],

$$\begin{gathered} \frac{\partial w(t,x)}{\partial t} + \sum_{|\beta| \le 2m} a_\beta(t,x,w,Dw) D^\beta w(t,x) \\ = f(t,x,w(t,x), w(h_1(w(t,x),t)),x), \quad t > 0,\ x \in \Omega, \\ D^\beta w(t,x) = 0, \quad |\beta| \le m, \quad 0 \le t \le T, \quad x \in \partial\Omega, \\ w(0,x) = w_0(x), \quad x \in \Omega, \end{gathered} \tag{4.1}$$

where

$$\begin{aligned} &f(t,x,w(t,x), w(h_1(w(t,x),t)),x) \\ &= \int_\Omega b(y,x) w\Big(y, \phi_1(t)\big|u\big(x, \phi_2(t)|u(x, \dots \phi_m(t)|u(t,x)|)|\big)\big|\Big) dy \quad \forall (t,x) \in \Omega_T, \end{aligned}$$

$\phi_j : \mathbb{R}_+ \to \mathbb{R}_+$, $j = 1, 2, 3, \dots, m$ are locally Hölder continuous with $\phi(0) = 0$, and $b \in C^1(\overline{\Omega} \times \overline{\Omega}; \mathbb{R})$. Here we assume the following two conditions [5]:

(i) $a_\beta(\cdot, \cdot, \cdot, \cdot)$ is a continuously differentiable real valued function in all variables for $|\beta| \le 2m$;

(ii) there exists constant $c > 0$ such that
$$(-1)^m \operatorname{Re} \Big\{ \sum_{|\beta|=2m} a_\beta(t, x, w, Dw)\zeta^\beta \Big\} \ge c|\zeta|^{2m} \tag{4.2}$$
for all $(t, x) \in \overline{\Omega}_T$ and $\zeta \in \mathbb{R}^n$.

We take $X_1 \equiv H^{2m}(\Omega) \cap H_0^m(\Omega)$, $X_{1/2} = H_0^m(\Omega)$, $X_{-1/2} = H^{-1}(\Omega)$ and define
$$A(t, u)u = \sum_{|\beta| \le 2m} a_\beta(t, x, u, Du)D^\beta u, \quad A_0 u = \sum_{|\beta| \le 2m} a_\beta(0, u_0, Du_0)D^\beta u,$$
where $u \in D(A_0)$ and
$$D^\beta u = \frac{\partial^{|\beta|} u}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \dots \partial x_n^{\beta_n}}$$
is the distributional derivative of u and β is a multi-index with $\beta = (\beta_1, \beta_2, \dots, \beta_n)$, $\beta_i \ge 0$ integers. It is clear from (4.2) that $-A(t)$ generates a strongly continuous analytic semi-group of bounded operators on $L^2(\Omega)$ and the assumptions (H1), (H2) are satisfied [5]. We define $u(t) = w(t, \cdot)$. Then (4.1) can be written as
$$\begin{gathered} \frac{du}{dt} + A(t, u(t))u(t) = f(t, u(t), u(w_1(t, u(t)))), \quad t > 0; \\ u(0) = u_0, \end{gathered} \tag{4.3}$$
where $w_1(t, u(t)) = h_1(t, u(h_2(t, \dots, u(h_m(t, u(t))) \dots)))$.

Let $\alpha = 1/2$ and $2m > n$. By Minkowski's integral inequality and imbedding theorem $H_0^m(\Omega) \subset C(\overline{\Omega})$, we obtain
$$\begin{aligned} \|f(x, \psi_1(x, \cdot)) - f(x, \psi_2(x, \cdot))\|_{L^2(\Omega)}^2 &\le \|b\|_\infty^2 \int_\Omega \int_\Omega |(\psi_1 - \psi_2)(y, \cdot)|^2 dx dy \\ &\le \|b\|_\infty^2 \int_\Omega |(\psi_1 - \psi_2)(y, \cdot)|^2 dy \\ &\le c\|b\|_\infty^2 \|\psi_1 - \psi_2\|_{H_0^m(\Omega)}^2 \end{aligned}$$
for a constant $c > 0$, for all $\psi_1, \psi_2 \in H_0^m(\Omega)$. This shows that f satisfies (2.6). We show that the functions $h_i : [0, T] \times H_0^m(\Omega) \to [0, T]$ defined by $h_i(t, \phi) = g_i(t)|\phi(x, \cdot)|$ for each $i = 1, 2, \dots, m$, satisfies the assumption (2.7). Let $t \in [0, T]$. Then using the embedding $H_0^m(\Omega) \subset C(\overline{\Omega})$, we obtain
$$\begin{aligned} |h_i(t, \chi)| &= |\phi_i(t)| \ |\chi(x, \cdot)| \\ &\le ||\phi_i||_\infty ||\chi||_{L^\infty(0,1)} \\ &\le C||\chi||_{H_0^m(\Omega)}, \end{aligned}$$
where C is a constant depending on bounds of ϕ_i. Let $t_1, t_2 \in [0, T]$ and $\chi_1, \chi_2 \in H_0^m(\Omega)$. Using the Hölder continuity of ϕ and the imbedding theorem $H_0^m(\Omega) \subset C(\overline{\Omega})$, we have
$$\begin{aligned} |h_i(t, \chi_1) - h_i(t, \chi_2)| &\le |\phi_i(t)|(|\chi_1(x, \cdot)| - |\chi_2(x, \cdot)|) + |(\phi_i(t) - \phi_i(s))||\chi_2(x, \cdot)| \\ &\le ||\phi_i||_\infty ||\chi_1 - \chi_2||_{L^\infty(0,1)} + L_{\phi_i}|t - s|^\theta ||\chi_2||_{L^\infty(0,1)} \end{aligned}$$

$$\leq C||\phi_i||_\infty ||\chi_1 - \chi_2||_{H_0^m(\Omega)} + L_{\phi_i}|t-s|^\theta ||\chi_2||_{H_0^m(\Omega)}$$
$$\leq \max\{C||\phi_i||_\infty, L_{\phi_i}||\chi_2||_\infty\}(||\chi_1 - \chi_2||_{H_0^m(\Omega)} + |t-s|^\theta).$$

Thus (2.7) is satisfied. We have the following theorem.

Theorem 4.1. *Let $\beta > 1/2$. If $u_0 \in X_\beta$, then Problem* (4.3) *has a unique solution in $L^2(\Omega)$.*

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