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LOCAL UNIQUENESS FOR SINGULARLY PERTURBED PERIODIC NONLINEAR TRACTION PROBLEMS

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Abstract. We present a limiting property and a local uniqueness result for converging families of solutions of a singularly perturbed nonlinear traction problem in an unbounded periodic domain with small holes.

1. Introduction

In this article, we use an argument based on functional analysis and potential theory to show a limiting property and a local uniqueness result for families of solutions of a singularly perturbed nonlinear traction problem in linearized elasticity. We fix once for all

$$n \in \mathbb{N} \setminus \{0,1\}\,, \quad q_{11}, \dots, q_{nn} \in]0,+\infty[\,, \quad Q \equiv \Pi_{j=1}^{n}]0, q_{jj}[\,.$$

Then we denote by q the $n \times n$ diagonal matrix with diagonal entries $q_{11}, \dots, q_{nn}$. We also assume that

$$\begin{aligned}&\alpha \in]0,1[\text{ and } \Omega^h \subseteq \mathbb{R}^n \text{ is bounded, open, connected, of class } C^{1,\alpha},\\ &\text{containing the origin 0, and with a connected exterior } \mathbb{R}^n \setminus \mathrm{cl}\, \Omega^h.\end{aligned}$$

Here cl denotes the closure and the letter 'h' stands for 'hole'. The set Ω^h will play the role of the shape of the perforation. Moreover, we fix

$$p \in Q \text{ and } \epsilon_0 \in]0,+\infty[\text{ such that } p + \epsilon\, \mathrm{cl}\, \Omega^h \subseteq Q \text{ for all } \epsilon \in]-\epsilon_0, \epsilon_0[\,.$$

To shorten our notation, we set

$$\Omega^h_{p,\epsilon} \equiv p + \epsilon \Omega^h$$

and we define the periodically perforated domain

$$\mathbb{S}[\Omega^h_{p,\epsilon}]^- \equiv \mathbb{R}^n \setminus \cup_{z \in \mathbb{Z}^n} \mathrm{cl}(\Omega^h_{p,\epsilon} + qz)$$

for all $\epsilon \in]-\epsilon_0, \epsilon_0[$. A function u defined on $\mathrm{cl}\, \mathbb{S}[\Omega^h_{p,\epsilon}]^-$ is said to be q-periodic if

$$u(x + qz) = u(x) \quad \forall x \in \mathrm{cl}\, \mathbb{S}[\Omega^h_{p,\epsilon}]^-\,, \quad \forall z \in \mathbb{Z}^n\,.$$

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We now introduce a nonlinear traction boundary value problem in $\mathbb{S}[\Omega^h_{p,\epsilon}]^-$. To do so, we denote by T the function from $]1-(2/n),+\infty[\times M_n(\mathbb{R})$ to $M_n(\mathbb{R})$ which takes the pair (ω, A) to

$$T(\omega, A) \equiv (\omega-1)(\operatorname{tr} A) I_n + (A + A^t)\,.$$

Here $M_n(\mathbb{R})$ denotes the space of $n\times n$ matrices with real entries, I_n denotes the $n\times n$ identity matrix, $\operatorname{tr} A$ and A^t denote the trace and the transpose matrix of A, respectively. We observe that $(\omega-1)$ plays the role of the ratio between the first and second Lamé constants and that the classical linearization of the Piola Kirchoff tensor equals the second Lamé constant times $T(\omega,\cdot)$ (cf., e. g., Kupradze et al [17]). We also note that

$$\operatorname{div} T(\omega, Du) = \Delta u + \omega \nabla \operatorname{div} u\,,$$

for all regular vector valued functions u. Now let G be a function from $\partial\Omega^h \times \mathbb{R}^n$ to $\mathbb{R}^n$, let $B \in M_n(\mathbb{R})$, and let $\epsilon \in]0,\epsilon_0[$. We introduce the nonlinear traction problem

$$\begin{gathered} \operatorname{div} T(\omega, Du) = 0 \quad \text{in } \mathbb{S}[\Omega^h_{p,\epsilon}]^-\,, \\ u(x+qe_j) = u(x) + Be_j \quad \forall x \in \operatorname{cl}\mathbb{S}[\Omega^h_{p,\epsilon}]^-, \forall j \in \{1,\dots,n\}, \\ T(\omega, Du(x))\nu_{\Omega^h_{p,\epsilon}}(x) = G\big((x-p)/\epsilon, u(x)\big) \quad \forall x \in \partial\Omega^h_{p,\epsilon}\,, \end{gathered} \tag{1.1}$$

where $\nu_{\Omega^h_{p,\epsilon}}$ denotes the outward unit normal to $\partial\Omega^h_{p,\epsilon}$ and $\{e_1,\dots,e_n\}$ denotes the canonical basis of $\mathbb{R}^n$. Because of the presence of a nonlinear term in the third equation of problem (1.1), we cannot claim in general the existence of a solution. However, we know by [12] that under suitable assumptions there exists $\epsilon_1 \in]0,\epsilon_0]$ such that the boundary value problem in (1.1) has a solution $u(\epsilon,\cdot)$ in $C^{1,\alpha}_{\mathrm{loc}}(\operatorname{cl}\mathbb{S}[\Omega^h_{p,\epsilon}]^-, \mathbb{R}^n)$ for all $\epsilon \in]0,\epsilon_1[$. Moreover, the family $\{u(\epsilon,\cdot)\}_{\epsilon\in]0,\epsilon_1[}$ is uniquely determined (for ϵ small) by its limiting behavior as ϵ tends to 0 and the dependence of $u(\epsilon,\cdot)$ upon the parameter ϵ can be described in terms of real analytic maps of ϵ defined in an open neighborhood of 0.

In this article, we study the limiting behavior and the local uniqueness of families of solutions of problem (1.1), under weaker assumptions than those in [12]. In particular, in Theorem 4.5, we show that if $\{\varepsilon_j\}_{j\in\mathbb{N}}$ is a sequence in $]0,\epsilon_0[$ converging to 0 and if $\{u_j\}_{j\in\mathbb{N}}$ is a family of functions such that u_j solves problem (1.1) for $\epsilon = \varepsilon_j$ and such that the restrictions to $\partial\Omega^h$ of the rescaled functions $u_j(p+\varepsilon_j\cdot)$ converge to a function v_* as j tends to $+\infty$, then v_* must be equal to a constant vector $\xi_* \in \mathbb{R}^n$ and u_j converges to $\xi_* + Bq^{-1}(\cdot - p)$ uniformly on bounded open subsets of $\mathbb{R}^n \setminus (p+q\mathbb{Z}^n)$. In Theorem 4.6, instead, we prove that, under suitable assumptions, if $\{\varepsilon_j\}_{j\in\mathbb{N}}$ is a sequence in $]0,\epsilon_0[$ converging to 0 and if $\{u_j\}_{j\in\mathbb{N}}$, $\{v_j\}_{j\in\mathbb{N}}$ are families of functions such that u_j and v_j solve problem (1.1) for $\epsilon=\varepsilon_j$ and such that the restrictions to $\partial\Omega^h$ of $u_j(p+\varepsilon_j\cdot)$ and of $v_j(p+\varepsilon_j\cdot)$ converge to the same function, then we must have $u_j = v_j$ for j big enough. We also note that the present article extends to the case of a nonlinear traction problem the results of [9], concerning a nonlinear Robin problem for the Laplace equation.

The functional analytic approach adopted in [12] and in the present paper for the investigation of the behavior of the solutions of problem (1.1) has been previously exploited by Lanza de Cristoforis and the authors to analyze singular perturbation problems for the Laplace operator in [10, 18], for the Lamé equations in [6, 7, 8], and for the Stokes system in [5]. Concerning problems in an infinite periodically perforated domain, we mention in particular [11, 12, 20, 24].

We note that singularly perturbed boundary value problems have been largely investigated with the methods of asymptotic analysis. As an example, we mention the works of Beretta et al [2], Bonnaillie-Noël et al [3], Iguernane et al [16], Maz'ya et al [21], Maz'ya et al [22], Nazarov et al [25], Nazarov and Sokolowski [26], and Vogelius and Volkov [27]. In particular, in connection with periodic problems, we mention, e. g., Ammari et al [1].

Moreover, for problems in periodic domains, we mention the method of functional equations and, for example, the works of Castro et al [4] and Drygas and Mityushev [13]

This article is organized as follows. Section 2 is a section of notation and preliminaries. In Section 3 we provide an integral formulation of problem (1.1). In Section 4 we prove our main results on the limiting behavior and the local uniqueness of a family of solutions of problem (1.1).

2. Notation and preliminaries

Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. We denote by $\mathcal{L}(\mathcal{X},\mathcal{Y})$ the space of linear and continuous maps from $\mathcal{X}$ to $\mathcal{Y}$, equipped with its usual norm of the uniform convergence on the unit sphere of $\mathcal{X}$. We denote by I the identity operator. The inverse function of an invertible function f is denoted $f^{(-1)}$, as opposed to the reciprocal of a real-valued function g, or the inverse of a matrix B, which are denoted g^{-1} and B^{-1}, respectively. If B is a matrix, then B_{ij} denotes the (i,j) entry of B. If $x \in \mathbb{R}^n$, then x_j denotes the j-th coordinate of x and $|x|$ denotes the Euclidean modulus of x. A dot '$\cdot$' denotes the inner product in $\mathbb{R}^n$. For all $R>0$ and all $x\in\mathbb{R}^n$ we denote by $\mathbb{B}_n(x,R)$ the ball $\{y\in\mathbb{R}^n : |x-y|<R\}$. Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$. Let $k\in\mathbb{N}$. The space of k times continuously differentiable real-valued functions on $\mathcal{O}$ is denoted by $C^k(\mathcal{O})$. Let $r\in\mathbb{N}\setminus\{0\}$. Let $f\equiv(f_1,\dots,f_r)\in\big(C^k(\mathcal{O})\big)^r$. Then Df denotes the Jacobian matrix $\big(\frac{\partial f_s}{\partial x_l}\big)_{(s,l)\in\{1,\dots,r\}\times\{1,\dots,n\}}$. Let $\eta\equiv(\eta_1,\dots,\eta_n)\in\mathbb{N}^n$, $|\eta|\equiv\eta_1+\dots+\eta_n$. Then $D^\eta f$ denotes $\frac{\partial^{|\eta|}f}{\partial x_1^{\eta_1}\dots\partial x_n^{\eta_n}}$. The subspace of $C^k(\mathcal{O})$ of those functions f whose derivatives $D^\eta f$ of order $|\eta|\le k$ can be extended with continuity to $\operatorname{cl}\mathcal{O}$ is denoted $C^k(\operatorname{cl}\mathcal{O})$. Let $\beta\in]0,1[$. The subspace of $C^k(\operatorname{cl}\mathcal{O})$ whose functions have k-th order derivatives that are uniformly Hölder continuous in $\operatorname{cl}\mathcal{O}$ with exponent β is denoted $C^{k,\beta}(\operatorname{cl}\mathcal{O})$ (cf., e.g., Gilbarg and Trudinger [14]). The subspace of $C^k(\operatorname{cl}\mathcal{O})$ of those functions f such that $f\big|_{\operatorname{cl}(\mathcal{O}\cap\mathbb{B}_n(0,R))}\in C^{k,\beta}(\operatorname{cl}(\mathcal{O}\cap\mathbb{B}_n(0,R)))$ for all $R\in]0,+\infty[$ is denoted $C^{k,\beta}_{\mathrm{loc}}(\operatorname{cl}\mathcal{O})$. Then $C^{k,\beta}(\operatorname{cl}\mathcal{O},\mathbb{R}^n)$ denotes $(C^{k,\beta}(\operatorname{cl}\mathcal{O}))^n$ and $C^{k,\beta}_{\mathrm{loc}}(\operatorname{cl}\mathcal{O},\mathbb{R}^n)$ denotes $(C^{k,\beta}_{\mathrm{loc}}(\operatorname{cl}\mathcal{O}))^n$. If $\mathcal{O}$ is a bounded open subset of $\mathbb{R}^n$, then $C^{k,\beta}(\operatorname{cl}\mathcal{O},\mathbb{R}^n)$ endowed with its usual norm is well known to be a Banach space. We say that a bounded open subset $\mathcal{O}$ of $\mathbb{R}^n$ is of class $C^{k,\beta}$, if its closure is a manifold with boundary imbedded in $\mathbb{R}^n$ of class $C^{k,\beta}$ (cf., e. g., Gilbarg and Trudinger [14, §6.2]). If $\mathcal{M}$ is a manifold imbedded in $\mathbb{R}^n$ of class $C^{k,\beta}$ with $k\ge1$, then one can define the Schauder spaces also on $\mathcal{M}$ by exploiting the local parametrization. In particular, if $\mathcal{O}$ is a bounded open set of class $C^{k,\beta}$ with $k\ge1$, then one can consider the space $C^{l,\beta}(\partial\mathcal{O},\mathbb{R}^n)$ with $l\in\{0,\dots,k\}$ and the trace operator from $C^{l,\beta}(\operatorname{cl}\mathcal{O},\mathbb{R}^n)$ to $C^{l,\beta}(\partial\mathcal{O},\mathbb{R}^n)$ is linear and continuous. If $\mathcal{S}_Q$ is an arbitrary subset of $\mathbb{R}^n$ such that $\operatorname{cl}\mathcal{S}_Q\subseteq Q$, then we define

$$\mathbb{S}[\mathcal{S}_Q]\equiv\cup_{z\in\mathbb{Z}^n}(qz+\mathcal{S}_Q)=q\mathbb{Z}^n+\mathcal{S}_Q\,,\quad \mathbb{S}[\mathcal{S}_Q]^-\equiv\mathbb{R}^n\setminus\operatorname{cl}\mathbb{S}[\mathcal{S}_Q]\,.$$

We note that if $\mathbb{R}^n\setminus\operatorname{cl}\mathcal{S}_Q$ is connected, then $\mathbb{S}[\mathcal{S}_Q]^-$ is also connected.

We now introduce some preliminaries of potential theory. We denote by S_n the function from $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}$ defined by

$$S_n(x) \equiv \begin{cases} \frac{1}{s_n} \log|x| & \forall x \in \mathbb{R}^n \setminus \{0\}, \text{ if } n = 2\,, \\ \frac{1}{(2-n)s_n}|x|^{2-n} & \forall x \in \mathbb{R}^n \setminus \{0\}, \text{ if } n > 2\,, \end{cases}$$

where s_n denotes the $(n-1)$-dimensional measure of $\partial\mathbb{B}_n(0,1)$. S_n is well-known to be the fundamental solution of the Laplace operator.

Let $\omega \in]1-(2/n), +\infty[$. We denote by $\Gamma_{n,\omega}$ the matrix valued function from $\mathbb{R}^n \setminus \{0\}$ to $M_n(\mathbb{R})$ which takes x to the matrix $\Gamma_{n,\omega}(x)$ with (j,k) entry defined by

$$\Gamma^k_{n,\omega,j}(x) \equiv \frac{\omega+2}{2(\omega+1)}\delta_{j,k} S_n(x) - \frac{\omega}{2(\omega+1)}\frac{1}{s_n}\frac{x_j x_k}{|x|^n} \quad \forall (j,k) \in \{1,\dots,n\}^2\,,$$

where $\delta_{j,k} = 1$ if $j = k$, $\delta_{j,k} = 0$ if $j \neq k$. As is well known, $\Gamma_{n,\omega}$ is the fundamental solution of the operator $L[\omega] \equiv \Delta + \omega \nabla \operatorname{div}$. We find also convenient to set

$$\Gamma^k_{n,\omega} \equiv \big(\Gamma^k_{n,\omega,j}\big)_{j\in\{1,\dots,n\}}\,,$$

which we think as a column vector for all $k \in \{1,\dots,n\}$. Now let $\alpha \in]0,1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then we set

$$v[\omega,\mu](x) \equiv \int_{\partial\Omega} \Gamma_{n,\omega}(x-y)\mu(y)\, d\sigma_y\,,$$

for all $x \in \mathbb{R}^n$ and for all $\mu \equiv (\mu_j)_{j\in\{1,\dots,n\}} \in C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$. Here $d\sigma$ denotes the area element on $\partial\Omega$. As is well known, the elastic single layer potential $v[\omega,\mu]$ is continuous in the whole of $\mathbb{R}^n$. We set $v^+[\omega,\mu] \equiv v[\omega,\mu]\big|_{\mathrm{cl}\,\Omega}$ and $v^-[\omega,\mu] \equiv v[\omega,\mu]\big|_{\mathbb{R}^n\setminus\Omega}$. We also find convenient to set

$$w_*[\omega,\mu](x) \equiv \int_{\partial\Omega} \sum_{l=1}^n \mu_l(y) T(\omega, D\Gamma^l_{n,\omega}(x-y))\nu_\Omega(x)\, d\sigma_y \quad \forall x \in \partial\Omega\,.$$

Here ν_Ω denotes the outward unit normal to $\partial\Omega$. For properties of elastic layer potentials, we refer, e. g., to [6, Appendix A].

We now introduce a periodic analogue of the fundamental solution of $L[\omega]$ (cf., e. g., Ammari et al [1, Lemma 3.2], [12, Thm. 3.1]). Let $\omega \in]1-(2/n), +\infty[$. We denote by $\Gamma^q_{n,\omega} \equiv (\Gamma^{q,k}_{n,\omega,j})_{(j,k)\in\{1,\dots,n\}^2}$ the matrix of distributions with (j,k) entry defined by

$$\Gamma^{q,k}_{n,\omega,j}(x) \equiv \sum_{z\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{4\pi^2|Q||q^{-1}z|^2}\Big[-\delta_{j,k} + \frac{\omega}{\omega+1}\frac{(q^{-1}z)_j(q^{-1}z)_k}{|q^{-1}z|^2}\Big] e^{2\pi i (q^{-1}z)\cdot x}$$

for all $(j,k) \in \{1,\dots,n\}^2$, where the series converges in the sense of distributions. Then

$$L[\omega]\Gamma^q_{n,\omega} = \sum_{z\in\mathbb{Z}^n} \delta_{qz} I_n - \frac{1}{|Q|} I_n\,,$$

where δ_{qz} denotes the Dirac measure with mass at qz for all $z \in \mathbb{Z}^n$. Moreover, $\Gamma^q_{n,\omega}$ is real analytic from $\mathbb{R}^n \setminus q\mathbb{Z}^n$ to $M_n(\mathbb{R})$ and the difference $\Gamma^q_{n,\omega} - \Gamma_{n,\omega}$ can be extended to a real analytic function from $(\mathbb{R}^n \setminus q\mathbb{Z}^n) \cup \{0\}$ to $M_n(\mathbb{R})$ which we denote by $R^q_{n,\omega}$. We find convenient to set

$$\Gamma^{q,k}_{n,\omega} \equiv \big(\Gamma^{q,k}_{n,\omega,j}\big)_{j\in\{1,\dots,n\}}\,, \quad R^{q,k}_{n,\omega} \equiv \big(R^{q,k}_{n,\omega,j}\big)_{j\in\{1,\dots,n\}}\,,$$

which we think as column vectors for all $k \in \{1, \dots, n\}$. Let Ω_Q be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$ such that $\mathrm{cl}\,\Omega_Q \subseteq Q$. Let $\mu \in C^{0,\alpha}(\partial\Omega_Q, \mathbb{R}^n)$. Then we denote by $v_q[\omega, \mu]$ the periodic single layer potential, namely the q-periodic function from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by

$$v_q[\omega, \mu](x) \equiv \int_{\partial\Omega_Q} \Gamma^q_{n,\omega}(x - y)\mu(y)\, d\sigma_y \quad \forall x \in \mathbb{R}^n .$$

We also find convenient to set

$$w_{q,*}[\omega, \mu](x) \equiv \int_{\partial\Omega_Q} \sum_{l=1}^{n} \mu_l(y) T(\omega, D\Gamma^{q,l}_{n,\omega}(x - y))\nu_{\Omega_Q}(x)\, d\sigma_y \quad \forall x \in \partial\Omega_Q .$$

Here ν_{Ω_Q} denotes the outward unit normal to $\partial\Omega_Q$. If $\mu \in C^{0,\alpha}(\partial\Omega_Q, \mathbb{R}^n)$, then the function $v_q^+[\omega, \mu] \equiv v_q[\omega, \mu]\big|_{\mathrm{cl}\,\mathbb{S}[\Omega_Q]}$ belongs to $C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\,\mathbb{S}[\Omega_Q], \mathbb{R}^n)$ and the function $v_q^-[\omega, \mu] \equiv v_q[\omega, \mu]\big|_{\mathrm{cl}\,\mathbb{S}[\Omega_Q]^-}$ belongs to $C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\,\mathbb{S}[\Omega_Q]^-, \mathbb{R}^n)$. For further properties of $v_q[\omega, \cdot]$ and $w_{q,*}[\omega, \cdot]$ we refer the reader to [12, Thm. 3.2].

3. An integral equation formulation of the nonlinear traction problem

In this section we provide an integral formulation of problem (1.1) (cf. [12, §5]). We use the following notation. If $G \in C^0(\partial\Omega^h \times \mathbb{R}^n, \mathbb{R}^n)$, then we denote by F_G the (nonlinear nonautonomous) composition operator from $C^0(\partial\Omega^h, \mathbb{R}^n)$ to itself which takes $v \in C^0(\partial\Omega^h, \mathbb{R}^n)$ to the function $F_G[v]$ from $\partial\Omega^h$ to $\mathbb{R}^n$ defined by

$$F_G[v](t) \equiv G(t, v(t)) \quad \forall t \in \partial\Omega^h .$$

Then we consider the following assumptions

$$G \in C^0(\partial\Omega^h \times \mathbb{R}^n, \mathbb{R}^n), \quad F_G \text{ maps } C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n) \text{ to itself.} \tag{3.1}$$

We also note here that if F_G is continuosly Fréchet differentiable from $C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)$ to itself, then the gradient matrix $D_u G(\cdot, \cdot)$ of $G(\cdot, \cdot)$ with respect to the variable in $\mathbb{R}^n$ exists. Moreover, $D_u G(\cdot, \xi) \in C^{0,\alpha}(\partial\Omega^h, M_n(\mathbb{R}))$ for all $\xi \in \mathbb{R}^n$, where $C^{0,\alpha}(\partial\Omega^h, M_n(\mathbb{R}))$ denotes the space of functions of class $C^{0,\alpha}$ from $\partial\Omega^h$ to $M_n(\mathbb{R})$ (cf. Lanza de Cristoforis [18, Prop. 6.3]).

We now transform problem (1.1) into an integral equation by means of the following (cf. [12, Prop. 5.2]). We find convenient to set $C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)_0 \equiv \{f \in C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n) \colon \int_{\partial\Omega^h} f\, d\sigma = 0\}$.

Proposition 3.1. *Let $\omega \in]1-(2/n), +\infty[$. Let $B \in M_n(\mathbb{R})$. Let G be as in assumption (3.1). Let Λ be the map from $]-\epsilon_0, \epsilon_0[\times C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)_0 \times \mathbb{R}^n$ to $C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)$, defined by*

$$\begin{aligned}
&\Lambda[\epsilon, \theta, \xi](t) \\
&\equiv \frac{1}{2}\theta(t) + w_*[\omega, \theta](t) + \epsilon^{n-1} \int_{\partial\Omega^h} \sum_{l=1}^{n} \theta_l(s) T(\omega, DR^{q,l}_{n,\omega}(\epsilon(t - s)))\nu_{\Omega^h}(t)\, d\sigma_s \\
&\quad + T(\omega, Bq^{-1})\nu_{\Omega^h}(t) - G\Big(t, \epsilon v[\omega, \theta](t) \\
&\quad + \epsilon^{n-1} \int_{\partial\Omega^h} R^q_{n,\omega}(\epsilon(t - s))\theta(s)\, d\sigma_s + \epsilon B q^{-1} t + \xi\Big) \quad \forall t \in \partial\Omega^h ,
\end{aligned}$$

for all $(\epsilon, \theta, \xi) \in]-\epsilon_0, \epsilon_0[\times C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)_0 \times \mathbb{R}^n$. *If* $\epsilon \in]0, \epsilon_0[$, *then the map* $u[\epsilon, \cdot, \cdot]$ *from the set of pairs* $(\theta, \xi) \in C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)_0 \times \mathbb{R}^n$ *that solve the equation*

$$\Lambda[\epsilon, \theta, \xi] = 0 \tag{3.2}$$

to the set of functions $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\,\mathbb{S}[\Omega^h_{p,\epsilon}]^-, \mathbb{R}^n)$ *which solve problem* (1.1), *which takes* (θ, ξ) *to the function defined by*

$$u[\epsilon, \theta, \xi](x) \equiv \epsilon^{n-1} \int_{\partial\Omega^h} \Gamma^q_{n,\omega}(x - p - \epsilon s)\theta(s)\, d\sigma_s - Bq^{-1}p + \xi + Bq^{-1}x$$

for all $x \in \mathrm{cl}\,\mathbb{S}[\Omega^h_{p,\epsilon}]^-$, *is a bijection.*

Hence we are reduced to analyze equation (3.2). To study (1.1) for ϵ small, we first observe that for $\epsilon = 0$ we obtain an equation which we address to as the *limiting equation* and which has the form

$$\frac{1}{2}\theta(t) + w_*[\omega, \theta](t) + T(\omega, Bq^{-1})\nu_{\Omega^h}(t) - G(t, \xi) = 0 \quad \forall t \in \partial\Omega^h\,. \tag{3.3}$$

Then we have the following Proposition, which shows, under suitable assumptions, the solvability of the limiting equation (cf. [12, Prop. 5.3]).

Proposition 3.2. *Let* $\omega \in]1 - (2/n), +\infty[$. *Let* $B \in M_n(\mathbb{R})$. *Let* G *be as in assumption* (3.1). *Assume that there exists* $\tilde{\xi} \in \mathbb{R}^n$ *such that*

$$\int_{\partial\Omega^h} G(t, \tilde{\xi})\, d\sigma_t = 0\,.$$

Then the integral equation

$$\frac{1}{2}\theta(t) + w_*[\omega, \theta](t) + T(\omega, Bq^{-1})\nu_{\Omega^h}(t) - G(t, \tilde{\xi}) = 0 \quad \forall t \in \partial\Omega^h$$

has a unique solution in $C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)_0$, *which we denote by* $\tilde{\theta}$. *As a consequence, the pair* $(\tilde{\theta}, \tilde{\xi})$ *is a solution in* $C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)_0 \times \mathbb{R}^n$ *of the limiting equation* (3.3).

Finally, by a straightforward modification of the proof of [12, Thm. 5.5], we deduce the validity of the following theorem, where we analyze equation (3.2) around the degenerate value $\epsilon = 0$.

Theorem 3.3. *Let* $\omega \in]1-(2/n), +\infty[$. *Let* $B \in M_n(\mathbb{R})$. *Let* G *be as in assumption* (3.1). *Assume that*

$$F_G \text{ is a continuosly Fréchet differentiable operator from } C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n) \text{ to itself.} \tag{3.4}$$

Assume that there exists $\tilde{\xi} \in \mathbb{R}^n$ *such that*

$$\int_{\partial\Omega^h} G(t, \tilde{\xi})\, d\sigma_t = 0 \quad \text{and} \quad \det\Big(\int_{\partial\Omega^h} D_u G(t, \tilde{\xi})\, d\sigma_t\Big) \neq 0. \tag{3.5}$$

Let Λ *be as in Proposition 3.1. Let* $\tilde{\theta}$ *be the unique function in* $C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)_0$ *such that* $\Lambda[0, \tilde{\theta}, \tilde{\xi}] = 0$ *(cf. Proposition 3.2). Then there exist* $\epsilon_1 \in]0, \epsilon_0]$, *an open neighborhood* $\mathcal{U}$ *of* $(\tilde{\theta}, \tilde{\xi})$ *in* $C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)_0 \times \mathbb{R}^n$, *and a continuously differentiable map* (Θ, Ξ) *from* $]-\epsilon_1, \epsilon_1[$ *to* $\mathcal{U}$, *such that the set of zeros of the map* Λ *in* $]-\epsilon_1, \epsilon_1[\times \mathcal{U}$ *coincides with the graph of* (Θ, Ξ). *In particular,* $(\Theta[0], \Xi[0]) = (\tilde{\theta}, \tilde{\xi})$.

Remark 3.4. Let the notation and assumptions of Theorem 3.3 hold. Let $u[\cdot,\cdot,\cdot]$ be as in Proposition 3.1. Let $u(\epsilon,x)\equiv u[\epsilon,\Theta[\epsilon],\Xi[\epsilon]](x)$ for all $x\in \mathrm{cl}\,\mathbb{S}[\Omega^h_{p,\epsilon}]^-$ and for all $\epsilon\in]0,\epsilon_1[$. Then for each $\epsilon\in]0,\epsilon_1[$ the function $u(\epsilon,\cdot)$ is a solution of problem (1.1).

4. Converging families of solutions

In this section we investigate some limiting and uniqueness properties of converging families of solutions of problem (1.1).

4.1. **Preliminary results.** We first need to study some auxiliary integral operators. In the following lemma, we introduce an operator which we denote by $M_\#$. The proof of the lemma can be done by using classical properties of the elastic layer potentials (see, *e.g.*, [6, Appendix A] and Maz'ya [23, p. 202]).

Lemma 4.1. *Let $\omega\in]1-(2/n),+\infty[$. Also let $M_\#$ denote the operator from $C^{0,\alpha}(\partial\Omega^h,\mathbb{R}^n)_0\times\mathbb{R}^n$ to $C^{1,\alpha}(\partial\Omega^h,\mathbb{R}^n)$, which takes a pair (θ,ξ) to the function $M_\#[\theta,\xi]$ defined by*

$$M_\#[\theta,\xi](t)\equiv v[\omega,\theta](t)+\xi\quad\forall t\in\partial\Omega^h\,.$$

Then $M_\#$ is a linear homeomorphism.

Then, if $\epsilon\in]0,\epsilon_0[$, we define the auxiliary integral operator M_ϵ and we prove its invertibility.

Lemma 4.2. *Let $\omega\in]1-(2/n),+\infty[$. Let $\epsilon\in]0,\epsilon_0[$. Let M_ϵ denote the operator from $C^{0,\alpha}(\partial\Omega^h,\mathbb{R}^n)_0\times\mathbb{R}^n$ to $C^{1,\alpha}(\partial\Omega^h,\mathbb{R}^n)$ which takes a pair (θ,ξ) to the function $M_\epsilon[\theta,\xi]$ defined by*

$$M_\epsilon[\theta,\xi](t)\equiv v[\omega,\theta](t)+\epsilon^{n-2}\int_{\partial\Omega^h}R^q_{n,\omega}(\epsilon(t-s))\theta(s)\,d\sigma_s+\xi\quad\forall t\in\partial\Omega^h\,.$$

Then M_ϵ is a linear homeomorphism.

Proof. We start by proving that M_ϵ is a Fredholm operator of index 0. We first note that

$$M_\epsilon[\theta,\xi](t)=M_\#[\theta,\xi](t)+\epsilon^{n-2}\int_{\partial\Omega^h}R^q_{n,\omega}(\epsilon(t-s))\theta(s)\,d\sigma_s\quad\forall t\in\partial\Omega^h\,,$$

for all $(\theta,\xi)\in C^{0,\alpha}(\partial\Omega^h,\mathbb{R}^n)_0\times\mathbb{R}^n$. By standard properties of integral operators with real analytic kernels and with no singularity (cf. [19, §4]), we deduce that the linear operator from $C^{0,\alpha}(\partial\Omega^h,\mathbb{R}^n)_0$ to $C^{2,\alpha}(\mathrm{cl}\,\Omega^h,\mathbb{R}^n)$ which takes θ to the function $\epsilon^{n-2}\int_{\partial\Omega^h}R^q_{n,\omega}(\epsilon(t-s))\theta(s)\,d\sigma_s$ of the variable $t\in\mathrm{cl}\,\Omega^h$ is continuous. Then by the compactness of the imbedding of $C^{2,\alpha}(\mathrm{cl}\,\Omega^h,\mathbb{R}^n)$ into $C^{1,\alpha}(\mathrm{cl}\,\Omega^h,\mathbb{R}^n)$, and by the continuity of the trace operator from $C^{1,\alpha}(\mathrm{cl}\,\Omega^h,\mathbb{R}^n)$ to $C^{1,\alpha}(\partial\Omega^h,\mathbb{R}^n)$, and by Lemma 4.1, we deduce that M_ϵ is a compact perturbation of the linear homeomorphism $M_\#$, and thus a Fredholm operator of index 0. Then, by the Fredholm theory, in order to prove that M_ϵ is a linear homeomorphism, it suffices to show that M_ϵ is injective. So let $(\theta,\xi)\in C^{0,\alpha}(\partial\Omega^h,\mathbb{R}^n)_0\times\mathbb{R}^n$ be such that $M_\epsilon[\theta,\xi]=0$. Then by the rule of change of variables in integrals, we have

$$M_\epsilon[\theta,\xi](\frac{x-p}{\epsilon})=v_q[\omega,\frac{1}{\epsilon}\theta\big(\frac{\cdot-p}{\epsilon}\big)](x)+\xi=0\quad\forall x\in\partial\Omega^h_{p,\epsilon}\,.$$

Then by the periodicity of $v_q[\omega, \frac{1}{\epsilon}\theta(\frac{\cdot - p}{\epsilon})]$ and by a straightforward modification of the argument of [12, Proof of Prop. 4.1], we deduce that

$$v_q[\omega, \frac{1}{\epsilon}\theta(\frac{\cdot - p}{\epsilon})](x) + \xi = 0 \quad \forall x \in \mathrm{cl}\, \mathbb{S}[\Omega^h_{p,\epsilon}]^- .$$

As a consequence,

$$0 = T(\omega, Dv_q^-[\omega, \frac{1}{\epsilon}\theta(\frac{\cdot - p}{\epsilon})](x))\nu_{\Omega^h_{p,\epsilon}}(x) = \frac{1}{2}\big(\frac{1}{\epsilon}\theta(\frac{x-p}{\epsilon})\big) + w_{q,*}[\omega, \frac{1}{\epsilon}\theta(\frac{\cdot - p}{\epsilon})](x)$$

for all $x \in \partial\Omega^h_{p,\epsilon}$. Then by [12, Prop. 4.4], we deduce that $\theta = 0$ and accordingly $\xi = 0$. □

We can now show that if $\{\varepsilon_j\}_{j\in\mathbb{N}}$ is a sequence in $]0, \epsilon_0[$ converging to 0, then $M^{(-1)}_{\varepsilon_j}$ converges to $M^{(-1)}_{\#}$ as $j \to +\infty$.

Lemma 4.3. *Let $\omega \in]1-(2/n), +\infty[$. Let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be a sequence in $]0, \epsilon_0[$ converging to 0. Then $\lim_{j\to+\infty} M^{(-1)}_{\varepsilon_j} = M^{(-1)}_{\#}$ in $\mathcal{L}(C^{1,\alpha}(\partial\Omega^h, \mathbb{R}^n), C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)_0 \times \mathbb{R}^n)$.*

Proof. Let N_j be the operator from $C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)_0 \times \mathbb{R}^n$ to $C^{1,\alpha}(\partial\Omega^h, \mathbb{R}^n)$ which takes (θ, ξ) to

$$N_j[\theta, \xi](t) \equiv \varepsilon_j^{n-2} \int_{\partial\Omega^h} R^q_{n,\omega}(\varepsilon_j(t-s))\theta(s)\, d\sigma_s \quad \forall t \in \partial\Omega^h\,,\ \forall j \in \mathbb{N}.$$

Let $\mathcal{U}_{\Omega^h}$ be an open bounded neighborhood of $\mathrm{cl}\,\Omega^h$. Let $\epsilon_\#$ be such that $\epsilon(t-s) \in (\mathbb{R}^n \setminus q\mathbb{Z}^n) \cup \{0\}$ for all $t, s \in \mathcal{U}_{\Omega^h}$ and all $\epsilon \in]-\epsilon_\#, \epsilon_\#[$. By the real analyticity of $R^q_{n,\omega}$ in $(\mathbb{R}^n \setminus q\mathbb{Z}^n) \cup \{0\}$ it follows that the map which takes (ϵ, t, s) to $R^q_{n,\omega}(\epsilon(t-s))$ is real analytic from $]-\epsilon_\#, \epsilon_\#[\times \mathcal{U}_{\Omega^h} \times \mathcal{U}_{\Omega^h}$ to $M_n(\mathbb{R})$. Hence, there exists a real analytic map $\tilde{R}^q_{n,\omega}$ from $]-\epsilon_\#, \epsilon_\#[\times \mathcal{U}_{\Omega^h} \times \mathcal{U}_{\Omega^h}$ to $M_n(\mathbb{R})$ such that $R^q_{n,\omega}(\epsilon(t-s)) - R^q_{n,\omega}(0) = \epsilon \tilde{R}^q_{n,\omega}(\epsilon, t, s)$ for all $t, s \in \mathcal{U}_{\Omega^h}$ and all $\epsilon \in]-\epsilon_\#, \epsilon_\#[$. Then, by the membership of θ in $C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)_0$, one has

$$N_j[\theta, \xi](t) = \varepsilon_j^{n-1} \int_{\partial\Omega^h} \tilde{R}^q_{n,\omega}(\varepsilon_j, t, s)\theta(s)\, d\sigma_s \quad \forall t \in \partial\Omega^h$$

for all j such that $\varepsilon_j \in]0, \epsilon_\#[$ and for all $(\theta, \xi) \in C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)_0 \times \mathbb{R}^n$. Then, by standard properties of integral operators with real analytic kernels and with no singularities (cf. [19, §4]), we deduce that $\lim_{j\to+\infty} N_j = 0$ in $\mathcal{L}(C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)_0 \times \mathbb{R}^n, C^{1,\alpha}(\partial\Omega^h, \mathbb{R}^n))$. Since $M_{\varepsilon_j} = M_\# + N_j$, it follows that $\lim_{j\to+\infty} M_{\varepsilon_j} = M_\sharp$ in $\mathcal{L}(C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)_0 \times \mathbb{R}^n, C^{1,\alpha}(\partial\Omega^h, \mathbb{R}^n))$. Then by the continuity of the mapping from the open subset of the invertible operators of $\mathcal{L}(C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)_0 \times \mathbb{R}^n, C^{1,\alpha}(\partial\Omega^h, \mathbb{R}^n))$ to $\mathcal{L}(C^{1,\alpha}(\partial\Omega^h, \mathbb{R}^n), C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)_0 \times \mathbb{R}^n)$ which takes an operator to its inverse, one deduces that $\lim_{j\to+\infty} M^{(-1)}_{\varepsilon_j} = M^{(-1)}_{\#}$ (cf. e. g., Hille and Phillips [15, Thms. 4.3.2 and 4.3.3]). □

4.2. **Limiting behavior of a converging family of solutions.** We are now ready to investigate in this subsection the limiting behavior of a converging family of solutions of problem (1.1). To begin with, in the following proposition we consider the limiting behavior of converging families of q-periodic displacement functions.

Proposition 4.4. *Let $\omega \in]1-(2/n), +\infty[$. Let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be a sequence in $]0, \epsilon_0[$ converging to 0 and let $\{u_{\#,j}\}_{j\in\mathbb{N}}$ be a sequence of functions such that for each*

$j \in \mathbb{N}$

$$u_{\#,j} \in C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\,\mathbb{S}[\Omega^h_{p,\varepsilon_j}]^-, \mathbb{R}^n)\,, \quad u_{\#,j} \text{ is } q\text{-periodic,}$$
$$\text{and } \operatorname{div} T(\omega, Du_{\#,j}) = 0 \text{ in } \mathbb{S}[\Omega^h_{p,\varepsilon_j}]^-\,.$$

Assume that there exists a function $v_\# \in C^{1,\alpha}(\partial\Omega^h, \mathbb{R}^n)$ such that

$$\lim_{j\to+\infty} u_{\#,j}(p+\varepsilon_j\cdot)\big|_{\partial\Omega^h} = v_\# \quad \text{in } C^{1,\alpha}(\partial\Omega^h, \mathbb{R}^n)\,. \tag{4.1}$$

Then there exists a pair $(u_\#, \xi_\#) \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega^h, \mathbb{R}^n) \times \mathbb{R}^n$ such that

$$v_\# = u_\#\big|_{\partial\Omega^h} + \xi_\#, \quad \operatorname{div} T(\omega, Du_\#) = 0 \text{ in } \mathbb{R}^n \setminus \mathrm{cl}\,\Omega^h\,,$$

and such that

$$\sup_{x\in\mathbb{R}^n\setminus\Omega^h} |x|^{n-2+\delta_{2,n}}|u_\#(x)| < \infty\,, \qquad \sup_{x\in\mathbb{R}^n\setminus\Omega^h} |x|^{n-1+\delta_{2,n}}|Du_\#(x)| < \infty\,.$$

Moreover,

$$\lim_{j\to+\infty} u_{\#,j}(p+\varepsilon_j\cdot)\big|_{\mathrm{cl}\,\mathcal{O}} = u_\#\big|_{\mathrm{cl}\,\mathcal{O}} + \xi_\# \quad \text{in } C^{1,\alpha}(\mathrm{cl}\,\mathcal{O}, \mathbb{R}^n) \tag{4.2}$$

for all open bounded subsets $\mathcal{O}$ of $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega^h$, and

$$\lim_{j\to+\infty} u_{\#,j}\big|_{\mathrm{cl}\,\tilde{\mathcal{O}}} = \xi_\# \quad \text{in } C^k(\mathrm{cl}\,\tilde{\mathcal{O}}, \mathbb{R}^n) \tag{4.3}$$

for all $k \in \mathbb{N}$ and for all open bounded subsets $\tilde{\mathcal{O}}$ of $\mathbb{R}^n$ such that $\mathrm{cl}\,\tilde{\mathcal{O}} \subseteq \mathbb{R}^n \setminus (p + q\mathbb{Z}^n)$.

Proof. Let

$$(\theta_j, \xi_j) \equiv M^{(-1)}_{\varepsilon_j}[u_{\#,j}(p+\varepsilon_j\cdot)\big|_{\partial\Omega^h}]$$

for all $j \in \mathbb{N}$ and $(\theta_\#, \xi_\#) \equiv M^{(-1)}_\#[v_\#]$. Since the evaluation mapping from $\mathcal{L}(C^{1,\alpha}(\partial\Omega^h, \mathbb{R}^n), C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)_0 \times \mathbb{R}^n) \times C^{1,\alpha}(\partial\Omega^h, \mathbb{R}^n)$ to $C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)_0 \times \mathbb{R}^n$, which takes a pair (A, v) to $A[v]$ is bilinear and continuous, the limiting relation (4.1) and Lemma 4.3 imply that

$$\lim_{j\to+\infty} (\theta_j, \xi_j) = \lim_{j\to+\infty} M^{(-1)}_{\varepsilon_j}[u_{\#,j}(p+\varepsilon_j\cdot)\big|_{\partial\Omega^h}] = M^{(-1)}_\#[v_\#] = (\theta_\#, \xi_\#) \tag{4.4}$$

in $C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)_0 \times \mathbb{R}^n$. Also, one has

$$u_{\#,j}(x) = \varepsilon_j^{n-2} \int_{\partial\Omega^h} \Gamma^q_{n,\omega}(x - p - \varepsilon_j s)\theta_j(s)\, d\sigma_s + \xi_j \quad \forall x \in \mathrm{cl}\,\mathbb{S}[\Omega^h_{p,\varepsilon_j}]^-\,, \forall j \in \mathbb{N}\,. \tag{4.5}$$

Then one has

$$u_{\#,j}(p+\varepsilon_j t) = v[\omega, \theta_j](t) + \varepsilon_j^{n-2} \int_{\partial\Omega^h} R^q_{n,\omega}(\varepsilon_j(t-s))\theta_j(s)\, d\sigma_s + \xi_j \tag{4.6}$$

for all $t \in \mathbb{R}^n \setminus \cup_{z\in\mathbb{Z}^n}(\varepsilon_j^{-1}qz + \mathrm{cl}\,\Omega^h)$. By continuity of the map from $C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)$ to $C^{1,\alpha}(\partial\Omega^h, \mathbb{R}^n)$ which takes θ to $v[\omega, \theta]\big|_{\partial\Omega^h}$, by standard properties of integral operators with real analytic kernels and with no singularities (cf. [20, §4]), by condition $\int_{\partial\Omega^h} \theta_\# d\sigma = 0$, and by (4.4), one verifies that

$$\lim_{j\to+\infty} u_{\#,j}(p+\varepsilon_j\cdot)\big|_{\partial\Omega^h} = v[\omega, \theta_\#]\big|_{\partial\Omega^h} + \xi_\# \quad \text{in } C^{1,\alpha}(\partial\Omega^h, \mathbb{R}^n)\,.$$

Hence, the limiting relation in (4.1) implies that $v_\# = v[\omega, \theta_\#]\big|_{\partial\Omega^h} + \xi_\#$. Now the validity of the proposition follows by setting $u_\#(t) \equiv v[\omega, \theta_\#](t)$ for all $t \in \mathbb{R}^n \setminus \Omega^h$.

Indeed, by classical results for elastic layer potentials and by condition $\int_{\partial\Omega^h} \theta_\# d\sigma = 0$, we have $u_\# \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega^h, \mathbb{R}^n)$, $\operatorname{div} T(\omega, Du_\#) = 0$ in $\mathbb{R}^n \setminus \operatorname{cl}\Omega^h$, and

$$\sup_{x\in\mathbb{R}^n\setminus\Omega^h} |x|^{n-2+\delta_{2,n}} |u_\#(x)| < \infty\,, \quad \sup_{x\in\mathbb{R}^n\setminus\Omega^h} |x|^{n-1+\delta_{2,n}} |Du_\#(x)| < \infty\,.$$

Finally, the validity of (4.2) for all open bounded subsets $\mathcal{O}$ of $\mathbb{R}^n \setminus \operatorname{cl}\Omega^h$ follows by equality (4.6), by the limiting relation in (4.4), by the continuity of the map from $C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)$ to $C^{1,\alpha}(\operatorname{cl}\mathcal{O}, \mathbb{R}^n)$ which takes θ to $v[\omega,\theta]\big|_{\operatorname{cl}\mathcal{O}}$, by standard properties of integral operators with real analytic kernels and with no singularities, and by $\int_{\partial\Omega^h} \theta_\# d\sigma = 0$. Similarly, the validity of (4.3) for all $k \in \mathbb{N}$ and for all open bounded subsets $\tilde{\mathcal{O}}$ of $\mathbb{R}^n$ such that $\operatorname{cl}\tilde{\mathcal{O}} \subseteq \mathbb{R}^n \setminus (p+q\mathbb{Z}^n)$ follows by equality (4.5), by the limiting relation in (4.4), by standard properties of integral operators with real analytic kernels and with no singularities (cf. [20, §4]), and by $\int_{\partial\Omega^h} \theta_\# d\sigma = 0$. $\square$

We are now ready to prove the main result of this subsection, where we study the limiting behavior of converging families of solutions of problem (1.1).

Theorem 4.5. *Let $\omega \in]1-(2/n), +\infty[$. Let $B \in M_n(\mathbb{R})$. Let $G \in C^0(\partial\Omega^h \times \mathbb{R}^n, \mathbb{R}^n)$ be such that F_G is continuous from $C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)$ to itself. Let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be a sequence in $]0, \epsilon_0[$ converging to 0 and let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence of functions such that u_j belongs to $C^{1,\alpha}_{\mathrm{loc}}(\operatorname{cl}\mathbb{S}[\Omega^h_{p,\varepsilon_j}]^-, \mathbb{R}^n)$ and is a solution of (1.1) with $\epsilon = \varepsilon_j$ for all $j \in \mathbb{N}$. Assume that there exists a function $v_* \in C^{1,\alpha}(\partial\Omega^h, \mathbb{R}^n)$ such that*

$$\lim_{j\to+\infty} u_j(p+\varepsilon_j\cdot)\big|_{\partial\Omega^h} = v_* \quad \text{in } C^{1,\alpha}(\partial\Omega^h, \mathbb{R}^n)\,.$$

Then there exists $\xi_* \in \mathbb{R}^n$ such that

$$v_* = \xi_* \ \text{on } \partial\Omega^h, \quad \int_{\partial\Omega^h} G(t,\xi_*)\, d\sigma_t = 0\,.$$

Moreover,

$$\lim_{j\to+\infty} u_j(p+\varepsilon_j\cdot)\big|_{\operatorname{cl}\mathcal{O}} = \xi_* \quad \text{in } C^{1,\alpha}(\operatorname{cl}\mathcal{O}, \mathbb{R}^n)$$

for all open bounded subsets $\mathcal{O}$ of $\mathbb{R}^n \setminus \operatorname{cl}\Omega^h$, and

$$\lim_{j\to+\infty} u_j\big|_{\operatorname{cl}\tilde{\mathcal{O}}} = \xi_* + Bq^{-1}(\cdot - p) \quad \text{in } C^k(\operatorname{cl}\tilde{\mathcal{O}}, \mathbb{R}^n)$$

for all $k \in \mathbb{N}$ and for all open bounded subsets $\tilde{\mathcal{O}}$ of $\mathbb{R}^n$ such that $\operatorname{cl}\tilde{\mathcal{O}} \subseteq \mathbb{R}^n \setminus (p + q\mathbb{Z}^n)$.

Proof. We set

$$u_{\#,j}(x) \equiv u_j(x) - Bq^{-1}x \quad \forall x \in \operatorname{cl}\mathbb{S}[\Omega^h_{p,\epsilon}]^-\,, \quad \forall j \in \mathbb{N}\,,$$
$$v_\#(x) \equiv v_*(x) - Bq^{-1}p \quad \forall x \in \partial\Omega^h\,.$$

Then for each $j \in \mathbb{N}$, the function $u_{\#,j} \in C^{1,\alpha}_{\mathrm{loc}}(\operatorname{cl}\mathbb{S}[\Omega^h_{p,\varepsilon_j}]^-, \mathbb{R}^n)$, $u_{\#,j}$ is q-periodic and $\operatorname{div} T(\omega, Du_{\#,j}) = 0$ in $\mathbb{S}[\Omega^h_{p,\varepsilon_j}]^-$. We have

$$v_\# \in C^{1,\alpha}(\partial\Omega^h, \mathbb{R}^n)\,, \quad \lim_{j\to+\infty} u_{\#,j}(p+\varepsilon_j\cdot)\big|_{\partial\Omega^h} = v_\# \quad \text{in } C^{1,\alpha}(\partial\Omega^h, \mathbb{R}^n)\,.$$

Hence, by Proposition 4.4, there exists a pair $(u_\#, \xi_\#) \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega^h, \mathbb{R}^n) \times \mathbb{R}^n$ such that

$$v_\# = u_\#\big|_{\partial\Omega^h} + \xi_\#\,, \quad \operatorname{div} T(\omega, Du_\#) = 0 \quad \text{in } \mathbb{R}^n \setminus \operatorname{cl}\Omega^h\,, \tag{4.7}$$

$$\sup_{x\in\mathbb{R}^n\setminus\Omega^h} |x|^{n-2+\delta_{2,n}}|u_\#(x)| < \infty\,, \qquad \sup_{x\in\mathbb{R}^n\setminus\Omega^h} |x|^{n-1+\delta_{2,n}}|Du_\#(x)| < \infty\,. \tag{4.8}$$

Moreover,

$$\lim_{j\to+\infty} u_{\#,j}(p+\varepsilon_j\cdot)\big|_{\mathrm{cl}\,\mathcal{O}} = u_\#\big|_{\mathrm{cl}\,\mathcal{O}} + \xi_\# \quad \text{in } C^{1,\alpha}(\mathrm{cl}\,\mathcal{O},\mathbb{R}^n) \tag{4.9}$$

for all open bounded subsets $\mathcal{O}$ of $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega^h$, and

$$\lim_{j\to+\infty} u_{\#,j}\big|_{\mathrm{cl}\,\tilde{\mathcal{O}}} = \xi_\# \quad \text{in } C^k(\mathrm{cl}\,\tilde{\mathcal{O}},\mathbb{R}^n)$$

for all $k \in \mathbb{N}$ and for all open bounded subsets $\tilde{\mathcal{O}}$ of $\mathbb{R}^n$ such that $\mathrm{cl}\,\tilde{\mathcal{O}} \subseteq \mathbb{R}^n \setminus (p + q\mathbb{Z}^n)$. Then we observe that

$$T(\omega, Du_{\#,j}(p+\varepsilon_j t) + Bq^{-1})\nu_{\Omega^h_{p,\varepsilon_j}}(p+\varepsilon_j t) = G(t, u_{\#,j}(p+\varepsilon_j t) + Bq^{-1}(p+\varepsilon_j t)) \tag{4.10}$$

for all $t \in \partial\Omega^h$ and all $j \in \mathbb{N}$, which implies

$$\begin{aligned} &T\Big(\omega, D_t\big(u_{\#,j}(p+\varepsilon_j t)\big)\Big)\nu_{\Omega^h}(t) \\ &= -\varepsilon_j T(\omega, Bq^{-1})\nu_{\Omega^h}(t) + \varepsilon_j G(t, u_{\#,j}(p+\varepsilon_j t) + Bq^{-1}p + \varepsilon_j Bq^{-1}t) \end{aligned} \tag{4.11}$$

for all $t \in \partial\Omega^h$, and all $j \in \mathbb{N}$. Then, by (4.9), by the continuity of F_G from $C^{0,\alpha}(\partial\Omega^h,\mathbb{R}^n)$ to itself, and by taking the limit as $j \to +\infty$ in (4.11), one obtains

$$T(\omega, Du_\#(t))\nu_{\Omega^h}(t) = 0 \quad \forall t \in \partial\Omega^h\,,$$

which, together with (4.7) and (4.8), implies $u_\# = 0$. In particular,

$$\lim_{j\to+\infty} u_{\#,j}(p+\varepsilon_j\cdot)\big|_{\mathrm{cl}\,\mathcal{O}} = \xi_\# \quad \text{in } C^{1,\alpha}(\mathrm{cl}\,\mathcal{O},\mathbb{R}^n) \tag{4.12}$$

for all open bounded subsets $\mathcal{O}$ of $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega^h$. Furthermore, by (4.10), by [12, Prop. 4.2], and by the equality $\int_{\partial\Omega^h_{p,\varepsilon_j}} T(\omega, Bq^{-1})\nu_{\Omega^h_{p,\varepsilon_j}}(x)\,d\sigma_x = 0$, one has

$$\begin{aligned} 0 &= \frac{1}{\varepsilon_j^{n-1}} \int_{\partial\Omega^h_{p,\varepsilon_j}} T(\omega, Du_{\#,j}(x) + Bq^{-1})\nu_{\Omega^h_{p,\varepsilon_j}}(x)\,d\sigma_x \\ &= \int_{\partial\Omega^h} G(t, u_{\#,j}(p+\varepsilon_j t) + Bq^{-1}p + \varepsilon_j Bq^{-1}t)\,d\sigma_t \end{aligned} \tag{4.13}$$

for all $j \in \mathbb{N}$. Then, by the continuity of F_G from $C^{0,\alpha}(\partial\Omega^h,\mathbb{R}^n)$ to itself, by the limiting relation in (4.12), and by letting $j \to +\infty$ in (4.13), one deduces

$$\int_{\partial\Omega^h} G(t, \xi_\# + Bq^{-1}p)\,d\sigma_t = 0\,.$$

Finally, by setting $\xi_* \equiv \xi_\# + Bq^{-1}p$, the validity of the theorem follows. □

4.3. **A local uniqueness result for converging families of solutions.** In this subsection we prove that a converging family of solutions of (1.1) is essentially unique in a local sense which we clarify in the following theorem.

Theorem 4.6. *Let $\omega \in]1-(2/n), +\infty[$. Let $B \in M_n(\mathbb{R})$. Let G be as in assumptions (3.1), (3.4). Let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be a sequence in $]0,\epsilon_0[$ converging to 0. Let $\{u_j\}_{j\in\mathbb{N}}$ and $\{v_j\}_{j\in\mathbb{N}}$ be sequences such that u_j and v_j belong to $C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\,\mathbb{S}[\Omega^h_{p,\varepsilon_j}]^-,\mathbb{R}^n)$ and* *both u_j and v_j are solutions of* (1.1) *with $\epsilon = \varepsilon_j$ for all $j \in \mathbb{N}$. Assume that there exists a function $v_* \in C^{1,\alpha}(\partial\Omega^h, \mathbb{R}^n)$ such that*

$$\lim_{j\to+\infty} u_j(p+\varepsilon_j\cdot)\big|_{\partial\Omega^h} = \lim_{j\to+\infty} v_j(p+\varepsilon_j\cdot)\big|_{\partial\Omega^h} = v_* \quad \text{in } C^{1,\alpha}(\partial\Omega^h, \mathbb{R}^n) \tag{4.14}$$

and that

$$\det\Big(\int_{\partial\Omega^h} D_u G(t, v_*(t))\, d\sigma_t\Big) \neq 0\,.$$

Then there exists a natural number $j_0 \in \mathbb{N}$ such that $u_j = v_j$ for all $j \geq j_0$.

Proof. We first observe that the family $\{u_j\}_{j\in\mathbb{N}}$ and the function v_* satisfy the conditions in Theorem 4.5. As a consequence, there exists $\tilde{\xi} \in \mathbb{R}^n$ such that

$$\lim_{j\to+\infty} u_j(p+\varepsilon_j\cdot)\big|_{\partial\Omega^h} = \tilde{\xi} \quad \text{in } C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)\,.$$

Then by (4.14) one also has

$$\lim_{j\to+\infty} v_j(p+\varepsilon_j\cdot)\big|_{\partial\Omega^h} = \tilde{\xi} \quad \text{in } C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)\,.$$

Moreover, we deduce that $\tilde{\xi}$ satisfies assumption (3.5). By Proposition 3.1, for each $j \in \mathbb{N}$ there exist and are unique two pairs $(\theta_{1,j}, \xi_{1,j})$, $(\theta_{2,j}, \xi_{2,j})$ in $C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)_0 \times \mathbb{R}^n$ such that

$$u_j(x) = u[\varepsilon_j, \theta_{1,j}, \xi_{1,j}](x)\,, \quad v_j(x) = u[\varepsilon_j, \theta_{2,j}, \xi_{2,j}](x)\,, \quad \forall x \in \mathrm{cl}\,\mathbb{S}[\Omega^h_{p,\varepsilon_j}]^-\,. \tag{4.15}$$

Let $\tilde{\theta}$, ϵ_1 be as in Theorem 3.3. Then to show the validity of the theorem, it will be enough to prove that

$$\lim_{j\to+\infty} (\theta_{1,j}, \xi_{1,j}) = (\tilde{\theta}, \tilde{\xi}) \quad \text{in } C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)_0 \times \mathbb{R}^n\,, \tag{4.16}$$

$$\lim_{j\to+\infty} (\theta_{2,j}, \xi_{2,j}) = (\tilde{\theta}, \tilde{\xi}) \quad \text{in } C^{0,\alpha}(\partial\Omega^h, \mathbb{R}^n)_0 \times \mathbb{R}^n\,. \tag{4.17}$$

Indeed, if we denote by $\mathcal{U}$ the neighborhood of Theorem 3.3, the limiting relations in (4.16), (4.17) imply that there exists $j_0 \in \mathbb{N}$ such that $(\varepsilon_j, \theta_{1,j}, \xi_{1,j}), (\varepsilon_j, \theta_{2,j}, \xi_{2,j}) \in]0, \epsilon_1[\times\mathcal{U}$ for all $j \geq j_0$ and thus Theorem 3.3 implies that $(\theta_{1,j}, \xi_{1,j}) = (\theta_{2,j}, \xi_{2,j}) = (\Theta[\varepsilon_j], \Xi[\varepsilon_j])$ for all $j \geq j_0$, and accordingly the theorem follows by (4.15). The proof of the limits in (4.16), (4.17) follows the lines of [12, Proof of Thm. 7.1] and is accordingly omitted. □

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