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SIGNED RADIAL SOLUTIONS FOR A WEIGHTED p-SUPERLINEAR PROBLEM

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Abstract. We study the existence of one-signed radial solutions for weighted semipositone problems where Δ_p operator is involved and the nonlinearity is p-superlinear at infinity and has only two zeros. We establish the existence of at least two one-signed solutions when the weight is small enough.

1. Introduction

Let us consider the existence of one-signed radial solutions for the boundary-value problem

$$\begin{gathered} -\Delta_p u = K(\|x\|)f(u), \quad x \in B_1(0) \\ u = 0, \quad \|x\| = 1, \end{gathered} \tag{1.1}$$

where $B_1(0) \subset \mathbb{R}^N$ is the unit ball and $2 \le p < N$.

The existence of one-signed radial solutions to (1.1) is equivalent to the existence of one-signed solutions to the ordinary differential equation

$$\begin{gathered} [\varphi_p(u'(r))]' + \frac{N-1}{r}\varphi_p(u'(r)) + K(r)f(u(r)) = 0, \quad 0 < r < 1 \\ u'(0) = 0, \quad u(1) = 0, \end{gathered} \tag{1.2}$$

where $\varphi_p : \mathbb{R} \to \mathbb{R}$ is defined as $\varphi_p(s) = |s|^{p-2}s$, $s \neq 0$, $\varphi_p(0) = 0$.

We assume that nonlinearity f satisfies the following hypotheses:

- (F1) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is differentiable, $f'(t) \ge 0$ for all $t \le v_0 < 0$ and all $t \ge u_0 > 0$, where v_0 and u_0 are the only zeros of f.
- (F2) $f(0^-) := \lim_{t\to 0^-} f(t)$ is a positive number and $f(0^+) := \lim_{t\to 0^+} f(t)$ is a negative number.
- (F3) $\lim_{|\alpha|\to\infty} \frac{f(\alpha)}{\varphi_p(\alpha)} = +\infty$.
- (F4) There exist constants $k \in (0,1)$ and $\theta > N-p$ such that for all $\delta \ge \theta$
$$\lim_{|\alpha|\to\infty} \Big(\frac{\varphi_p(\alpha)}{f(\alpha)}\Big)^{\frac{N}{p}} \Big(\delta F(k\alpha) - \frac{N-p}{p}\alpha f(\alpha)\Big) = +\infty,$$
where $F(t) = \int_0^t f(s)ds$.

For the weight K, hereafter we will assume that

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(K1) $K \in C([0,1], \mathbb{R}^+)$ and K is differentiable in $(0,1)$.

(K2) $r \mapsto N + r\frac{K'(r)}{K(r)} > \theta$, is non-increasing in $(0,1)$ and $\lim_{r\to 0^+} \frac{K'(r)}{K(r)}$ exists in $\mathbb{R}$.

A function satisfying the statements (K1) and (K2) will be called an admissible weight.

The aim of this article is to prove that under hypotheses (F1)–(F4) for the nonlinearity and (K1)–(K2) for the weight, problem (1.2) has at least two solutions, provided that the weight is small enough. Moreover, one of them is positive and the other one is negative. In order to prove the existence of positive solution, we modify our nonlinearity in the following way

$$f^+(t) := \begin{cases} f(t) & \text{if } t > 0 \\ f(0^+) & \text{if } t = 0 \\ 0 & \text{if } t < 0. \end{cases}$$

Similar modifications vanishing the positive part of f give us a negative solution.

The case $p = 2$ was studied by Castro and Shivaji in [2]. Recently, Hakimi and Zertiti, [6], following the ideas in [2] obtained existence of positive solutions for a more general nonlinearity than Castro and Shivaji. Both works considered a constant weight in the semilinear case. We emphasize that in this work we deal with the case $2 \le p < N$ and a nonconstant weight. Therefore, it is a generalization of the previous results.

There are several studies related to radial solutions involving p-Laplacian problems and a lot of techniques have been used. Most of them are interested in the existence, nonexistence, and multiplicity of positive radial solutions. For instance, in [7] the authors considered the equation of problem (1.1) with $f(u) = u^q, q > p-1$, in exterior domains. They employed a global continuation theorem and fixed point index theory based on a weighted space as the underlying space. By using this approach they obtained multiplicity of positive solutions depending on a certain real parameter μ. A one dimensional weighted p-Laplacian problem is presented in [8]. Sharp conditions for the existence of solutions with prescribed numbers of zeros in terms of the ratio $f(s)/s^{p-1}$ at zero and at infinity were established there. Their technique was based on the shooting method together with the qualitative theory for half-linear differential equations. Other results can be found in [1, 3, 4, 5, 9] and some references therein.

Our main tool for solving problem (1.2) is the shooting method. Hence, we start considering the auxiliary initial value problem

$$\begin{gathered} [\varphi_p(u'(r))]' + \frac{N-1}{r}\varphi_p(u'(r)) + K(r)f(u(r)) = 0, \quad 0 < r < 1 \\ u'(0) = 0, \quad u(0) = \alpha. \end{gathered} \tag{1.3}$$

It can be shown (see appendix) that this problem has a unique solution, usually denoted by $u(\cdot, \alpha, K)$, which depends continuously on the initial data α.

We have the following equivalent integral formulas to problem (1.3):

$$u(s) = \alpha - \int_0^s \varphi_{p'}\Big(r^{1-N}\int_0^r t^{N-1}K(t)f(u(t))dt\Big)dr, \quad 0 \le s \le 1, \tag{1.4}$$

$$\varphi_p(u')(r) = -r^{1-N}\int_0^r t^{N-1}K(t)f(u(t))dt, \quad 0 \le r \le 1. \tag{1.5}$$

Our main result reads as follows.

Theorem 1.1. *Under hypotheses* (F1)–(F4) *and* (K1)–(K2)*, there exists a positive number* λ_0 *such that if* $\|K\|_\infty < \lambda_0$*, then problem* (1.1) *has at least two radial solutions. Moreover, one of them is positive, decreasing and has negative radial derivative in* $\|x\| = 1$*. The second solution is negative, increasing and has positive radial derivative in* $\|x\| = 1$*.*

This paper is organized in the following way: In Section 2 we present some facts which are useful for showing Theorem 1.1. Next, in Section 3, we prove the main result and we exhibit some examples of functions f and K satisfying conditions (F1)–(F4) and (K1)–(K2). Finally, for convenience of the reader, we present in the appendix a Pozohaev type identity and some proofs about the existence and uniqueness of (1.3).

2. Preliminary results

For the rest of this article, we consider problem (1.2) with nonlinearity f^+ instead of f. Let $u_1 \in (u_0, +\infty)$ be the unique zero of F^+ where $F^+(t) := \int_0^t f^+(s)\,ds$. Fix an admissible weight K and set

$$\eta := \min_{r\in[0,1]} K(r); \quad \bar\lambda := \max_{r\in[0,1]} K(r),$$

$$\delta_0 := \min_{r\in[0,1]} \big\{N + r\frac{K'(r)}{K(r)}\big\} = N + \frac{K'(1)}{K(1)},$$

$$\delta_1 := \max_{r\in[0,1]} \big\{N + r\frac{K'(r)}{K(r)}\big\} = N.$$

For fixed $\alpha \ge u_1/k$, there exists $t_0 = t_0(\alpha) \in (0,1)$ such that $u(t_0,\alpha,K) = k\alpha$ and $k\alpha \le u(t,\alpha,K) \le \alpha$ for all $t \in [0,t_0)$, where $u(\cdot,\alpha,K)$ is the unique solution of problem (1.3). Moreover, t_0 satisfies the estimate (2.1) which, in particular, shows that $t_0(\alpha) \to 0$ as $\alpha \to +\infty$. Indeed, as f^+ is increasing on $[u_0,+\infty)$, then for $t \in [0,t_0]$

$$t^{N-1}K(t)f^+(k\alpha) \le t^{N-1}K(t)f^+(u(t,\alpha,K)) \le t^{N-1}K(t)f^+(\alpha).$$

Hence,

$$-r^{1-N}\int_0^r t^{N-1}K(t)f^+(\alpha)dt \le \varphi_p(u'(t,\alpha,k)) \le -r^{1-N}\int_0^r t^{N-1}K(t)f^+(k\alpha)dt,$$

which in turn, implies that

$$-\varphi_{p'}\Big(\bar\lambda\frac{f^+(\alpha)}{N}\Big)r^{p'-1} \le u'(t,\alpha,k) \le -\varphi_{p'}\Big(\eta\frac{f^+(k\alpha)}{N}\Big)r^{p'-1},$$

for all $t \in [0,t_0]$. Here p' denotes the conjugate of p. After integration on $[0,t_0]$ we obtain

$$-\varphi_{p'}\Big(\bar\lambda\frac{f^+(\alpha)}{N}\Big)\frac{t_0^{p'}}{p'} \le \alpha(k-1) \le -\varphi_{p'}\Big(\eta\frac{f^+(k\alpha)}{N}\Big)\frac{t_0^{p'}}{p'}.$$

These inequalities yield the following estimate for t_0

$$C_1\Big(\frac{\varphi_p(\alpha)}{\bar\lambda f^+(\alpha)}\Big)^{1/p} \le t_0 \le C_1\Big(\frac{\varphi_p(\alpha)}{\eta f^+(k\alpha)}\Big)^{1/p}, \tag{2.1}$$

where $C_1 := (p'(1-k)N^{p'-1})^{1/p'} > 0$.

2.1. **Pohozaev Identity.** In this subsection, we present a Pohozaev identity as well as some consequences. The proof of this identity is quite standard but it is presented in the appendix for the sake of completeness. Let us define the Energy associated to problem (1.3) by

$$E(t,\alpha,K) := \frac{|u'(t,\alpha,K)|^p}{p'K(t)} + F^+(u(t,\alpha,K)).$$

Also, we define

$$H(t,\alpha,K) := tK(t)E(t,\alpha,K) + \frac{N-p}{p}\varphi_p(u'(t,\alpha,K))u(t,\alpha,K).$$

Suppose that $u(\cdot,\alpha,K)$ is a solution of (1.3). Then, a Pohozaev type identity takes place

$$\begin{aligned} &t^{N-1}H(t,\alpha,K) - s^{N-1}H(s,\alpha,K) \\ &= \int_s^t r^{N-1}K(r)\Big[\Big(N + r\frac{K'(r)}{K(r)}\Big)F^+(u) - \frac{N-p}{p}f^+(u)u\Big]dr, \end{aligned} \tag{2.2}$$

for $0 \le s \le t \le 1$. We shall use this version of Pohozaev identity as follows. For $s = 0$ and $t = t_0$, we get

$$t_0^{N-1}H(t_0,\alpha,K) = \int_0^{t_0} r^{N-1}K(r)\Big[\Big(N + r\frac{K'(r)}{K(r)}\Big)F^+(u) - \frac{N-p}{p}f^+(u)u\Big]dr.$$

Since f^+ and F^+ are nonnegative and increasing functions on the interval $[u_1,\infty)$, then for all $r \in [0,t_0]$,

$$\begin{aligned} \Big(N + r\frac{K'(r)}{K(r)}\Big)F^+(u) - \frac{N-p}{p}uf^+(u) &\ge \delta_0 F^+(u) - \frac{N-p}{p}uf^+(u) \\ &\ge \delta_0 F^+(k\alpha) - \frac{N-p}{p}\alpha f^+(\alpha). \end{aligned}$$

In consequence,

$$t_0^{N-1}H(t_0,\alpha,K) \ge \frac{\eta}{N}\Big(\delta_0 F^+(k\alpha) - \frac{N-p}{p}\alpha f^+(\alpha)\Big)t_0^N.$$

From this and (2.1) we find that

$$t_0^{N-1}H(t_0,\alpha,K) \ge \frac{\eta C_1^N}{N\bar{\lambda}^{N/p}}\Big(\delta_0 F^+(k\alpha) - \frac{N-p}{p}\alpha f^+(\alpha)\Big)\Big(\frac{\varphi_p(\alpha)}{f^+(\alpha)}\Big)^{N/p}. \tag{2.3}$$

We claim that for each number $\delta \ge \theta$, there is a positive constant B_δ, such that

$$\delta F^+(s) - \frac{N-p}{p}sf^+(s) \ge -B_\delta, \quad \text{for all } s \in \mathbb{R}. \tag{2.4}$$

In fact, (F4) guarantees existence of $C_\delta > 0$ satisfying

$$\delta F^+(s) - \frac{N-p}{p}sf^+(s) \ge \delta F^+(ks) - \frac{N-p}{p}sf^+(s) \ge 0,$$

for all $s > C_\delta$ and all $s < 0$. On the other hand, if we set $M_\delta := \sup_{s\in[0,C_\delta]}|f^+(s)|$, then

$$\begin{aligned} \delta F^+(s) - \frac{N-p}{p}sf^+(s) &= \int_0^s \Big(\delta f^+(t) - \frac{N-p}{p}f^+(s)\Big)dt \\ &\ge -M_\delta\int_0^s \Big(\delta + \frac{N-p}{p}\Big)dt \end{aligned}$$

$$
\begin{aligned}
&= -M_\delta s(\delta + \frac{N-p}{p}) \\
&\geq -M_\delta C_\delta \Big(\delta + \frac{N-p}{p}\Big) =: -B_\delta.
\end{aligned}
$$

Now, replacing t_0 by s in Pohozaev identity (2.2), and using the estimate (2.4) with δ_0 and δ_1 we obtain

$$
\begin{aligned}
&t^{N-1}H(t,\alpha,K) \\
&= t_0^{N-1}H(t_0,\alpha,K) + \int_A r^{N-1}K(r)\Big[(N + r\frac{K'(r)}{K(r)})F^+(u) - \frac{N-p}{p}f^+(u)u\Big]dr \\
&\quad + \int_B r^{N-1}K(r)\Big[(N + r\frac{K'(r)}{K(r)})F^+(u) - \frac{N-p}{p}f^+(u)u\Big]dr \\
&\geq t_0^{N-1}H(t_0,\alpha,K) + \int_A r^{N-1}K(r)\Big[\delta_0 F^+(u) - \frac{N-p}{p}f^+(u)u\Big]dr \\
&\quad + \int_B r^{N-1}K(r)\Big[\delta_1 F^+(u) - \frac{N-p}{p}f^+(u)u\Big]dr \\
&\geq t_0^{N-1}H(t_0,\alpha,K) - B_{\delta_0}\int_A r^{N-1}K(r)dr - B_{\delta_1}\int_B r^{N-1}K(r)dr \\
&\geq t_0^{N-1}H(t_0,\alpha,K) - 2\frac{\bar\lambda B_{\delta_1}}{N}.
\end{aligned}
$$

Here $A := \{r \in [t_0,t] : F^+(u(r,\alpha,K) \geq 0)\}$ and $B := \{r \in [t_0,t] : F^+(u(r,\alpha,K) < 0)\}$. Then we reach the estimate

$$
t^{N-1}H(t,\alpha,K) \geq t_0^{N-1}H(t_0,\alpha,K) - 2\bar\lambda M_N C_N\Big(1 + \frac{N-p}{Np}\Big), \tag{2.5}
$$

where we have used the fact that $\delta_1 = N$ (it is remarkable that δ_1 does not depend on K).

3. Main Result

In this section we shall prove our main theorem. Before, we will establish three preliminary Lemmas.

Lemma 3.1. *There exists a positive real number λ_2 with the following property. For every admissible weight K, with $\|K\|_\infty \equiv \bar\lambda < \lambda_2$, there is a real number $\underline{\alpha} > u_1/k$, such that for all $\alpha \geq \underline{\alpha}$ and all $t \in [0,1]$, $|u(t,\alpha,K)|^p + |u'(t,\alpha,K)|^p > 0$.*

Proof. Given a weight K with the properties (K1) and (K2), there is $\underline{\alpha} > u_1/k$ such that for all $\alpha \geq \underline{\alpha}$

$$
\Big(\delta_0 F^+(k\alpha) - \frac{N-p}{p}\alpha f^+(\alpha)\Big)\Big(\frac{\varphi_p(\alpha)}{f^+(\alpha)}\Big)^{N/p} \geq \frac{1}{\eta}.
$$

From (2.3) and (2.5) we have, for $t \geq t_0$

$$
\begin{aligned}
t^{N-1}H(t,\alpha,K) &\geq t_0^{N-1}H(t_0,\alpha,K) - 2\bar\lambda M_{\delta_1}C_{\delta_1}\Big(1 + \frac{N-p}{Np}\Big) \\
&\geq \frac{\eta C_1^N}{N\bar\lambda^{N/p}}\frac{1}{\eta} - 2\bar\lambda M_N C_N\Big(1 + \frac{N-p}{Np}\Big) \qquad (3.1)\\
&= \bar\lambda\Big(\frac{C_1^N}{N\bar\lambda^{(N-p)/p}} - 2M_N C_N(1 + \frac{N-p}{Np})\Big).
\end{aligned}
$$

Now, there exists $\lambda_2 > 0$ such that

$$\frac{C_1^N}{N\lambda^{(N-p)/p}} - 2M_N C_N(1 + \frac{N-p}{Np}) > 0, \tag{3.2}$$

for every $\lambda \in (0, \lambda_2)$. We fix a weight K so that $||K||_\infty := \bar{\lambda} < \lambda_2$. Thus, from (3.1) and (3.2) we have $H(t, \alpha, K) > 0$ for $t \in [t_0, 1]$. Then, for all $t \in [t_0, 1]$, $|u(t, \alpha, K)|^p + |u'(t, \alpha, K)|^p > 0$. Clearly, $|u(t, \alpha, K)|^p + |u'(t, \alpha, K)|^p > 0$ holds for $t \in [0, t_0]$. $\square$

Lemma 3.2. *There exists a real number $\lambda_1 \in (0, \lambda_2]$ with the following property. For every admissible weight K, with $\bar{\lambda} < \lambda_1$, we have $u(t, \underline{\alpha}, K) \geq u_0$ for all $t \in [0, 1]$. Here, $\underline{\alpha}$ is a number obtained in the preceding Lemma.*

Proof. Given a weight K, set

$$t_1 := \sup\{t \in [0, 1] : u(r, \underline{\alpha}, K) \geq u_0 \text{ for all } r \in (0, t)\}.$$

We observe that t_0 belongs to the previous set. Since f^+ is nonnegative on the interval $[u_0, \infty)$, we see from (1.5), that

$$\varphi_p(u'(t, \underline{\alpha}, K)) = -t^{1-N} \int_0^t r^{N-1} K(r) f^+(u(r)) dr \leq 0, \quad \text{for all } t \in [0, t_1].$$

Therefore, u is decreasing on $[0, t_1]$. Besides, for all $t \in [0, t_1]$,

$$\begin{aligned}
|\varphi_p(u'(t, \underline{\alpha}, K))| &\leq t^{1-N} \int_0^t r^{N-1} K(r) f^+(u(r, \underline{\alpha}, K)) dr \\
&\leq \bar{\lambda} f^+(u(0, \underline{\alpha}, K)) t^{1-N} \int_0^t r^{N-1} dr \\
&\leq \frac{\bar{\lambda} f^+(\underline{\alpha})}{N} t \\
&\leq \frac{\bar{\lambda} f^+(\underline{\alpha})}{N}.
\end{aligned}$$

Hence

$$|u'(t, \underline{\alpha}, K)| \leq \varphi_{p'}\Big(\frac{\bar{\lambda} f^+(\underline{\alpha})}{N}\Big).$$

Now, fix $\lambda_1 \leq \min\{\lambda_2, \frac{N}{f^+(\underline{\alpha})}\varphi_p(\underline{\alpha} - u_0)\}$, then $\varphi_{p'}(\lambda_1) \leq \varphi_{p'}(\frac{N}{f^+(\underline{\alpha})})(\underline{\alpha} - u_0)$. It follows that if $\bar{\lambda} < \lambda_1$, we have $|u'(r, \underline{\alpha}, K)| \leq \underline{\alpha} - u_0$ for all $t \in [0, t_1]$. An application of the mean value theorem, allows us to choose a real number $\xi \in (0, t_1)$ such that

$$u(t_1, \underline{\alpha}, K) - u(0, \underline{\alpha}, K) = u'(\xi, \underline{\alpha}, K) t_1 \geq -(\underline{\alpha} - u_0) t_1.$$

If we assume that $t_1 < 1$, then $u(t_1, \underline{\alpha}, K) > u_0$, contradicting the definition of t_1. This completes the proof. $\square$

Lemma 3.3. *For a given admissible weight K with $\bar{\lambda} < \lambda_1$, there exists $\alpha_1 \geq \underline{\alpha}$ such that $u(t, \alpha_1, K) < 0$ for some $t \in [0, 1]$.*

Proof. We argue by contradiction. Suppose that there exists a suitable weight K, such that for all $\alpha \geq \underline{\alpha}$ and all $t \in [0, 1]$, $u(t, \alpha, K) \geq 0$. Without lost of generality we can assume that $u(t, \alpha, K) > 0$ for all $t \in [0, 1)$. Let $\bar{t} = \bar{t}(\alpha)$ be the supremum of the set

$$V := \{t \in [0, 1] : u(\cdot, \alpha, K) \text{ is decreasing on } [0, t]\}.$$

V is a nonempty set because it contains t_0. On the other hand, in view of the inequalities (2.3), (2.5) and hypothesis (F4), we can fix a real number α_1 such that for all $\alpha \geq \alpha_1$ and all $t \in [0,1]$, $t^{N-1}H(t,\alpha,K) > 0$. Now, we claim that $u'(t,\alpha,K) \neq 0$ for all $t \in (0,1]$. For if $u'(t_1,\alpha,K) = 0$ for some $t_1 \in (0,1]$ then the differential equation in (1.2) would imply that $f^+(u(t_1,\alpha,K)) = 0$. Hence $u(t_1,\alpha,K) = u_0$, but

$$0 < t_1^{N-1}H(t_1,\alpha,K) = t_1^N K(t_1)F^+(u(t_1,\alpha,K)) = t_1^N K(t_1)F^+(u_0) < 0,$$

which is a contradiction. We have stated that if $\alpha \geq \alpha_1$ then $\bar{t} = 1$. Let v be a positive solution to problem

$$\begin{gathered} (\varphi_p(v'))' + \frac{N-1}{r}\varphi_p(v') = -\mu\varphi_p(v), \quad 0 < r < \rho := \frac{1}{p+1} \\ v(0) = 1, \quad v'(0) = 0, \quad v(\rho) = 0, \quad v(r) > 0 \quad \text{for } 0 < r < \rho. \end{gathered} \tag{3.3}$$

Notice that $v'(\rho) < 0$, since

$$\varphi_p(v'(\rho)) = -\mu\rho^{1-N}\int_0^\rho r^{N-1}\varphi_p(v)dr < 0.$$

Hypothesis (F3) is now used to assure the existence of $\alpha_0 > u_1/k$ such that

$$\frac{f^+(x)}{\varphi_p(x)} \geq \mu/\bar{\lambda}, \quad \text{for all } x \geq \alpha_0. \tag{3.4}$$

From the corresponding integral formulas for the solutions u and v (cf. (1.5) and (3.3)) we have

$$[r^{N-1}\varphi_p(u')]'\varphi_p(v) = -r^{N-1}K(r)f^+(u)\varphi_p(v), \tag{3.5}$$

$$[r^{N-1}\varphi_p(v')]'\varphi_p(u) = -\mu r^{N-1}\varphi_p(v)\varphi_p(u). \tag{3.6}$$

Let t_1 be the supremum of the set $A := \{t \in [0,\rho] : v'u(r) \leq u'v(r) \text{ for all } r \in (t,\rho]\}$, which is a nonempty set due to the fact that $0 < -v'u(\rho) + u'v(\rho)$. Certainly

$$v'u(t_1) = u'v(t_1). \tag{3.7}$$

We claim that there exists a real number t on the interval $[t_1,\rho]$ that satisfies the inequality $u(t,\alpha,K) < \alpha_0$. The proof of this claim will be carried out arguing by contradiction. Assume that $u(t,\alpha,K) \geq \alpha_0$ for all $t \in [t_1,\rho]$. Integrating by parts (3.5) and (3.6), subtracting the resulting equations and taking into account (3.7), we see that

$$\begin{aligned} -\rho^{N-1}\varphi_p(v'u)(\rho) &= (p-1)\int_{t_1}^\rho r^{N-1}(|u'v|^{p-2} - |v'u|^{p-2})u'v'dr \\ &\quad + \int_{t_1}^\rho r^{N-1}\Big(\mu - K(r)\frac{f^+(u)}{\varphi_p(u)}\Big)\varphi_p(uv)dr. \end{aligned} \tag{3.8}$$

Since $|u'v|(t) \leq |uv'|(t)$ for all $t \in (t_1,\rho)$ and $p-2 \geq 0$, then

$$|u'v|^{p-2}(t) - |uv'|^{p-2}(t) \leq 0.$$

In consequence, the first term of (3.8) is nonpositive. On the other hand, from (3.4) we see that

$$\mu - \bar{\lambda}\frac{f^+(u)}{\varphi_p(u)} \leq 0.$$

Hence, the second term of (3.8) is also nonpositive. This is impossible since $v'u(\rho) < 0$. This proves the claim. Therefore, $u(t_2, \alpha, K) = \alpha_0$ for some $t_2 \in (0, \rho)$. Since $\alpha > u_0$ and u is decreasing, the estimate

$$u(t, \alpha, K) \leq \alpha_0, \text{ for all } t \in [t_2, 1] \tag{3.9}$$

holds. Because F^+ is increasing on $[u_0, \infty)$ and $u(t, \alpha, K) \geq \alpha k > u_1$ for all $t \in [0, t_0]$ then

$$E(t, \alpha, K) \geq F^+(u(t, \alpha, K)) \geq F^+(k\alpha).$$

On the other hand, for $t \in (t_0, 1]$, since $u(t)u'(t) \leq 0$, then

$$\begin{aligned} t^N K(t) E(t, \alpha, K) &\geq t^{N-1} H(t, \alpha, K) \\ &\geq \frac{\eta C_1^N}{N \bar{\lambda}^{N/p}} \Big(\delta_0 F^+(k\alpha) - \frac{N-p}{p} \alpha f^+(\alpha) \Big) \Big(\frac{\varphi_p(\alpha)}{f^+(\alpha)} \Big)^{\frac{N}{p}} \\ &\quad - 2\bar{\lambda} M_N C_N \Big(1 + \frac{N-p}{Np} \Big). \end{aligned}$$

Thus, $K(t)E(t, \alpha, K) \to +\infty$ as $\alpha \to +\infty$, uniformly in $t \in [0, 1]$. Set $\alpha \geq \alpha_0$ such that

$$K(t)E(t, \alpha, K) \geq \bar{\lambda} F^+(\alpha_0) + (p')^{p-1} \alpha_0^p \quad \text{for all } t \in [0, 1].$$

Relation (3.9) implies $F^+(\alpha_0) - F^+(u(t, \alpha, K)) \geq 0$ for all t in $[t_2, 1]$, thus

$$|u'(t, \alpha, K)| \geq p' \alpha_0, \quad \text{for all } t \in [t_2, 1].$$

Define $\tau = t_2 + \frac{1}{p'}$ and observe that $(t_2, \tau) \subseteq (0, 1)$. The mean value theorem applied to u on the interval $[t_2, \tau]$, leads to the equation

$$u(\tau, \alpha, K) - u(t_2, \alpha, K) = u'(\xi, \alpha, K) \frac{1}{p'},$$

for some $\xi \in (t_2, \tau)$. On the other hand, $u(t_2, \alpha, K) = \alpha_0$ and $u'(\xi, \alpha, K)\frac{1}{p'} \leq -\alpha_0$, thus $u(\tau, \alpha, K) \leq 0$, which is absurd. □

Proposition 3.4. *Under hypotheses* (F1)–(F4) *and* (K1)–(K2)*, there exists a positive number λ_0 such that if $\|K\|_\infty < \lambda_0$ then problem* (1.1) *has at least one positive decreasing radial solution, with radial negative derivative in $\|x\| = 1$.*

Proof. Set $\lambda_0 = \lambda_1$ from Lemma 3.2 and $\underline{\alpha}$ given by Lemma 3.1. According to Lemma 3.3, there exists $\alpha_1 > \underline{\alpha}$ with its corresponding solution being negative in some point on $[0, 1]$. Let $\tilde{\alpha}$ be the supremum of the set $\mathcal{A}$ defined as

$$\{\alpha \in [\underline{\alpha}, \alpha_1] : u(t, \alpha, K) \geq 0 \text{ for all } t \in [0, 1]\}.$$

This supremum makes sense because of $\underline{\alpha}$ belongs to $\mathcal{A}$ (which is implied by Lemma 3.2). Due to the continuous dependence of u on α, $\mathcal{A}$ is closed. Thus, $\tilde{\alpha}$ belongs to $\mathcal{A}$ and therefore $u(t, \tilde{\alpha}, K) \geq 0$, for all $t \in [0, 1]$. Moreover, $\tilde{\alpha} < \alpha_1$. Now, we will see that $u(\cdot, \tilde{\alpha}, K)$ is a solution needed.

(i) $u(t, \tilde{\alpha}, K) > 0$ for all $t \in [0, 1)$. Arguing by contradiction, if $u(\tau, \tilde{\alpha}, K) = 0$ for some $\tau \in (0, 1)$, then by Lemma 3.1, $u'(\tau, \tilde{\alpha}, K) \neq 0$. Hence, there exists a $\tau_1 \in (0, 1)$, such that $u(\tau_1, \tilde{\alpha}, K) < 0$. This is not possible.

(ii) $u(1, \tilde{\alpha}, K) = 0$. If $u(1, \tilde{\alpha}, K) > 0$, then $u(\cdot, \tilde{\alpha}, K) > 0$ on the compact set $[0, 1]$. For the continuous dependence in the initial data, there is some α, $\tilde{\alpha} < \alpha < \alpha_1$, such that $u(\cdot, \alpha, K) > 0$ on $[0, 1]$. This contradicts the definition of $\tilde{\alpha}$.

(iii) $u'(1, \tilde{\alpha}, K) < 0$. Due to the previous steps, $u'(1, \tilde{\alpha}, K) \leq 0$. Now, the fact that $u'(1, \tilde{\alpha}, K)$ is nonzero follows from Lemma 3.1.

(iv) The proof that $u(\cdot,\alpha,K)$ is decreasing is contained in the proof of Lemma 3.3.

Therefore the proposition is proved. $\square$

Proof of theorem 1.1. The existence of the positive solution is a consequence of previous proposition. Let

$$f^-(t) := \begin{cases} -f(-t) & \text{if } t > 0 \\ -f(0^-) & \text{if } t = 0 \\ 0 & \text{if } t < 0. \end{cases}$$

A straightforward application of the Proposition 3.4 with f^- gives us a negative solution with the desired properties. $\square$

Now, we exhibit some examples of functions f and K satisfying conditions (F1)–(F4) and (K1)–(K2). Let $f(t) = t^q - t^{q-1} - 1$, for $t > 0$ and $f(t) = 1 + (-t)^{q-1} - (-t)^q$ for $t < 0$, where $1 < q < p < q+1 < p^* := \frac{Np}{N-p}$ and K any positive constant weight. Choosing $\frac{(N-p)(q+1)}{p} < \theta < N$ we can find $k \in (0,1)$ such that

$$\theta \frac{k^{q+1}}{q+1} > \frac{N-p}{p}, \tag{3.10}$$

from which follows (F4). If we consider the same nonlinearity f with $1 < q < p < q+1 < \frac{(N-1)p}{N-p}$ and $K(r) = \varepsilon e^{-r}$, where $\varepsilon > 0$ is fixed and we choose $\frac{(N-p)(q+1)}{p} < \theta < N-1$, we can find $k \in (0,1)$ such that (3.10) holds, from which follows (F4). Since $N + r\frac{K'(r)}{K(r)} = N - r \geq N - 1 > \theta$, we obtain (K2). Other example is given by $f(t) = t^q \ln t - 1$ for for $t > 0$ and $f(t) = 1 - (-t)^q \ln(-t)$ for $t < 0$, where $2 < p < q+1 < p^*$ and K any positive constant weight. Reasoning as before we can show conditions (F1)–(F4) and (K1)–(K2). Assuming $p < q+1 < \frac{(N-1)p}{N-p}$, as before $K(r) = \varepsilon e^{-r}$, where $\varepsilon > 0$ is fixed, is an admissible weight.

4. Appendix

This section we establish the Pohozaev identity as well as the existence of local solution to problem (1.3).

Proposition 4.1 (Pohozaev Identity)**.** *Assume that $u(t,\alpha,K)$ is a solution of the initial value problem (1.3), then for all $0 \leq s \leq t \leq 1$,*

$$\begin{aligned} &t^{N-1}H(t,\alpha,K) - s^{N-1}H(s,\alpha,K) \\ &= \int_s^t r^{N-1}K(r)\Big[\Big(N + r\frac{K'(r)}{K(r)}\Big)F(u) - \frac{N-p}{p}f(u)u\Big]\,dr\,. \end{aligned} \tag{4.1}$$

Proof. It is easy to see that our ordinary differential equation can be written as

$$[r^{N-1}\varphi_p(u')]' = -r^{N-1}K(r)f(u). \tag{4.2}$$

Multiplying (4.2) by u and integrating on $[s,t]$, by parts, we obtain

$$t^{N-1}\varphi_p(u')u - s^{N-1}\varphi_p(u')u = \int_s^t r^{N-1}\varphi_p(u')u'dr - \int_s^t r^{N-1}K(r)f(u)u\,dr.$$

Then

$$\int_s^t r^{N-1}|u'|^p dr = b(s,t) + \int_s^t r^{N-1}K(r)f(u)u\,dr, \tag{4.3}$$

where $b(s,t)=t^{N-1}\varphi_p(u'(t))u(t)-s^{N-1}\varphi_p(u'(s))u(s)$. Now, multiplying (4.2) by ru' and integrating by parts, we have

$$t^N\varphi_p(u')u'-s^N\varphi_p(u')u'=\int_s^t r^{N-1}\varphi_p(u')(ru''+u')dr-\int_s^t r^N K(r)f(u)u'dr. \tag{4.4}$$

From (4.2), we realize that

$$\varphi_p(u')(ru''+u')=\frac{p-N}{p-1}|u'|^p-\frac{K(r)}{p-1}rf(u)u'.$$

Then, from (4.4), it follows that

$$a(s,t)-\int_s^t r^{N-1}\frac{p-N}{p-1}|u'|^p dr-\int_s^t r^N\frac{K(r)}{p-1}f(u)u'dr=-\int_s^t r^N K(r)f(u)u'dr,$$

where $a(s,t):=t^N\varphi_p(u'(t))u'(t)-s^N\varphi_p(u'(s))u'(s)$. Therefore,

$$a(s,t)+\frac{N-p}{p-1}\int_s^t r^{N-1}|u'|^p dr=-p'\int_s^t r^N K(r)f(u)u'dr.$$

This equation and (4.3) imply that

$$a(s,t)+\frac{N-p}{p-1}\Big(b(s,t)+\int_s^t r^{N-1}K(r)f(u)udr\Big)=-p'\int_s^t r^N K(r)f(u)u'dr.$$

Now, integrating by parts the right hand side of the last equation we obtain

$$\begin{aligned}
&a(s,t)+\frac{N-p}{p-1}b(s,t)\\
&=\frac{p-N}{p-1}\int_s^t r^{N-1}K(r)f(u)udr-p'(t^N K(t)F(u)-s^N K(s)F(u))\\
&\quad+p'\int_s^t r^{N-1}(NK(r)+rK'(r))F(u)dr.
\end{aligned}$$

Taking into account the definitions of $a(s,t)$ and $b(s,t)$, we can see that

$$\begin{aligned}
&t^N|u'|^p-s^N|u'|^p+\frac{N-p}{p-1}(t^{N-1}\varphi_p(u')u-s^{N-1}\varphi_p(u')u)\\
&+p'(t^N K(t)F(u)-s^N K(s)F(u))\\
&=\int_s^t r^{N-1}\Big[\frac{p-N}{p-1}K(r)f(u)u+\frac{p}{p-1}(NK(r)+rK'(r))F(u)\Big]dr.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\frac{p-1}{p}t^{N-1}\Big[\Big(t|u'|^p+\frac{N-p}{p-1}\varphi_p(u')u+p'tK(t)F(u)\Big)\\
&-s^{N-1}\Big(s|u'|^p+\frac{N-p}{p-1}\varphi_p(u')u+p'sK(s)F(u)\Big)\Big]\\
&=\int_s^t r^{N-1}K(r)\Big[\Big(N+r\frac{K'(r)}{K(r)}\Big)F(u)-\frac{N-p}{p}f(u)u\Big]dr.
\end{aligned}$$

However,

$$\begin{aligned}
&t^{N-1}\Big(\frac{t}{p'}\{|u'|^p+p'K(t)F(u)\}+\frac{N-p}{p}\varphi_p(u')u\Big)\\
&-s^{N-1}(\frac{s}{p'}\{|u'|^p+p'K(s)F(u)\}+\frac{N-p}{p}\varphi_p(u')u)
\end{aligned}$$

$$= t^{N-1}\Big[tK(t)E(t) + \frac{N-p}{p}\varphi_p(u')u\Big] - s^{N-1}\Big[sK(s)E(s) + \frac{N-p}{p}\varphi_p(u')u\Big].$$

Hence (4.1) holds. □

Proposition 4.2. *Let $p \geq 2$, $\alpha > 4u_1/3$ and K be a suitable weight. There exists a positive real number ε such that* (1.3) *has a unique solution $u(\cdot, \alpha, K)$ on the interval $[0, \varepsilon]$.*

Proof. For $\varepsilon > 0$ and $R = \alpha/4$, set $Y := \{u \in C([0, \varepsilon] : \mathbb{R}) : \|u - \alpha\|_\infty \leq R\}$. Define $T : Y \to Y$ by

$$(Tu)(s) := \alpha - \int_0^s \varphi_{p'}\Big(r^{1-N}\int_0^r t^{N-1}K(t)f(u(t))dt\Big)dr, \quad 0 \leq s \leq \varepsilon.$$

This was suggested by (1.4). T is well defined if ε i s small enough, since f is Lipschitz continuous on $[3\alpha/4, 5\alpha/4]$. Now, we will see that T is a contraction. Let u, v be elements of Y and $s \in [0, \varepsilon]$. Applying the mean value theorem we obtain the estimate

$$\begin{aligned}
&|(Tu)(s) - (Tv)(s)| \\
&\leq \int_0^s \varphi_{p'}(r^{1-N})\Big|\varphi_{p'}\Big(\int_0^r t^{N-1}K(t)f(u)dt\Big) - \varphi_{p'}\Big(\int_0^r t^{N-1}K(t)f(v)dt\Big)\Big|dr \\
&\leq (p'-1)\int_0^s \varphi_{p'}(r^{1-N})|\xi_r|^{p'-2}\Big(\int_0^r t^{N-1}K(t)|f(u) - f(v)|dt\Big)dr,
\end{aligned}$$

where ξ_r is a value between the two positive numbers $\int_0^r t^{N-1}K(t)f(u)dt$ and $\int_0^r t^{N-1}K(t)f(v)dt$. Assume, without loss of generality that $\int_0^r t^{N-1}K(t)f(u)dt \leq \xi_r$. Then, we have $\frac{\eta}{N}f(\frac{3}{4}\alpha)r^N \leq \xi_r$. Because of $p' - 2 \leq 0$,

$$|\xi_r|^{p'-2} \leq [\frac{\eta}{N}r^N f(\frac{3}{4}\alpha)]^{p'-2}.$$

Thus,

$$|(Tu)(s) - (Tv)(s)| \leq C_\alpha\|u - v\|_\infty \int_0^s r^{p'-1}dr < L_\alpha\|u - v\|_\infty$$

where $L_\alpha < 1$ and C_α is a suitable constant. Hence, T is a contraction. Thereupon T has a unique fixed point, which is the unique local solution of the IVP (1.3). □

Now, our goal is to extend this solution to the interval $[0, 1]$.

Lemma 4.3. *Suppose that $u(\cdot, \alpha, K)$ is a solution of* (1.3)*, restricted to an interval $[0, \varepsilon)$. Assume that $\limsup_{t\to\varepsilon^-}|u(t, \alpha, K)| < \infty$. Then, there exist a positive number $\bar\varepsilon$ and a function $\bar u(\cdot, \alpha, K) : [0, \varepsilon + \bar\varepsilon) \to \mathbb{R}$, that solves the IVP and such that $\bar u|_{[0,\varepsilon)} \equiv u$.*

Proof. Let $\{t_n\}_n \subseteq [0, \varepsilon)$ be an increasing sequence such that $t_n \to \varepsilon$. Then, for $n < m$ we have

$$\begin{aligned}
|u(t_n, \alpha, K) - u(t_m, \alpha, K)| &\leq \int_{t_n}^{t_m}\Big|\varphi_{p'}\Big(r^{1-N}\int_0^r t^{N-1}K(t)f(u)dt\Big)\Big|dr \\
&\leq \bar\lambda C\int_{t_n}^{t_m}\varphi_{p'}(r)dr \leq C|t_m - t_n|,
\end{aligned}$$

for some constant C. Here, we have used that $|u(\cdot, \alpha, K)|$ is bounded on $[0, \varepsilon)$, according to the hypothesis, $\limsup_{t\to\varepsilon^-}|u(t, \alpha, K)| < \infty$. This chain of inequalities shows that $\{u(t_n,\alpha,K)\}_n$ is a Cauchy sequence. Thus, we can define $u(\varepsilon,\alpha,K) := \lim_{n\to\infty} u(t_n,\alpha,K)$. On the other hand

$$\begin{aligned} |t_n^{N-1}\varphi_p(u'(t_n,\alpha,K)) - t_m^{N-1}\varphi_p(u'(t_m,\alpha,K))| &\le \int_{t_n}^{t_m} |t^{N-1}K(t)f(u)dt| \\ &\le M|t_m - t_n|, \end{aligned}$$

where M is a suitable constant. Hence, $\{t_n^{N-1}\varphi_p(u'(t_n,\alpha,K))\}_n$ is convergent and so is $\{u'(t_n,\alpha,K)\}_n$. Then, we can write $u'(\varepsilon,\alpha,K) := \lim_{n\to\infty} u'(t_n,\alpha,K)$. Now, we consider the problem

$$\begin{gathered} [\varphi_p(\bar u'(r))]' + \frac{N-1}{r}\varphi_p(\bar u'(r)) + K(r)f(\bar u) = 0, \quad \varepsilon < r < \varepsilon + \tilde\varepsilon \\ \bar u'(\varepsilon) = u'(\varepsilon), \quad \bar u(\varepsilon) = u(\varepsilon), \end{gathered}$$

for small $\tilde\varepsilon$. Proceeding in the same manner as we did with the former IVP, we can establish the existence and uniqueness of solution of this problem in the interval $[0,\varepsilon+\bar\varepsilon)$. Due to the uniqueness property, $\bar u|_{[0,\varepsilon)} \equiv u$. □

Lemma 4.4. *The local solution to the IVP can be extended to* $[0,1]$.

Proof. We argue by contradiction. As a consequence of the previous Lemma, if $u(\cdot,\alpha,K)$ cannot be extended, then it blows up. Therefore, there exists a sequence $\{t_n\}_n$ such that $\lim_{n\to\infty} u(t_n,\alpha,K) = +\infty$. By the mean value theorem, without loss of generality, we can assume that $\lim_{n\to\infty} u'(t_n,\alpha,K) = +\infty$. From the energy we obtain that

$$\frac{\partial}{\partial t}E(t,\alpha,K) = -\frac{|u'(t,\alpha,K)|}{p'tK(t)}\Big(\frac{N-1}{p-1}p + t\frac{K'(t)}{K(t)}\Big) \le 0.$$

Then

$$E(t,\alpha,K) \le E(0,\alpha,K) = F(\alpha).$$

This is a contradiction since $\lim_{n\to\infty} u(t_n,\alpha,K) = +\infty$. □

Remark 4.5. The continuous dependence of $u(\cdot,\alpha,K)$ whit respect to α is a consequence of the estimate

$$|u(t,\alpha_1,K) - u(t,\alpha_2,K)| \le |\alpha_1 - \alpha_2| + L_\alpha\|u(\cdot,\alpha_1,K) - u(\cdot,\alpha_2,K)\|_\infty,$$

for all $t \in [0,1]$. Thus

$$(1 - L_\alpha)\|u(\cdot,\alpha_1,K) - u(\cdot,\alpha_2,K)\|_\infty \le |\alpha_1 - \alpha_2|.$$

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