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# SPECTRUM FOR ANISOTROPIC EQUATIONS INVOLVING WEIGHTS AND VARIABLE EXPONENTS

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Abstract. We study the problem

$$-\sum_{i=1}^{N}\Big[\partial_{x_i}\Big(|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\Big)+|u|^{p_i(x)-2}u\Big]+|u|^{q(x)-2}u=\lambda g(x)|u|^{r(x)-2}u$$

in $\Omega$, $u=0$ on $\partial\Omega$, where $\Omega$ is a bounded domain in $\mathbb{R}^N$ $(N\geq 3)$, with smooth boundary, $\lambda$ is a positive real number, the functions $p_i,q,r:\overline{\Omega}\to[2,\infty)$ are Lipschitz continuous, $g:\overline{\Omega}\to[0,\infty)$ is measurable and these fulfill certain conditions. The main result of this paper establish the existence of two positive constants $\lambda_0$ and $\lambda_1$ with $0<\lambda_0\leq\lambda_1$ such that any $\lambda\in[\lambda_1,\infty)$ is an eigenvalue, while any $\lambda\in(0,\lambda_0)$ is not an eigenvalue of our problem.

## 1. Introduction

The purpose of this paper is to study the eigenvalue problem

$$\begin{gathered}-\sum_{i=1}^{N}\big[\partial_{x_i}(|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u)+|u|^{p_i(x)-2}u\big]+|u|^{q(x)-2}u=\lambda g(x)|u|^{r(x)-2}u\quad\text{in }\Omega,\\ u=0\quad\text{on }\partial\Omega,\end{gathered}\tag{1.1}$$

where $\Omega$ is a bounded domain with smooth boundary in $\mathbb{R}^N$ $(N\geq 3)$. The functions $p_i,q,r:\overline{\Omega}\to[2,\infty)$ are Lipschitz continuous, while $g:\overline{\Omega}\to[0,\infty)$ is a measurable function for which there exists an open subset $\Omega_0\subset\Omega$ such that $g(x)>0$ for any $x\in\Omega_0$, and $\lambda\geq 0$ is a real number.

A motivation for the study of problem (1.1) is given in [8, 9]. In [8] the problem studied involves the Laplace operator and $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth boundary, while in [9] the authors deal with a problem involving the $p(\cdot)$-Laplace operator and $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is a smooth exterior domain.

We emphasize the presence of $\vec{p}(\cdot)$-Laplace operator in problem (1.1). This is a natural extension of the $p(\cdot)$-Laplace operator. Both $p(\cdot)$-Laplace operator and $\vec{p}(\cdot)$-Laplace operator are nonhomogeneous, unlike the $p$-Laplace operator, where $p$ is a positive constant. The study of nonlinear elliptic equations involving quasilinear homogeneous type operators like the $p$-Laplace operator is based on the theory

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of standard Sobolev spaces to find weak solutions, while in the case of operators $p(\cdot)$-Laplace and $\vec{p}(\cdot)$-Laplace the natural setting is the use of the isotropic variable exponent Sobolev spaces and anisotropic variable exponent Sobolev spaces respectively (for our approach).

Thanks to the applicability to diverse fields of variable Sobolev spaces, in the past decades appeared many papers which involve such spaces. These are used to model various phenomena in image restoration (see [2]), in elastic mechanics (see [15]) and for the modelling of electrorheological fluids (or smart fluids). The first major discovery on electrorheological fluids was in 1949 due to Winslow [14]. These fluids have the interesting property that their viscosity can undergoes a significant change (namely can raise by up to five orders of magnitude) which depends on the electric field in the fluid. This phenomenon is known as the Winslow effect. Electrorheological fluids have been used in robotics and space technology. The experimental research has been done mainly in the USA, for instance in NASA laboratories.

## 2. Abstract framework

First, we introduce briefly a variable exponent Lebesgue-Sobolev setting. For more information on properties of variable exponent Lebesgue-Sobolev spaces we refer to [3, 4, 5, 6, 10, 11].

Throughout this paper, for any Lipschitz continuous function $p : \overline{\Omega} \to (1, \infty)$ we define

$$p^+ = \operatorname{ess\,sup}_{x\in\Omega} p(x) \quad \text{and} \quad p^- = \operatorname{ess\,inf}_{x\in\Omega} p(x).$$

We define the *variable exponent Lebesgue space*

$$L^{p(\cdot)}(\Omega) = \Big\{u; u\text{is a measurable real-valued function and } \int_\Omega |u|^{p(x)}dx < \infty\Big\},$$

endowed with the so-called *Luxemburg norm*

$$|u|_{p(\cdot)} = \inf\Big\{\mu > 0; \int_\Omega \Big|\frac{u(x)}{\mu}\Big|^{p(x)}dx \le 1\Big\},$$

which is a separable and reflexive Banach space. If $0 < |\Omega| < \infty$ and $p_1$, $p_2$ are variable exponents such that $p_1(x) \le p_2(x)$ almost everywhere in $\Omega$, then the embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous.

We denote by $L^{p'(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ the following Hölder type inequality

$$\Big|\int_\Omega uv\,dx\Big| \le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)|u|_{p(\cdot)}|v|_{p'(\cdot)} \le 2|u|_{p(\cdot)}|v|_{p'(\cdot)} \tag{2.1}$$

holds.

An important role in handling the generalized Lebesgue spaces is played by the $p(\cdot)$-*modular* of $L^{p(\cdot)}(\Omega)$ space, which is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{(\cdot)}(u) = \int_\Omega |u|^{p(x)}dx.$$

If $(u_n), u \in L^{p(\cdot)}(\Omega)$, then the following relations hold:

$$|u|_{p(\cdot)} > 1 \Rightarrow |u|_{p(\cdot)}^{p^-} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^+}, \tag{2.2}$$

$$|u|_{p(\cdot)} < 1 \Rightarrow |u|_{p(\cdot)}^{p^+} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^-}, \tag{2.3}$$

$$|u_n - u|_{p(\cdot)} \to 0 \Leftrightarrow \rho_{p(\cdot)}(u_n - u) \to 0. \tag{2.4}$$

We denote by $W_0^{1,p(\cdot)}$ the *variable exponent Sobolev space* defined by

$$W_0^{1,p(\cdot)}(\Omega) = \big\{u; u_{|\partial\Omega} = 0,\ u \in L^{p(\cdot)}(\Omega) \text{ and } |\nabla u| \in L^{p(\cdot)}(\Omega)\big\},$$

endowed with the equivalent norms

$$\|u\|_{p(\cdot)} = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)}$$

and

$$\|u\| = \inf\Big\{\mu > 0;\ \int_\Omega \Big(\Big|\frac{\nabla u(x)}{\mu}\Big|^{p(x)} + \Big|\frac{u(x)}{\mu}\Big|^{p(x)}\Big)dx \le 1\Big\},$$

where, in the definition of $\|u\|_{p(\cdot)}$, $|\nabla u|_{p(\cdot)}$ is the Luxemburg norm of $|\nabla u|$. We remember that $W_0^{1,p(\cdot)}(\Omega)$ is a separable and reflexive Banach space. Also, we note that if $p, s : \overline{\Omega} \to (1,\infty)$ are Lipschitz continuous functions with $p^+ < N$ and $p(x) \le s(x) \le p^*(x)$ for all $x \in \overline{\Omega}$, then the there exists the continuous embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)$, where $p^*(x) = \frac{Np(x)}{N-p(x)}$.

Next, we present the *anisotropic variable exponent Sobolev space* $W_0^{1,\vec{p}(\cdot)}(\Omega)$, where $\vec{p} : \overline{\Omega} \to \mathbb{R}^N$ is the vectorial function $\vec{p}(\cdot) = (p_1(\cdot), \dots, p_N(\cdot))$ and the components $p_i$, $i \in \{1, \dots, N\}$, are logarithmic Hölder continuous, that is, there exists $M > 0$ such that $|p_i(x) - p_i(y)| \le -M/\log(|x-y|)$ for any $x, y \in \Omega$ with $|x-y| \le 1/2$ and $i \in \{1, \dots, N\}$. Also, we define $W_0^{1,\vec{p}(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{\vec{p}(\cdot)} = \sum_{i=1}^N \big(|\partial_{x_i} u|_{p_i(\cdot)} + |u|_{p_i(\cdot)}\big),$$

and is a reflexive Banach space (see [7]).

Now, we introduce $\vec{P}_+, \vec{P}_- \in \mathbb{R}^N$ as

$$\vec{P}_+ = (p_1^+, \dots, p_N^+), \quad \vec{P}_- = (p_1^-, \dots, p_N^-),$$

and $P_+^+, P_-^+, P_-^- \in \mathbb{R}^+$ as

$$P_+^+ = \max\{p_1^+, \dots, p_N^+\}, \quad P_-^+ = \max\{p_1^-, \dots, p_N^-\}, \quad P_-^- = \min\{p_1^-, \dots, p_N^-\}.$$

We also always assume that

$$\sum_{i=1}^N \frac{1}{p_i^-} > 1,$$

and define $P_-^*,\ P_{-,\infty} \in \mathbb{R}^+$ by

$$P_-^* = \frac{N}{\sum_{i=1}^N 1/p_i^- - 1}, \quad P_{-,\infty} = \max\{P_-^+, P_-^*\}.$$

## 3. Main result

We study the problem (1.1) assuming that the functions $p_i$, $q$ and $r$ satisfy the hypotheses

$$2 \le P_-^- \le P_+^+ < N, \tag{3.1}$$

$$P_-^+ \le P_+^+ < r^- \le r^+ < q^- \le q^+ < P_-^* \le p_i^*(x) \quad \forall x \in \overline{\Omega},\ \forall i \in \{1,\dots,N\}. \tag{3.2}$$

Furthermore, we assume that the weight function $g(x)$ satisfies the hypotheses

$$\int_\Omega (\lambda g(x))^{\frac{q(x)}{q(x)-r(x)}}\,dx < \infty, \tag{3.3}$$

$$g \in L^\infty(\Omega) \cap L^{p_i^0(\cdot)}(\Omega), \tag{3.4}$$

where $p_i^0(x) = \frac{p_i^*(x)}{p_i^*(x)-r^-}$ for any $x \in \overline{\Omega}$ and any $i \in \{1,\dots,N\}$.

We look for weak solutions for problem (1.1) in the space $W_0^{1,\vec{p}(\cdot)}(\Omega)$. We say that $\lambda \in \mathbb{R}$ is an *eigenvalue* of problem (1.1) if there exists a $u \in W_0^{1,\vec{p}(\cdot)}(\Omega) \setminus \{0\}$ such that

$$\begin{aligned}&\int_\Omega \Big[\sum_{i=1}^N \big(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u \partial_{x_i} v + |u|^{p_i(x)-2}uv\big) + |u|^{q(x)-2}uv\Big]dx\\ &- \lambda \int_\Omega g(x)|u|^{r(x)-2}uv\,dx = 0,\end{aligned}$$

for all $v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$. We point out that if $\lambda$ is an eigenvalue of problem (1.1) then the corresponding $u \in W_0^{1,\vec{p}(\cdot)}(\Omega) \setminus \{0\}$ is a *weak solution* of problem (1.1).

Define

$$\lambda_1 := \inf_{u \in W_0^{1,\vec{p}(\cdot)}(\Omega)\setminus\{0\}} \frac{\int_\Omega \sum_{i=1}^N \Big(\frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} + \frac{|u|^{p_i(x)}}{p_i(x)}\Big)dx + \int_\Omega \frac{|u|^{q(x)}}{q(x)}dx}{\int_\Omega \frac{g(x)}{r(x)}|u|^{r(x)}dx},$$

$$\lambda_0 := \inf_{u \in W_0^{1,\vec{p}(\cdot)}(\Omega)\setminus\{0\}} \frac{\int_\Omega \sum_{i=1}^N \big(|\partial_{x_i} u|^{p_i(x)} + |u|^{p_i(x)}\big)dx + \int_\Omega |u|^{q(x)}dx}{\int_\Omega g(x)|u|^{r(x)}dx}.$$

Our main result is given by the following theorem.

**Theorem 3.1.** *Assume that conditions* (3.1)–(3.4) *are satisfied. Then*

$$0 < \lambda_0 \le \lambda_1.$$

*In addition, any $\lambda \in (0, \lambda_0)$ is not an eigenvalue of problem* (1.1)*, while each $\lambda \in [\lambda_1, \infty)$ is an eigenvalue of our problem.*

## 4. Proof of the main result

In what follows we denote by $E$ the generalized Sobolev space $W_0^{1,\vec{p}(\cdot)}(\Omega)$. We need to define the functionals $J_1, I_1, J_0, I_0 : E \to \mathbb{R}$ by

$$J_1(u) = \int_\Omega \sum_{i=1}^N \Big(\frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} + \frac{|u|^{p_i(x)}}{p_i(x)}\Big)dx + \int_\Omega \frac{|u|^{q(x)}}{q(x)}dx,$$

$$I_1(u) = \int_\Omega \frac{g(x)}{r(x)}|u|^{r(x)}dx,$$

$$J_0(u)=\int_\Omega\sum_{i=1}^N\big(|\partial_{x_i}u|^{p_i(x)}+|u|^{p_i(x)}\big)dx+\int_\Omega|u|^{q(x)}dx,$$
$$I_0(u)=\int_\Omega g(x)|u|^{r(x)}dx.$$

[7, Theorem 1] assures that $J_1,I_1\in C^1(E,\mathbb{R})$ and the Fréchet derivatives are given by

$$\langle J_1'(u),v\rangle=\int_\Omega\Big[\sum_{i=1}^N\big(|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\partial_{x_i}v+|u|^{p_i(x)-2}uv\big)+|u|^{q(x)-2}uv\Big]dx,$$
$$\langle I_1'(u),v\rangle=\int_\Omega g(x)|u|^{r(x)-2}uv\,dx.$$

Also, we define for any $\lambda>0$ the functional

$$T^1_\lambda(u)=J_1(u)-\lambda\cdot I_1(u)\quad\forall u\in E.$$

We point out that $\lambda$ is an eigenvalue of problem (1.1) if and only if there is an element $u_\lambda\in E\setminus\{0\}$, which is a critical point of the functional $T^1_\lambda$.

To give a clear view of what needs to be proved, we divide the proof of the theorem in four steps.

**Step 1.** We show that $\lambda_0,\lambda_1>0$. It should be noticed that from the condition (3.2), we have $p_i(x)<r(x)<q(x)$ for any $x\in\overline{\Omega}$ and any $i\in\{1,\dots,N\}$, and therefore

$$|u(x)|^{r(x)}\le|u(x)|^{p_i(x)}+|u(x)|^{q(x)}\quad\forall u\in E,\ \forall x\in\overline{\Omega},\ \forall i\in\{1,\dots,N\}.$$

Thus,

$$\int_\Omega\big(|u|^{p_i(x)}+|u|^{q(x)}\big)dx\ge\frac{1}{|g|_\infty}\cdot\int_\Omega g(x)|u|^{r(x)}dx\quad\forall u\in E,\ \forall i\in\{1,\dots,N\}.\tag{4.1}$$

It is obvious that

$$\int_\Omega\sum_{i=1}^N\big(|\partial_{x_i}u|^{p_i(x)}+|u|^{p_i(x)}\big)dx+\int_\Omega|u|^{q(x)}dx\ge\int_\Omega\big(|u|^{p_i(x)}+|u|^{q(x)}\big)dx,$$

which together with relation (4.1) we can deduce that

$$\frac{\int_\Omega\sum_{i=1}^N\big(|\partial_{x_i}u|^{p_i(x)}+|u|^{p_i(x)}\big)dx+\int_\Omega|u|^{q(x)}dx}{\int_\Omega g(x)|u|^{r(x)}dx}\ge\frac{1}{|g|_\infty}>0.$$

Hence we obtain that $\lambda_0>0$.

Next, using (4.1), by a simple computation we arrive at

$$\int_\Omega\frac{|u|^{p_i(x)}}{p_i(x)}dx+\int_\Omega\frac{|u|^{q(x)}}{q(x)}dx\ge\frac{r^-}{q^+\cdot|g|_\infty}\int_\Omega\frac{g(x)}{r(x)}|u|^{r(x)}dx.$$

It is clear that

$$\int_\Omega\sum_{i=1}^N\Big(\frac{|\partial_{x_i}u|^{p_i(x)}}{p_i(x)}+\frac{|u|^{p_i(x)}}{p_i(x)}\Big)dx+\int_\Omega\frac{|u|^{q(x)}}{q(x)}dx\ge\int_\Omega\frac{|u|^{p_i(x)}}{p_i(x)}dx+\int_\Omega\frac{|u|^{q(x)}}{q(x)}dx,$$

and considering the previous inequality we derive that

$$\frac{\int_\Omega\sum_{i=1}^N\Big(\frac{|\partial_{x_i}u|^{p_i(x)}}{p_i(x)}+\frac{|u|^{p_i(x)}}{p_i(x)}\Big)dx+\int_\Omega\frac{|u|^{q(x)}}{q(x)}dx}{\int_\Omega\frac{g(x)}{r(x)}|u|^{r(x)}dx}\ge\frac{r^-}{q^+\cdot|g|_\infty}>0,$$

wherefrom $\lambda_1 > 0$. Step 1 is verified.

**Step 2.** We prove that any $\lambda \in (0, \lambda_0)$ is not an eigenvalue of problem (1.1).

We argue indirectly. So, suppose that there is $\lambda \in (0, \lambda_0)$, an eigenvalue of problem (1.1). Thereby we can deduce the existence of an element $u_\lambda \in E \setminus \{0\}$ such that

$$\begin{aligned}&\int_\Omega \Big[\sum_{i=1}^N \big(|\partial_{x_i} u_\lambda|^{p_i(x)-2}\partial_{x_i} u_\lambda \partial_{x_i} v + |u_\lambda|^{p_i(x)-2} u_\lambda v\big) + |u_\lambda|^{q(x)-2}u_\lambda v\Big]dx \\ &= \lambda \int_\Omega g(x)|u_\lambda|^{r(x)-2}u_\lambda v\, dx \quad \forall v \in E.\end{aligned}$$

Taking $v = u_\lambda$ in the above equality we obtain

$$J_0(u_\lambda) = \lambda \cdot I_0(u_\lambda). \tag{4.2}$$

By $u_\lambda \in E \setminus \{0\}$ we have $J_0(u_\lambda) > 0$ and $I_0(u_\lambda) > 0$. On the other hand

$$\frac{J_0(u_\lambda)}{I_0(u_\lambda)} = \frac{\int_\Omega \sum_{i=1}^N \big(|\partial_{x_i} u_\lambda|^{p_i(x)} + |u_\lambda|^{p_i(x)}\big)dx + \int_\Omega |u_\lambda|^{q(x)}dx}{\int_\Omega g(x)|u_\lambda|^{r(x)}dx} \geq \lambda_0.$$

This, together with (4.2) yield

$$J_0(u_\lambda) \geq \lambda_0 \cdot I_0(u_\lambda) > \lambda \cdot I_0(u_\lambda) = J_0(u_\lambda),$$

which is a contradiction. This proves the Step 2.

**Step 3.** We verify that each $\lambda \in (\lambda_1, \infty)$ is an eigenvalue for problem (1.1). With an eye to show what we proposed in this step, we start by proving the following three lemmas.

**Lemma 4.1.** *Assume that conditions* (3.1)–(3.4) *are fulfilled and $s$ is a real number such that $r^+ < s < P_-^*$. Then $g \in L^{\frac{s}{s-r^-}}(\Omega) \cap L^{\frac{s}{s-r^+}}(\Omega)$ and*

$$\int_\Omega g(x)|u|^{r(x)}dx \leq |g|_{\frac{s}{s-r^-}}|u|_s^{r^-} + |g|_{\frac{s}{s-r^+}}|u|_s^{r^+} \quad \forall u \in E. \tag{4.3}$$

*Proof.* In the first instance we highlight the inequalities

$$\frac{s}{s-r^+} \geq \frac{s}{s-r^-} > \frac{P_-^*}{P_-^* - r^-} \geq \frac{p_i^*(x)}{p_i^*(x) - r^-} = p_i^0(x) \tag{4.4}$$

for all $x \in \overline{\Omega}$ and all $i \in \{1, \dots, N\}$. Also, we have

$$(p_i^0)^- \leq p_i^0(x) \leq \frac{P_-^*}{P_-^* - r^-} \quad \forall x \in \overline{\Omega},\ \forall i \in \{1, \dots, N\}.$$

So we arrive at

$$|g|_\infty^{\frac{s}{s-r^-} - (p_i^0)^-} + |g|_\infty^{\frac{s}{s-r^-} - \frac{P_-^*}{P_-^* - r^-}} \geq |g|_\infty^{\frac{s}{s-r^-} - p_i^0(x)} \quad \forall x \in \overline{\Omega},\ \forall i \in \{1, \dots, N\}. \tag{4.5}$$

By (3.4), (4.4) and (4.5) we can easily see that

$$\begin{aligned}\int_\Omega [g(x)]^{\frac{s}{s-r^-}}dx &= \int_\Omega [g(x)]^{p_i^0(x)} \cdot [g(x)]^{\frac{s}{s-r^-} - p_i^0(x)}dx \\ &\leq \int_\Omega [g(x)]^{p_i^0(x)} \cdot |g|_\infty^{\frac{s}{s-r^-} - p_i^0(x)}dx \\ &\leq \Big(|g|_\infty^{\frac{s}{s-r^-} - (p_i^0)^-} + |g|_\infty^{\frac{s}{s-r^-} - \frac{P_-^*}{P_-^* - r^-}}\Big)\int_\Omega [g(x)]^{p_i^0(x)}dx < \infty,\end{aligned}$$

that is $g \in L^{\frac{s}{s-r^-}}(\Omega)$. In a similar fashion, we can show that $g \in L^{\frac{s}{s-r^+}}(\Omega)$.

From

$$|u(x)|^{r^-} + |u(x)|^{r^+} \geq |u(x)|^{r(x)} \quad \forall u \in E,\ \forall x \in \overline{\Omega}, \tag{4.6}$$

and Hölder type inequality (2.1), we deduce

$$\begin{aligned}\int_\Omega g(x)|u|^{r(x)}dx &\leq \int_\Omega g(x)|u|^{r^-}dx + \int_\Omega g(x)|u|^{r^+}dx \\ &\leq |g|_{\frac{s}{s-r^-}}|u|_s^{r^-} + |g|_{\frac{s}{s-r^+}}|u|_s^{r^+} \quad \forall u \in E\,.\end{aligned}$$

The proof of Lemma 4.1 is complete. □

**Lemma 4.2.** *For each $\lambda > 0$ we have*

$$\lim_{\|u\|_{\vec{p}(\cdot)}\to\infty} T_\lambda^1(u) = \infty.$$

*Proof.* Let $s \in \mathbb{R}$ be such that

$$r^+ < s < q^- < P_-^*. \tag{4.7}$$

Without loss of generality we assume that $\|u\|_{\vec{p}(\cdot)} > 1$ for each $u \in E$. By (3.2) and (4.7) we have

$$|u(x)|^{p_i(x)} + |u(x)|^{q(x)} \geq |u(x)|^s \quad \forall u \in E,\ \forall x \in \overline{\Omega},\ \forall i \in \{1,\dots,N\}\,.$$

This implies

$$\int_\Omega \Big(\sum_{i=1}^N |u|^{p_i(x)} + |u|^{q(x)}\Big)dx \geq \int_\Omega |u|^s dx. \tag{4.8}$$

Now, using (4.8) and Lemma 4.1 we have

$$\begin{aligned}T_\lambda^1(u) &= \int_\Omega \sum_{i=1}^N \Big(\frac{|\partial_{x_i}u|^{p_i(x)}}{p_i(x)} + \frac{|u|^{p_i(x)}}{p_i(x)}\Big)dx + \int_\Omega \frac{|u|^{q(x)}}{q(x)}dx - \lambda\int_\Omega \frac{g(x)}{r(x)}|u|^{r(x)}dx \\ &\geq \Big(\frac{1}{P_+^+}\int_\Omega \sum_{i=1}^N |\partial_{x_i}u|^{p_i(x)}dx + \frac{1}{2P_+^+}\int_\Omega \sum_{i=1}^N |u|^{p_i(x)}dx\Big) \\ &\quad + \Big(\frac{1}{2P_+^+}\int_\Omega \sum_{i=1}^N |u|^{p_i(x)}dx + \frac{1}{q^+}\int_\Omega |u|^{q(x)}dx\Big) - \frac{\lambda}{r^-}\int_\Omega g(x)|u|^{r(x)}dx \\ &\geq \frac{1}{2P_+^+}\int_\Omega \sum_{i=1}^N \big(|\partial_{x_i}u|^{p_i(x)} + |u|^{p_i(x)}\big)dx + \frac{1}{\max\{2P_+^+, q^+\}}\int_\Omega |u|^s dx \\ &\quad - C_1|u|_s^{r^-} - C_2|u|_s^{r^+},\end{aligned}$$

where $C_1 = \frac{\lambda}{r^-}|g|_{\frac{s}{s-r^-}}$ and $C_2 = \frac{\lambda}{r^-}|g|_{\frac{s}{s-r^+}}$. To go further we need to define for each $i \in \{1,\dots,N\}$ the following:

$$\alpha_i = \begin{cases} P_+^+ & \text{for } |\partial_{x_i}u|_{p_i(\cdot)} < 1 \\ P_-^- & \text{for } |\partial_{x_i}u|_{p_i(\cdot)} > 1, \end{cases} \qquad \beta_i = \begin{cases} P_+^+ & \text{for } |u|_{p_i(\cdot)} < 1 \\ P_-^- & \text{for } |u|_{p_i(\cdot)} > 1. \end{cases}$$

From that fact and applying the Jensen's inequality to the convex function $a : \mathbb{R}^+ \to \mathbb{R}^+$, $a(t) = t^{P_-^-}$, $P_-^- \geq 2$, we can write

$$
\begin{aligned}
T_\lambda^1(u) &\geq \frac{1}{2P_+^+} \sum_{i=1}^N \big( |\partial_{x_i} u|_{p_i(\cdot)}^{\alpha_i} + |u|_{p_i(\cdot)}^{\beta_i} \big) \\
&\quad + \frac{1}{\max\{2P_+^+, q^+\}} \int_\Omega |u|^s dx - C_1 |u|_s^{r^-} - C_2 |u|_s^{r^+} \\
&\geq \frac{1}{2P_+^+} \sum_{i=1}^N |\partial_{x_i} u|_{p_i(\cdot)}^{P_-^-} - \frac{1}{2P_+^+} \sum_{\{i;\, \alpha_i = P_+^+\}} \big( |\partial_{x_i} u|_{p_i(\cdot)}^{P_-^-} - |\partial_{x_i} u|_{p_i(\cdot)}^{P_+^+} \big) \\
&\quad + \frac{1}{2P_+^+} \sum_{i=1}^N |u|_{p_i(\cdot)}^{P_-^-} - \frac{1}{2P_+^+} \sum_{\{i;\, \beta_i = P_+^+\}} \big( |u|_{p_i(\cdot)}^{P_-^-} - |u|_{p_i(\cdot)}^{P_+^+} \big) \\
&\quad + \frac{1}{\max\{2P_+^+, q^+\}} \int_\Omega |u|^s dx - C_1 |u|_s^{r^-} - C_2 |u|_s^{r^+} \\
&\geq \frac{\|u\|_{\vec{p}(\cdot)}^{P_-^-}}{2P_+^+ (2N)^{P_-^- - 1}} - \frac{N}{P_+^+} + (C_3 |u|_s^s - C_1 |u|_s^{r^-}) + (C_3 |u|_s^s - C_2 |u|_s^{r^+}),
\end{aligned}
\tag{4.9}
$$

where $C_3 = \frac{1}{2\max\{2P_+^+, q^+\}}$. We are going to show that for each $u \in E$ there are two positive constants $L_1 = L_1(r^-, s, C_1, C_3)$ and $L_2 = L_2(r^+, s, C_2, C_3)$ such that

$$
C_3 |u|_s^s - C_1 |u|_s^{r^-} \geq -L_1, \tag{4.10}
$$

$$
C_3 |u|_s^s - C_2 |u|_s^{r^+} \geq -L_2. \tag{4.11}
$$

For this purpose, we define the functional $\Upsilon : (0, \infty) \to \mathbb{R}$ as

$$
\Upsilon(t) = \alpha t^a - \beta t^b,
$$

where $\alpha$, $\beta$, $a$, $b$ are positive constants with $a > b$. By a usual computation we find that $\Upsilon$ achieves its negative global minimum

$$
\Upsilon(t_0) = -(a-b) \Big( \frac{b^b}{a^a} \Big)^{\frac{1}{a-b}} \alpha^{\frac{b}{b-a}} \cdot \beta^{\frac{a}{a-b}},
$$

where $t_0 = \big( \frac{\beta b}{\alpha a} \big)^{\frac{1}{a-b}} > 0$. Consequently,

$$
\alpha t^a - \beta t^b \geq -(a-b) \Big( \frac{b^b}{a^a} \Big)^{\frac{1}{a-b}} \alpha^{\frac{b}{b-a}} \cdot \beta^{\frac{a}{a-b}} \quad \forall t > 0. \tag{4.12}
$$

Taking in (4.12) $a = s$, $b = r^-$, $\alpha = C_3$ and $\beta = C_1$ we find

$$
L_1 = C(s,\, r^-) \alpha^{\frac{r^-}{r^- - s}} \beta^{\frac{s}{s - r^-}}.
$$

In a similar manner, taking in (4.12) $a = s$, $b = r^+$, $\alpha = C_3$ and $\beta = C_2$ we deduce that (4.11) holds for

$$
L_2 = C(s,\, r^+) \alpha^{\frac{r^+}{r^+ - s}} \beta^{\frac{s}{s - r^+}}.
$$

Finally, putting together (4.9)–(4.11) we conclude Lemma 4.2. □

**Lemma 4.3.** *For any $\lambda > 0$, the functional $T_\lambda^1$ is weakly lower semicontinuous on $E$.*

*Proof.* Let $(u_n) \subset E$ be such that $u_n \rightharpoonup u_0$ in $E$. We define

$$F(x,u) = \frac{1}{q(x)}|u|^{q(x)} - \frac{\lambda g(x)}{r(x)}|u|^{r(x)},$$
$$f(x,u) = F_u(x,u) = |u|^{q(x)-2}u - \lambda g(x)|u|^{r(x)-2}u.$$

Using ordinary rule of the derivation we find

$$f_u(x,u) = (q(x)-1)|u|^{q(x)-2} - \lambda g(x)(r(x)-1)|u|^{r(x)-2}. \tag{4.13}$$

We shall employ in what follows the following inequality: for any $k_1$, $k_2 > 0$ and $0 < q < r$ we have

$$k_1|t|^q - k_2|t|^r \le Ck_1\Big(\frac{k_1}{k_2}\Big)^{\frac{q}{r-q}} \quad \forall t \in \mathbb{R},$$

where $C = C(q, r) > 0$ is a constant depending on $q$ and $r$.

If we make the substitutions $k_1 = q(x) - 1$, $k_2 = \lambda g(x)(r(x) - 1)$, $q = q(x) - 2$ and $r = r(x) - 2$, then (4.13) becomes

$$f_u(x,u) \le C(q(x)-1)\Big(\frac{q(x)-1}{r(x)-1}\Big)^{\frac{q(x)-2}{r(x)-q(x)}}(\lambda g(x))^{\frac{q(x)-2}{q(x)-r(x)}}.$$

As a result of the fact that $C(q(x)-1)\big(\frac{q(x)-1}{r(x)-1}\big)^{\frac{q(x)-2}{r(x)-q(x)}}$ is a bounded expression, we arrive at

$$f_u(x,u) \le C_1(\lambda g(x))^{\frac{q(x)-2}{q(x)-r(x)}}, \tag{4.14}$$

where $C_1$ is a positive constant. Also, the equalities

$$\begin{aligned}\int_0^s f_u(x, u_0 + t(u_n - u_0))dt &= \frac{f(x, u_0 + s(u_n - u_0)) - f(x, u_0)}{u_n - u_0} \\ &= \frac{F_u(x, u_0 + s(u_n - u_0)) - F_u(x, u_0)}{u_n - u_0}\end{aligned}$$

hold. Integrating over $[0, 1]$ it results that

$$\begin{aligned}&\int_0^1 \int_0^s f_u(x, u_0 + t(u_n - u_0))dt\,ds \\ &= \frac{\int_0^1 \big[F_u(x, u_0 + s(u_n - u_0)) - F_u(x, u_0)\big]ds}{u_n - u_0} \\ &= \frac{F(x, u_n) - F(x, u_0)}{(u_n - u_0)^2} - \frac{f(x, u_0)}{u_n - u_0},\end{aligned}$$

which can be also written in the equivalent form

$$\begin{aligned}F(x, u_n) - F(x, u_0) = (u_n - u_0)^2 &\int_0^1 \int_0^s f_u(x, u_0 + t(u_n - u_0))dt\,ds \\ &+ (u_n - u_0)f(x, u_0).\end{aligned} \tag{4.15}$$

Taking into account (4.14), (4.15) and using the definition of $T^1_\lambda$ it follows that

$$
\begin{aligned}
&T^1_\lambda(u_0) - T^1_\lambda(u_n)\\
&= \int_\Omega \sum_{i=1}^N \Big(\frac{|\partial_{x_i} u_0|^{p_i(x)}}{p_i(x)} - \frac{|u_0|^{p_i(x)}}{p_i(x)}\Big)dx - \int_\Omega \sum_{i=1}^N \Big(\frac{|\partial_{x_i} u_n|^{p_i(x)}}{p_i(x)} - \frac{|u_n|^{p_i(x)}}{p_i(x)}\Big)dx\\
&\quad + \int_\Omega [F(x,u_n) - F(x,u_0)]dx\\
&\le \int_\Omega \sum_{i=1}^N \Big(\frac{|\partial_{x_i} u_0|^{p_i(x)}}{p_i(x)} - \frac{|u_0|^{p_i(x)}}{p_i(x)}\Big)dx - \int_\Omega \sum_{i=1}^N \Big(\frac{|\partial_{x_i} u_n|^{p_i(x)}}{p_i(x)} - \frac{|u_n|^{p_i(x)}}{p_i(x)}\Big)dx\\
&\quad + \int_\Omega (u_n - u_0)^2 \int_0^1 \int_0^s f_u(x, u_0 + t(u_n - u_0))dt\, ds\, dx\\
&\quad + \int_\Omega (u_n - u_0) f(x,u_0)dx\\
&\le \int_\Omega \sum_{i=1}^N \Big(\frac{|\partial_{x_i} u_0|^{p_i(x)}}{p_i(x)} - \frac{|u_0|^{p_i(x)}}{p_i(x)}\Big)dx - \int_\Omega \sum_{i=1}^N \Big(\frac{|\partial_{x_i} u_n|^{p_i(x)}}{p_i(x)} - \frac{|u_n|^{p_i(x)}}{p_i(x)}\Big)dx\\
&\quad + C_2 \int_\Omega (u_n - u_0)^2 (\lambda g(x))^{\frac{q(x)-2}{q(x)-r(x)}}\, dx + \int_\Omega (u_n - u_0) f(x,u_0)dx,
\end{aligned}
\tag{4.16}
$$

where $C_2$ is a positive constant. We intend to prove that the last two integrals converge to 0 as $n \to \infty$.

Relying on [7, Theorem 1] we find that $E$ is compactly embedded in $L^{q(\cdot)}(\Omega)$, and since $u_n \rightharpoonup u_0$ in $E$ we obtain $u_n \to u_0$ in $L^{q(\cdot)}(\Omega)$. This implies

$$
\int_\Omega |u_n - u_0|^{q(x)} dx \to 0,
$$

yielding $(u_n - u_0)^2 \in L^{\frac{q(\cdot)}{2}}(\Omega)$. Based on Hölder type inequality (2.1) and the hypothesis (3.3) we derive that

$$
\int_\Omega (u_n - u_0)^2 \cdot (\lambda g(x))^{\frac{q(x)-2}{q(x)-r(x)}}\, dx \le 2 \Big|(\lambda g(x))^{\frac{q(x)-2}{q(x)-r(x)}}\Big|_{\frac{q(\cdot)}{q(\cdot)-2}} \big|(u_n - u_0)^2\big|_{\frac{q(\cdot)}{2}}.
$$

On the other hand,

$$
\rho_{\frac{q(\cdot)}{2}}\big((u_n - u_0)^2\big) = \int_\Omega \big|(u_n - u_0)^2\big|^{\frac{q(x)}{2}} dx = \int_\Omega |u_n - u_0|^{q(x)} dx \to 0.
$$

Thereupon, relation (2.4) implies $\big|(u_n - u_0)^2\big|_{\frac{q(\cdot)}{2}} \to 0$, and for this reason we obtain

$$
\int_\Omega (u_n - u_0)^2 \cdot (\lambda g(x))^{\frac{q(x)-2}{q(x)-r(x)}}\, dx \to 0. \tag{4.17}
$$

Next, we define $\Theta : E \to \mathbb{R}$ by

$$
\Theta(v) = \int_\Omega f(x,u_0) v\, dx.
$$

In the first instance, it is clear that $\Theta$ is linear. On the other hand,

$$\begin{aligned} |\Theta(v)| &\le \int_\Omega |f(x,u_0)v|dx \\ &= \int_\Omega \big||u_0|^{q(x)-2}u_0 - \lambda g(x)|u_0|^{r(x)-2}u_0\big|\,|v|dx \\ &\le \int_\Omega |u_0|^{q(x)-1}|v|dx + \lambda \int_\Omega g(x)|u_0|^{r(x)-1}|v|dx. \end{aligned} \tag{4.18}$$

In accordance with the Hölder type inequality (2.1) we obtain

$$\int_\Omega |u_0|^{q(x)-1}|v|dx \le 2\big||u_0|^{q(x)-1}\big|_{\frac{q(\cdot)}{q(\cdot)-1}}|v|_{q(\cdot)}.$$

We know that the embedding $E \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous; that is, there is a positive constant $C$ such that

$$|v|_{q(\cdot)} \le C\|v\|_{\vec{p}(\cdot)} \quad \forall v \in E.$$

The last two inequalities lead us to

$$\int_\Omega |u_0|^{q(x)-1}|v|dx \le C_1\|v\|_{\vec{p}(\cdot)},$$

where $C_1 > 0$ is a constant. Also, reasoning as above we have

$$\begin{aligned} \int_\Omega g(x)|u_0|^{r(x)-1}|v|dx &\le |g|_\infty \int_\Omega |u_0|^{r(x)-1}|v|dx \\ &\le 2|g|_\infty\big||u_0|^{r(x)-1}\big|_{\frac{r(\cdot)}{r(\cdot)-1}}|v|_{r(\cdot)} \le C_2\|v\|_{\vec{p}(\cdot)}, \end{aligned}$$

where $C_2 > 0$ is a constant.

In light of the above, (4.18) becomes

$$|\Theta(v)| \le \overline{C}\|v\|_{\vec{p}(\cdot)} \quad \forall v \in E$$

(where $\overline{C} > 0$ is a constant); that is to say, $\Theta$ is continuous. Accordingly, we conclude that $\Theta(u_n) \to \Theta(u_0)$, and therefore

$$\int_\Omega f(x,u_0)(u_n - u_0)dx \to 0. \tag{4.19}$$

To complete the proof of lemma, we must prove that the functional $\Xi_1 : E \to \mathbb{R}$,

$$\Xi_1(u) = \int_\Omega \sum_{i=1}^N \frac{|u|^{p_i(x)}}{p_i(x)}dx$$

is convex. Considering that the function $[0,\infty) \ni t \mapsto t^\gamma$ is convex for each $\gamma > 1$, for any $x \in \Omega$ fixed we can say that

$$\Big|\frac{\alpha+\beta}{2}\Big|^{p_i(x)} \le \Big|\frac{|\alpha|+|\beta|}{2}\Big|^{p_i(x)} \le \frac{1}{2}|\alpha|^{p_i(x)} + \frac{1}{2}|\beta|^{p_i(x)} \tag{4.20}$$

for all $\alpha, \beta \in \mathbb{R}$ and all $i \in \{1,\dots,N\}$. If we take $\alpha = u$ and $\beta = v$ in (4.20), multiply by $1/p_i(x)$, sum from 1 to $N$ and intergate over $\Omega$, we obtain

$$\Xi_1\Big(\frac{u+v}{2}\Big) \le \frac{1}{2}\Xi_1(u) + \frac{1}{2}\Xi_1(v) \quad \forall u, v \in E.$$

In the same manner we can prove that the functional $\Xi_2 : E \to \mathbb{R}$ defined by

$$\Xi_2(u) = \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} dx$$

is convex. Thereby $\Xi_1 + \Xi_2$ is convex on $E$. Next, we propose to show that the functional $\Xi_1 + \Xi_2$ is weakly lower semicontinuous on $E$. Making use of Corollary III.8 in [1] we ascertain that is enough to demonstrate the lower semicontinuity of $\Xi_1 + \Xi_2$. Therefor, we fix $u \in E$ and $\varepsilon > 0$. Let $v \in E$ be arbitrary. By convexity of $\Xi_1 + \Xi_2$ and Hölder type inequality (2.1) we have

$$\begin{aligned}
&\Xi_1(v) + \Xi_2(v)\\
&\geq \Xi_1(u) + \Xi_2(u) + \langle \Xi_1'(u) + \Xi_2'(u), v - u\rangle\\
&= \Xi_1(u) + \Xi_2(u) + \int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \partial_{x_i}(v-u) dx\\
&\quad + \int_\Omega \sum_{i=1}^N |u|^{p_i(x)-2} u(v-u) dx\\
&\geq \Xi_1(u) + \Xi_2(u) - \int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)-1} |\partial_{x_i}(v-u)| dx - \int_\Omega \sum_{i=1}^N |u|^{p_i(x)-1} |v-u| dx\\
&\geq \Xi_1(u) + \Xi_2(u) - 2\Big( \sum_{i=1}^N \big| |\partial_{x_i} u|^{p_i(x)-1} \big|_{\frac{p_i(\cdot)}{p_i(\cdot)-1}} |\partial_{x_i}(v-u)|_{p_i(\cdot)}\\
&\quad + \sum_{i=1}^N \big| |u|^{p_i(x)-1} \big|_{\frac{p_i(\cdot)}{p_i(\cdot)-1}} |v-u|_{p_i(\cdot)} \Big)\\
&= \Xi_1(u) + \Xi_2(u) - 2\sum_{i=1}^N \Big( \big| |\partial_{x_i} u|^{p_i(x)-1} \big|_{\frac{p_i(\cdot)}{p_i(\cdot)-1}} |\partial_{x_i}(v-u)|_{p_i(\cdot)}\\
&\quad + \big| |u|^{p_i(x)-1} \big|_{\frac{p_i(\cdot)}{p_i(\cdot)-1}} |v-u|_{p_i(\cdot)} \Big)\\
&\geq \Xi_1(u) + \Xi_2(u) - 2\sum_{i=1}^N \Big( \big| |\partial_{x_i} u|^{p_i(x)-1} \big|_{\frac{p_i(\cdot)}{p_i(\cdot)-1}}\\
&\quad + \big| |u|^{p_i(x)-1} \big|_{\frac{p_i(\cdot)}{p_i(\cdot)-1}} \Big) \big( |\partial_{x_i}(v-u)|_{p_i(\cdot)} + |v-u|_{p_i(\cdot)} \big)\\
&\geq \Xi_1(u) + \Xi_2(u) - C\sum_{i=1}^N \big( |\partial_{x_i}(v-u)|_{p_i(\cdot)} + |v-u|_{p_i(\cdot)} \big)\\
&= \Xi_1(u) + \Xi_2(u) - C\|v-u\|_{\vec{p}(\cdot)}
\end{aligned}$$

for all $v \in E$ with $\|v-u\|_{\vec{p}(\cdot)} \leq \varepsilon/C$, where $C > 0$ is a constant, whence we obtain the weakly lower semicontinuity of $\Xi_1 + \Xi_2$ on $E$; that is,

$$\liminf_{n\to\infty} (\Xi_1 + \Xi_2)(u_n) \geq (\Xi_1 + \Xi_2)(u_0). \tag{4.21}$$

Passing to the limit in (4.16) and making use of (4.17), (4.19) and (4.21) it follows that

$$\liminf_{n\to\infty} T_\lambda^1(u_n) \geq T_\lambda^1(u_0)$$

meaning that Lemma 4.3 holds. $\square$

Then on the basis of these three lemmas above mentioned, we are going to show what we have proposed to Step 3. We fix $\lambda \in (\lambda_1, \infty)$. In the light of coercivity and weakly lower semicontinuity of $T^1_\lambda$ we can use [13, Theorem 1.2] to obtain the existence of a global minimum point of $T^1_\lambda$, $u_\lambda \in E$. Ultimately, to complete Step 3 we have to show only that $u_\lambda$ is not trivial. In truth, we have $\lambda_1 = \inf_{u\in E\setminus\{0\}} \frac{J_1(u)}{I_1(u)}$ and $\lambda_1 < \lambda$ whence we obtain that there is a $v_\lambda \in E$ so that $T^1_\lambda(v_\lambda) < 0$. Thus

$$\inf_E T^1_\lambda < 0,$$

and so we can conclude that $u_\lambda$ is a nontrivial critical point of $T^1_\lambda$ or, in other words, $\lambda$ is an eigenvalue of problem (1.1) leading to Step 3 is verified.

**Step 4.** In this last step we show that $\lambda_1$ is an eigenvalue of problem (1.1). First of all we prove two lemmas.

**Lemma 4.4.** *We have that*

$$\lim_{\|u\|_{\vec{p}(\cdot)}\to 0} \frac{J_0(u)}{I_0(u)} = +\infty.$$

*Proof.* We fix $s \in \mathbb{R}$ such that

$$r^+ < s < q^- < P^*_-.$$

It should be noticed that from the condition (3.2), we have $P_{-,\infty} = P^*_-$. Thereby $s < P_{-,\infty}$ and so $E \hookrightarrow L^s(\Omega)$ continuously, whence we obtain the existence of a positive constant $C$ such that

$$|u|_s \le C\|u\|_{\vec{p}(\cdot)} \quad \forall u \in E. \tag{4.22}$$

Without loss of generality we consider that $\|u\|_{\vec{p}(\cdot)} < 1$ for any $u \in E$. By applying the Jensen's inequality to the convex function $a : \mathbb{R}^+ \to \mathbb{R}^+$, $a(t) = t^{P^-_-}$, $P^-_- \ge 2$, using $\alpha_i$ and $\beta_i$ defined in Lema 4.2, and by (4.3) and (4.22) we infer that

$$\begin{aligned}
\frac{J_0(u)}{I_0(u)} &= \frac{\int_\Omega \sum_{i=1}^N \left(|\partial_{x_i} u|^{p_i(x)} + |u|^{p_i(x)}\right)dx + \int_\Omega |u|^{q(x)}dx}{\int_\Omega g(x)|u|^{r(x)}dx} \\
&\ge \frac{\frac{\|u\|^{P^-_-}_{\vec{p}(\cdot)}}{(2N)^{P^-_- -1}} - 2N}{|g|_{\frac{s}{s-r^-}}|u|^{r^-}_s + |g|_{\frac{s}{s-r^+}}|u|^{r^+}_s} \\
&\ge \frac{\|u\|^{P^-_-}_{\vec{p}(\cdot)} - (2N)^{P^-_-}}{(2N)^{P^-_- -1}\left(|g|_{\frac{s}{s-r^-}}C^{r^-}\|u\|^{r^-}_{\vec{p}(\cdot)} + |g|_{\frac{s}{s-r^+}}C^{r^+}\|u\|^{r^+}_{\vec{p}(\cdot)}\right)}.
\end{aligned}$$

Given that $r^+ \ge r^- > P^-_-$ and passing to the limit in the above inequality it is obvious that $\lim_{\|u\|_{\vec{p}(\cdot)}\to\infty} \frac{J_0(u)}{I_0(u)} = +\infty$ occurs, and so the Lemma 4.4 is proved. $\square$

**Lemma 4.5.** *Suppose that $(u_n)$ converges weakly to $u$ in $E$. Then we have*

$$\lim_{n\to\infty} \langle I'_1(u_n), u_n - u\rangle = 0. \tag{4.23}$$

*Proof.* We define $\Phi : E \to \mathbb{R}$ by

$$\Phi(v) = \int_\Omega g(x)|u_n|^{r(x)-2}u_n v\,dx.$$

Is easily seen that $\Phi$ is linear and we want to show that is also continuous. Indeed, by Hölder type inequality (2.1) we have

$$\begin{aligned} |\Phi(v)| &= \Big|\int_\Omega g(x)|u_n|^{r(x)-2}u_n v\,dx\Big| \le \int_\Omega \Big|g(x)|u_n|^{r(x)-2}u_n v\Big|\,dx \\ &= \int_\Omega g(x)|u_n|^{r(x)-1}|v|dx \le |g|_\infty \int_\Omega |u_n|^{r(x)-1}|v|dx \\ &\le 2|g|_\infty \big||u_n|^{r(x)-1}\big|_{\frac{r(\cdot)}{r(\cdot)-1}} |v|_{r(\cdot)}. \end{aligned} \tag{4.24}$$

We have $E \hookrightarrow L^{r(\cdot)}(\Omega)$ continuously, thus there exists a constant $C > 0$ such that

$$|v|_{r(\cdot)} \le C\|v\|_{\vec{p}(\cdot)} \quad \forall v \in E.$$

By the above inequality and (4.24) we obtain the continuity of $\Phi$. Then $\Phi(u_n) \to \Phi(u)$, or

$$\lim_{n\to\infty} \int_\Omega g(x)|u_n|^{r(x)-2}u_n(u_n - u)dx = 0$$

which is exactly (4.23). $\square$

Now, we return to the proof of Step 4. Let $\lambda_n \searrow \lambda_1$. Considering the Step 3 we infer that for any $n$ there exists $u_n \in E \setminus \{0\}$ so that

$$\langle J_1'(u_n), v\rangle = \lambda_n \cdot \langle I_1'(u_n), v\rangle \quad \forall v \in E. \tag{4.25}$$

Making the substitution $v = u_n$ in (4.25) we obtain

$$J_0(u_n) = \lambda_n \cdot I_0(u_n), \tag{4.26}$$

and passing to the limit as $n \to \infty$ we find that

$$\lim_{n\to\infty} (J_0(u_n) - \lambda_n \cdot I_0(u_n)) = 0. \tag{4.27}$$

Now, if we suppose that $\|u_n\|_{\vec{p}(\cdot)} \to \infty$, then reasoning as in the proof of Lemma 4.2 we reach a contradiction with (4.27). Hence, the sequence $(u_n)$ is bounded in $E$. On the other hand, we know that $E$ is a reflexive Banach space, and due to this reason we deduce that there is an element $u \in E$ so that, up to a subsequence, labeled again $(u_n)$, we have that $u_n \rightharpoonup u$ in $E$. Therefore, (4.23) occurs.

To proceed we use the inequality

$$\left(|\xi_i|^{r_i-2}\xi_i - |\psi_i|^{r_i-2}\psi_i\right)(\xi_i - \psi_i) \ge 2^{-r_i}|\xi_i - \psi_i|^{r_i} \quad \forall \xi_i, \psi_i \in \mathbb{R},\ \forall r_i \ge 2 \tag{4.28}$$

(see [12, inequality (2.2)]). Replacing in the above inequality $\xi_i$ by $\partial_{x_i}u_n$, $\psi_i$ by $\partial_{x_i}u$ and $r_i$ by $p_i(x)$ , and then $\xi_i$ by $u_n$, $\psi_i$ by $u$ and $r_i$ by $p_i(x)$ respectively, for each $i \in \{1, \dots, N\}$ and $x \in \Omega$, then adding the two inequalities obtained, and taking into account that $2^{p_i(x)}$ is bounded, it results that there exists $L_1 > 0$ such that

$$
\begin{aligned}
& L_1 \int_\Omega \Big( |\partial_{x_i} u_n - \partial_{x_i} u|^{p_i(x)} + |u_n - u|^{p_i(x)} \Big) dx \\
& \le \int_\Omega \Big( |\partial_{x_i} u_n|^{p_i(x)-2} \partial_{x_i} u_n - |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \Big) (\partial_{x_i} u_n - \partial_{x_i} u) dx \\
& \quad + \int_\Omega \Big( |u_n|^{p_i(x)-2} u_n - |u|^{p_i(x)-2} u \Big) (u_n - u) dx \quad \forall i \in \{1, \dots, N\}.
\end{aligned} \tag{4.29}
$$

Also, using again inequality (4.28), we find that there is $L_2 > 0$ such that

$$
L_2 \int_\Omega |u_n - u|^{q(x)} dx \le \int_\Omega \Big( |u_n|^{q(x)-2} u_n - |u|^{q(x)-2} u \Big) (u_n - u) dx. \tag{4.30}
$$

Summing from 1 to $N$ in (4.29) and adding the inequality which we obtain with (4.30) we can see that

$$
\begin{aligned}
& L_1 \int_\Omega \sum_{i=1}^N \Big( |\partial_{x_i} u_n - \partial_{x_i} u|^{p_i(x)} + |u_n - u|^{p_i(x)} \Big) dx \\
& \le \int_\Omega \sum_{i=1}^N \Big( |\partial_{x_i} u_n|^{p_i(x)-2} \partial_{x_i} u_n - |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \Big) (\partial_{x_i} u_n - \partial_{x_i} u) dx \\
& \quad + \int_\Omega \sum_{i=1}^N \Big( |u_n|^{p_i(x)-2} u_n - |u|^{p_i(x)-2} u \Big) (u_n - u) dx \\
& \quad + \int_\Omega \Big( |u_n|^{q(x)-2} u_n - |u|^{q(x)-2} u \Big) (u_n - u) dx \\
& = \langle J_1'(u_n) - J_1'(u), u_n - u \rangle.
\end{aligned}
$$

Taking into account (4.23) and (4.25) and that $(u_n)$ converges weakly to $u$ in $E$, we arrive at

$$
\begin{aligned}
& L_1 \int_\Omega \sum_{i=1}^N \Big( |\partial_{x_i} u_n - \partial_{x_i} u|^{p_i(x)} + |u_n - u|^{p_i(x)} \Big) dx \\
& \le \langle J_1'(u_n) - J_1'(u), u_n - u \rangle \\
& = \langle J_1'(u_n), u_n - u \rangle - \langle J_1'(u), u_n - u \rangle \\
& \le |\langle J_1'(u_n), u_n - u \rangle| + |\langle J_1'(u), u_n - u \rangle| \\
& = \lambda_n |\langle I_1'(u_n), u_n - u \rangle| + |\langle J_1'(u), u_n - u \rangle| \to 0,
\end{aligned}
$$

as $n \to \infty$. By (2.4) we deduce that

$$
\sum_{i=1}^N \big( |\partial_{x_i} u_n - \partial_{x_i} u|_{p_i(\cdot)} + |u_n - u|_{p_i(\cdot)} \big) \to 0
$$

or equivalently

$$
\|u_n - u\|_{p_i(\cdot)} \to 0,
$$

that is, $u_n \to u$ in $E$. Passing to the limit, as $n \to \infty$ in (4.25), yields

$$
\langle {T^1_{\lambda_1}}'(u), v \rangle = 0 \quad \forall v \in E,
$$

which means that $u$ is a critical point for $T^1_{\lambda_1}$. We intend to show that $u \neq 0$ and this fact would lead us to $\lambda_1$ is an eigenvalue for (1.1). To this end we suppose that

$u = 0$. Then $u_n \to 0$ in $E$, that is to say, $\|u_n\|_{\vec{p}(\cdot)} \to 0$. Applying Lemma 4.4 we obtain

$$\lim_{\|u_n\|_{\vec{p}(\cdot)} \to 0} \frac{J_0(u_n)}{I_0(u_n)} = +\infty. \tag{4.31}$$

But, if we pass to the limit as $n \to \infty$ in (4.26) we obtain

$$\lim_{n\to\infty} \frac{J_0(u_n)}{I_0(u_n)} = \lambda_1,$$

which is a contradiction to (4.31). So the assumption made is false, accordingly, $u \neq 0$ and thus $\lambda_1$ is an eigenvalue for problem (1.1) and Step 4 is verified.

From Steps 2–4 we obtain $\lambda_0 \leq \lambda_1$ and thereby the proof of Theorem 3.1 is complete.

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