Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 237, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

MULTIPLE POSITIVE SOLUTIONS FOR QUASILINEAR ELLIPTIC EQUATIONS OF $p(x)$-LAPLACIAN TYPE WITH SIGN-CHANGING NONLINEARITY

KY HO, CHAN-GYUN KIM, INBO SIM

Abstract. We establish sufficient conditions for the existence of multiple positive solutions to nonautonomous quasilinear elliptic equations with $p(x)$-Laplacian and sign-changing nonlinearity. For solving the Dirichlet boundary-value problem we use variational and topological methods. The nonexistence of positive solutions is also studied.

1. Introduction

We are concerned with the existence of multiple positive solutions for the problem

$$\begin{gathered} -\Delta_{p(x)}u = \lambda f(x,u), \quad x\in\Omega, \\ u(x)=0, \quad x\in\partial\Omega, \end{gathered} \tag{1.1}$$

where $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ (is called $p(x)$-Laplacian), $\Omega\subset\mathbb{R}^N$ a bounded domain with smooth boundary $\partial\Omega$ for $N\geq 1$, $p\in C^1(\overline{\Omega})$ with $p(x)>1$ for all $x\in\overline{\Omega}$, $f\in C(\overline{\Omega}\times\mathbb{R},\mathbb{R})$, and λ is a positive parameter.

The problems related to the $p(x)$-Laplacian have been intensively studied. We refer the reader to [15] for motivations from electrorheological fluids, and to [3, 4, 5, 6, 7, 8, 9, 12] for basic definitions, properties, and standard results associated with the $p(x)$-Laplacian and the variable exponent Lebesgue-Sobolev space. As far as the authors know, most studies are related to the positive nonlinearity $f(x,u)$, and very few are related to the existence of positive solutions for the sign-changing nonlinearity.

Throughout this article, unless otherwise stated, we assume that for $k,l,m\in\mathbb{N}$ and $m\geq 2$. We use the following assumptions:

- (F1) $f(x,0)\geq 0$ for all $x\in\overline{\Omega}$;
- (F2) there exist $a_k, b_l\in C(\overline{\Omega})$ and positive constants c_l, where $1\leq k\leq m,\ 1\leq l\leq m-1$ such that

$$0\leq a_1(x)<c_1\leq b_1(x)<a_2(x)<c_2\leq b_2(x)<\cdots<c_{m-1}\leq b_{m-1}(x)<a_m(x),$$

2000 *Mathematics Subject Classification.* 35J20, 35J60, 35J70, 47J10, 46E35.
Key words and phrases. $p(x)$-Laplacian; variable exponent; sign-changing nonlinearity; positive solutions; multiplicity.

Submitted April 15, 2014. Published November 13, 2014.

and for all $k \in \{1, 2, \dots, m-1\}$,

$$f(x,s) \begin{cases} \le 0, & \text{for all } x \in \overline{\Omega} \text{ and all } s \in [a_k(x), b_k(x)] \cup [a_m(x), c_m], \\ \ge 0, & \text{for all } x \in \overline{\Omega} \text{ and all } s \in [b_k(x), a_{k+1}(x)] \end{cases}$$

where $c_m := \max_{x\in\overline{\Omega}} a_m(x)$;

(F3) there exists a nonnegative constant d such that $f(x,s) \ge -ds^{p(x)-1}$ for all $x \in \overline{\Omega}$ and all $s \in [0,\delta]$ for some $\delta > 0$;

(F4k) $k \in \{2,\dots,m\}$, $a_k \in C^1(\overline{\Omega})$, $\int_\Omega \alpha_k(x)\,dx > 0$, where

$$\alpha_k(x) := F(x, a_k(x)) - \max\{F(x,s) : 0 \le s \le a_{k-1}(x),\ x \in \overline{\Omega}\},$$

where $F(x,s) := \int_0^s f(x,\tau)d\tau$ for $(x,s) \in \Omega \times \mathbb{R}$.

In spite of the fact that (F3) implies (F1), the reason we assumed (F1) is to compare the conditions which the researchers mentioned below used. Indeed let us briefly review the previous conditions and results which are related to (1.1). When $p(x) \equiv 2$, that is, for the Laplacian case, Hess [10] initiated the study about sufficient conditions for sign-changing nonlinearity to get at least $2m-1$ positive solutions for sufficiently large λ. Actually, his conditions was $f(x,u) = f(u)$ and $f \in C^1([0,\infty),\mathbb{R})$ with $f(0) > 0$ and (F2) and (F4k) with a_k, b_l constants. It is worth noting that if $f \in C^1([0,\infty),\mathbb{R})$ and $f(0) > 0$ then (F3) holds automatically. The p-Laplacian version was established by Loc-Schmitt [13] with $f(0) \ge 0$ (not $f(0) > 0$), Hess' assumptions, and some different condition from (F4k). They only showed the existence of at least $m-1$ non-negative solutions but also discussed the necessary conditions. We emphasize that non-negativity of solutions comes from $f(0) \ge 0$ (see, Proposition 2.3 and Remark 5.1). Let us note that in the above two papers the nonlinearity was autonomous.

For the nonautonomous case, when $p(x) \equiv p, m = 2$, Kim-Shi [11] showed that (1.1) has at least two positive solutions for sufficiently large λ, under the assumptions $f(x, a_1(x)) = 0$, (F2), (F3) and a condition weaker than (F4k), with $k = 2$,

(F5) there exists an open ball B_1 of Ω such that $a_2 \in C^1(\overline{B}_1)$ and

$$F(x, a_2(x)) > 0, \quad x \in B_1.$$

They also showed the nonexistence of positive solutions of (1.1) for sufficiently small λ.

Motivated by the above results, we shall consider the case of $p(x)$-Laplacian, $m \ge 2$ and sign-changing nonautonomous nonlinearity which are weaker than conditions of Hess, Loc-Schmitt and Kim-Shi and obtain some results which contain their results as a special case in a unified way.

2. Preliminaries

In this section we establish a basic setup and some preliminary results concerning the $p(x)$-Laplacian problems.

Let $C_+(\overline{\Omega}) := \{h \in C(\overline{\Omega}) : h(x) > 1 \text{ for all } x \in \overline{\Omega}\}$, and for $h \in C_+(\overline{\Omega})$, we denote $h^+ = \max_{\overline{\Omega}} h(x)$ and $h^- = \min_{\overline{\Omega}} h(x)$. For any $p \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space by $L^{p(x)}(\Omega) := \{u : u$ is a measurable real valued function, $\int_\Omega |u(x)|^{p(x)}\,dx < \infty\}$ with the norm

$$\|u\|_{p(x)} = \inf\big\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\,dx \le 1\big\}.$$

The space $(L^{p(x)}(\Omega), \|\cdot\|_{p(x)})$ is a separable, uniformly convex Banach space, and its conjugate space is $L^{q(x)}(\Omega)$, where $1/p(x) + 1/q(x) = 1$ for all $x \in \overline{\Omega}$.

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\|_1 = \|u\|_{p(x)} + \||\nabla u|\|_{p(x)}.$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Then $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable reflexive Banach spaces. Moreover, we have the compact imbedding $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ if $q \in C_+(\overline{\Omega})$ with $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, where

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & p(x) < N, \\ \infty, & p(x) \geq N, \end{cases}$$

(see, e.g., [3, 4, 5]).

By Poincaré type inequality [5, Theorem 2.7], we can define a norm

$$\|u\| = \||\nabla u|\|_{p(x)}$$

which is equivalent to the norm $\|\cdot\|_1$ on $W_0^{1,p(x)}(\Omega)$. In what follows, we will use $\|\cdot\|$ instead of $\|\cdot\|_1$ on $W_0^{1,p(x)}(\Omega)$.

Definition 2.1. A function $u \in W_0^{1,p(x)}(\Omega)$ is called a (weak) solution to (1.1) if

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi \, dx = \lambda \int_\Omega f(x,u)\varphi \, dx \quad \text{for all } \varphi \in W_0^{1,p(x)}(\Omega).$$

The next two propositions have a key role in the proofs of the main results.

Proposition 2.2 ([8, 9])**.** *For each $h \in L^\infty(\Omega)$ the problem*

$$\begin{cases} -\Delta_{p(x)} u = h, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega \end{cases}$$

has a unique solution $u := K(h) \in W_0^{1,p(x)}(\Omega)$. Moreover the mapping $K : L^\infty(\Omega) \to C^{1,\alpha}(\overline{\Omega})$ is bounded for some $\alpha \in (0,1)$, and hence the mapping $K : L^\infty(\Omega) \to C^1(\overline{\Omega})$ is completely continuous.

Proposition 2.3 ([7, 9])**.** *Suppose that $u \in W^{1,p(x)}(\Omega)$, $u \geq 0$ and $u \not\equiv 0$ in Ω. If $-\Delta_{p(x)} u + d(x) u^{q(x)-1} \geq 0$ in Ω, where $d \in L^\infty(\Omega), d \geq 0, p(x) \leq q(x) \leq p^*(x)$, then $u > 0$ in Ω, and when $u \in C^1(\overline{\Omega}), \partial u/\partial \nu < 0$ on $\partial\Omega$ where ν is the outward unit normal on $\partial\Omega$.*

The following lemma gives estimates for a solution of $p(x)$-Laplacian which has a cut-off type nonlinear term.

Lemma 2.4. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that there exists $\bar{s} > 0$ such that $g(x,s) \geq 0$ if $(x,s) \in \Omega \times (-\infty, 0]$ and $g(x,s) \leq 0$ if $(x,s) \in \Omega \times [\bar{s}, \infty)$. If u is a weak solution to problem*

$$\begin{gathered} -\Delta_{p(x)} u = g(x,u), \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega, \end{gathered}$$

then $0 \leq u(x) \leq \bar{s}$ for almost all $x \in \overline{\Omega}$.

Proof. Putting $\phi = (u-\bar{s})^+ = \max\{u-\bar{s}, 0\} \in W_0^{1,p(x)}(\Omega)$, we have

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\phi \, dx = \int_{\{u(x)>\bar{s}\}} g(x,u(x))\phi \, dx \le 0.$$

Since

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\phi \, dx = \int_\Omega |\nabla (u-\bar{s})^+|^{p(x)} \, dx \ge 0,$$

$\nabla(u-\bar{s})^+ = 0$ a.e. in Ω, and thus $u \le \bar{s}$. In a similar manner, taking $\phi = \max\{-u, 0\} \in W_0^{1,p(x)}(\Omega)$, we have $u \ge 0$ almost all $x \in \overline{\Omega}$. The proof is complete. □

3. Main results

In this section, we state the main theorems and compare the conditions and results in [10, 13, 11]. First, for any $\lambda \ge 0$, we define the functional $I(\lambda, \cdot) : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ by

$$I(\lambda, u) := \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)} \, dx - \lambda \int_\Omega F(x, u(x)) \, dx, \quad u \in W_0^{1,p(x)}(\Omega).$$

Theorem 3.1. *Assume that* (F2), (F3), (F4k) *(with $k = 2, \dots, m$) hold. Then, for sufficiently large $\lambda > 0$,* (1.1) *has at least m solutions $u_1(\lambda), \dots, u_m(\lambda)$ in which $u_1(\lambda)$ is a non-negative solution and $u_2(\lambda), \dots, u_m(\lambda)$ are positive solutions such that $0 \le \|u_1(\lambda)\|_\infty \le c_1 < \|u_2(\lambda)\|_\infty \le c_2 < \dots < c_{m-1} < \|u_m(\lambda)\|_\infty \le c_m$ and $I(\lambda, u_m(\lambda)) < \dots < I(\lambda, u_2(\lambda)) < I(\lambda, u_1(\lambda)) \le 0$. Moreover, if $f(x, 0) \not\equiv 0$ then u_1 is also a positive solution.*

To obtain more positive solutions, we need to assume:

(F6) $p(x) \le 2$ for all $x \in \overline{\Omega}$ and there exists a positive constant L such that $f(x,s) + Ls$ is nondecreasing in $s \in [0, c_m]$.

Theorem 3.2. *Assume that* (F2), (F3), (F4k) *(with $k = 2, \dots, m$),* (F6) *hold. Then, for sufficiently large $\lambda > 0$, equation* (1.1) *has other $m-1$ positive solutions $\hat{u}_2(\lambda), \dots, \hat{u}_m(\lambda)$ such that $\|\hat{u}_k(\lambda)\|_\infty \in (c_{k-1}, c_k)$ and $\hat{u}_k(\lambda) \ne u_k(\lambda)$ for $k = 2, \dots, m$.*

Remark 3.3. Since the existence of L in (F6) is guaranteed, when $f \in C^1$, Theorem 3.2 is just Hess' conclusion.

We have a similar result even in the case that we replace (F4k), with $k = 2$, by the weaker condition (F5).

Theorem 3.4. *Assume that* (F2), (F3), (F5) *for $m = 2$, or* (F2), (F3), (F5), (F4k) *(with $k = 3, \dots, m$), for $m \ge 3$ hold. Then, for sufficiently large $\lambda > 0$,* (1.1) *has at least $m-1$ positive solutions $u_2(\lambda), \dots, u_m(\lambda)$ such that $\|u_k(\lambda)\|_\infty \in (c_{k-1}, c_k]$ and $I(\lambda, u_m(\lambda)) < \dots < I(\lambda, u_2(\lambda)) < 0$. Moreover, if we also assume that* (F6) *holds, then there exists other $m-2$ positive solutions $\hat{u}_3(\lambda), \dots, \hat{u}_m(\lambda)$ such that $\|\hat{u}_k(\lambda)\|_\infty \in (c_{k-1}, c_k)$ and $\hat{u}_k(\lambda) \ne u_k(\lambda)$ for $k = 3, \dots, m$.*

When $a_1(x) \equiv 0$ in Ω, $f(x, 0) \equiv 0$ in Ω, and we can show that problem (1.1) has a positive Mountain pass type solution under the additional assumption:

(F7) $a_1(x) \equiv 0$, and $p^+ < p^*(x)$ for all $x \in \overline{\Omega}$.

Theorem 3.5. *Assume that* (F2), (F3), (F5), (F7) *hold. Then* (1.1) *has a positive solution* $\hat{u}_1(\lambda)$*, which is different from* $u_2(\lambda), \dots, u_m(\lambda)$*,* $\hat{u}_3(\lambda), \dots, \hat{u}_m(\lambda)$ *obtained in Theorem 3.4 such that* $\|\hat{u}_1(\lambda)\|_\infty < c_2$ *and* $I(\lambda, \hat{u}_1(\lambda)) > 0$ *for sufficiently large* $\lambda > 0$.

Remark 3.6. This theorem extends Kim-Shi's result of p-Laplacian into the case of $p(x)$-Laplacian with more humps (for this terminology, see [10]).

For the nonexistence result we need only a simple assumption.

Theorem 3.7. *Assume that there exists positive constants* C_1 *and* C_2 *such that* $f(x, s) \le 0$ *for all* $(x, s) \in \Omega \times ((0, C_1) \cup (C_2, \infty))$*. Then* (1.1) *has no positive solutions for small* $\lambda > 0$.

Remark 3.8. The property of the first eigenvalue of p-Laplacian problem and Picone's identity were used in [11], but both are not expected in $p(x)$-Laplacian problem.

By Theorems 3.4, 3.5 and 3.7, we have the following corollary.

Corollary 3.9. *Assume that* (F2), (F3), (F5), (F7) *for* $m = 2$*, or* (F2), (F3), (F5), (F4k) *(with* $k = 3, \dots, m$*),* (F7) *for* $m \ge 3$ *hold. If* $f(x, s)$ *satisfies* $f(x, s) \le 0$ *for* $(x, s) \in \Omega \times [c_m, \infty)$*, then problem* (1.1) *has at least* m *positive solutions for sufficiently large* λ*, and it has no positive solutions for small* $\lambda > 0$*. Moreover, if we also assume that* $(F6)$ *holds, then problem* (1.1) *has at least* $2m - 2$ *positive solutions for sufficiently large* λ.

4. Lemmas

For each $k = 1, 2, \dots, m$, let us consider the truncation of the nonlinearity $f(x, s)$ as follows;

$$f_k(x, s) := \begin{cases} f(x, 0), & (x, s) \in \overline{\Omega} \times (-\infty, 0], \\ f(x, s), & (x, s) \in \overline{\Omega} \times (0, c_k], \\ f(x, c_k), & (x, s) \in \overline{\Omega} \times (c_k, \infty). \end{cases}$$

Then $f_k(x, s) \ge 0$ for $(x, s) \in \overline{\Omega} \times (-\infty, 0]$ and $f_k(x, s) \le 0$ for $(x, s) \in \overline{\Omega} \times [c_k, \infty)$. For any $\lambda \ge 0$, we define the functional $I_k(\lambda, \cdot) : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ by

$$I_k(\lambda, u) := \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)}\, dx - \lambda \int_\Omega F_k(x, u(x))\, dx, \quad u \in W_0^{1,p(x)}(\Omega),$$

where $F_k(x, s) := \int_0^s f_k(x, \tau) d\tau$ for $(x, s) \in \Omega \times \mathbb{R}$.

Lemma 4.1. *Assume that* $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$*. Then* $I_k(\lambda, \cdot)$ *is continuously Fréchet differentiable on* $W_0^{1,p(x)}(\Omega)$*, and* $I_k'(\lambda, \cdot)$ *is of* (S_+) *type operator. Moreover* $I_k(\lambda, \cdot)$ *is sequentially weakly lower-semicontinuous, coercive on* $W_0^{1,p(x)}(\Omega)$ *and satisfies the Palais-Smale condition.*

Proof. Let $I_k(\lambda, \cdot) = J - \lambda J_k$, where $J(u) = \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)}\, dx$ and $J_k(u) = \int_\Omega F_k(x, u(x))\, dx$. Since $f_k(x, s)$ is bounded, it is well known that $I_k(\lambda, \cdot)$ is continuously Fréchet differentiable, sequentially weakly lower-semicontinuous and coercive on $W_0^{1,p(x)}(\Omega)$ (see, e.g., [6]). The (S_+)-property of $I_k'(\lambda, \cdot)$ comes from (S_+)-property of J' (see [6]) and the sequentially weak continuity of J_k'. Since $I_k'(\lambda, \cdot)$ is of (S_+) type operator, to show that $I_k(\lambda, \cdot)$ satisfies (PS) condition it is enough to show every (PS) sequence is bounded. Let $\{u_n\}_{n=1}^{\infty}$ be any (PS) sequence of $I_k(\lambda,\cdot)$ in $W_0^{1,p(x)}(\Omega)$; i.e., there exists a constant $M>0$ such that $|I_k(\lambda,u_n)|\le M$, for all n and $I_k'(\lambda,u_n)\to 0$ as $n\to\infty$. It follows from the boundedness of f_k and the relation between modular and norm (see [5, Theorem 1.3]) that for n large, we have

$$\begin{aligned} M+\|u_n\| &\ge I_k(\lambda,u_n)-\frac{1}{2p^+}I_k'(\lambda,u_n)u_n\\ &\ge \frac{1}{2p^+}\big(\|u_n\|^{p^-}-1\big)-C\int_\Omega |u_n|\,dx\\ &\ge \frac{1}{2p^+}\|u_n\|^{p^-}-CC_1\|u_n\|-\frac{1}{2p^+}, \end{aligned}$$

where C is some positive constant and C_1 is the imbedding constant for $\|u_n\|_{L^1(\Omega)}\le C_1\|u_n\|$. Thus $\{u_n\}_{n=1}^{\infty}$ is bounded in $W_0^{1,p(x)}(\Omega)$ since $p^->1$. ☐

Lemma 4.2. *Assume that* (F1), (F2) *hold. Let u be any critical point of $I_k(\lambda,\cdot)$ for some $k\in\{1,2,\dots,m\}$. Then $u\in C_0^{1,\alpha}(\overline{\Omega})$ for some $\alpha\in(0,1)$ and $0\le u(x)\le c_k$ for all $x\in\Omega$. Assume in addition that* (F3) *holds, then $u>0$ in Ω and $\partial u/\partial\nu<0$ on $\partial\Omega$ if $u\not\equiv 0$ in Ω, where ν is the outward unit normal on $\partial\Omega$.*

Proof. Let u be any critical point of $I_k(\lambda,\cdot)$. By Lemma 2.4, $0\le u(x)\le c_k$ for a.e $x\in\Omega$. Since u is a nonnegative bounded solution of (1.1), $u\in C_0^{1,\alpha}(\overline{\Omega})$ for some $\alpha\in(0,1)$ in view of $C^{1,\alpha}$-regularity result in the Proposition 2.2. Hence, $0\le u(x)\le c_k$ for all $x\in\Omega$. Assume in addition that (F3) is satisfied, it follows from Proposition 2.3 that $u>0$ in Ω and $\partial u/\partial\nu<0$ on $\partial\Omega$ if $u\not\equiv 0$ in Ω. ☐

Fix $k\in\{1,\dots,m\}$ and denote by $\mathcal{C}_k(\lambda)$ the set of critical points of $I_k(\lambda,\cdot)$. Note that $u\in\mathcal{C}_k(\lambda)$ if and only if u is a solution of

$$\begin{gathered} -\Delta_{p(x)}u=\lambda f_k(x,u),\quad x\in\Omega,\\ u=0,\quad x\in\partial\Omega. \end{gathered}\tag{4.1}$$

Since $I_k(\lambda,\cdot)$ is sequentially weakly lower-semicontinuous and coercive on the space $W_0^{1,p(x)}(\Omega)$, it follows that $I_k(\lambda,\cdot)$ has a global minimizer $u_k(\lambda)\in\mathcal{C}_k(\lambda)$ for any $\lambda>0$.

Lemma 4.3. *Assume that* (F1), (F2), (F5) *hold. Then there exists $\lambda_2>0$ such that for all $\lambda>\lambda_2$,*

$$I(\lambda,u_2(\lambda))<0.$$

Proof. We shall show that, for large λ, there exists $v\in W_0^{1,p(x)}(\Omega)$ such that $0\le v(x)\le a_2(x)$ for all $x\in\Omega$ and $I(\lambda,v)<0=I(\lambda,0)$, which implies that $I(\lambda,u_2(\lambda))<0$.

Let us define $v_\epsilon(x)$ for small $\epsilon>0$ and B_1 in $(F5)$ as follows:

$$v_\epsilon(x):=\begin{cases} 0, & x\in\Omega\setminus B_1^\epsilon\\ a_2^\epsilon(x), & x\in B_1^\epsilon\setminus\overline{B}_1\\ a_2(x), & x\in B_1, \end{cases}$$

where $B_1^\epsilon:=\{x\in\Omega:\operatorname{dist}(x,B_1)\le\epsilon\}$, $a_2(x)$ is the function in (F2) and $a_2^\epsilon(x)$ is an appropriate function such that $0\le v_\epsilon(x)\le a_2(x),\ x\in\Omega$ and $v_\epsilon\in C_0^1(\overline{\Omega})$. Then

$F_2(x, v_\epsilon(x)) = F(x, v_\epsilon(x))$, $x \in \Omega$ and

$$\begin{aligned} & I(\lambda, v_\epsilon) \\ &= \int_\Omega \frac{1}{p(x)} |\nabla v_\epsilon(x)|^{p(x)}\, dx - \lambda \int_\Omega F(x, v_\epsilon(x))\, dx \\ &= \int_\Omega \frac{1}{p(x)} |\nabla v_\epsilon(x)|^{p(x)}\, dx - \lambda \int_{B_1} F(x, a_2(x))\, dx - \lambda \int_{B_1^\epsilon \setminus \overline{B}_1} F(x, a_2^\epsilon(x))\, dx \\ &\le \int_\Omega \frac{1}{p(x)} |\nabla v_\epsilon(x)|^{p(x)}\, dx - \lambda \int_{B_1} F(x, a_2(x))\, dx + \lambda M |B_1^\epsilon \setminus \overline{B}_1|, \end{aligned} \tag{4.2}$$

where $M := \max\{|F(x,u)| : 0 \le u \le a_2(x),\ x \in \overline{\Omega}\}$. By (F5), $\int_{B_1} F(x, a_2(x))\, dx > 0$, and we can choose a sufficiently small constant $\epsilon_0 > 0$ so that

$$0 < M |B_1^{\epsilon_0} \setminus \overline{B}_1| \le \frac{1}{2} \int_{B_1} F(x, a_2(x))\, dx.$$

From (4.2), we infer

$$\begin{aligned} I(\lambda, v_{\epsilon_0}) &\le \int_\Omega \frac{1}{p(x)} |\nabla v_{\epsilon_0}(x)|^{p(x)}\, dx - \lambda \int_{B_1} F(x, a_2(x))\, dx + \lambda M |B_1^{\epsilon_0} \setminus \overline{B}_1| \\ &\le \int_\Omega \frac{1}{p(x)} |\nabla v_{\epsilon_0}(x)|^{p(x)}\, dx - \frac{\lambda}{2} \int_{B_1} F(x, a_2(x))\, dx, \end{aligned}$$

which implies that $I(\lambda, v_{\epsilon_0}) < 0$ for sufficiently large λ. Consequently, $I(\lambda, u_2(\lambda)) < 0$ for all large λ. This completes the proof. □

Lemma 4.4. *Fix k in $\{2, \dots, m\}$ and assume that* (F1), (F2) *and* (F4k) *hold. Then there exists $\lambda_k > 0$ such that for all $\lambda > \lambda_k$, $u_k(\lambda) \notin \mathcal{C}_{k-1}(\lambda)$ and $I(\lambda, u_k(\lambda)) < I(\lambda, u_{k-1}(\lambda))$.*

Proof. It is sufficient to show that there exist $\lambda_k > 0$ and $w_k \in W_0^{1,p(x)}(\Omega)$ such that $w_k \ge 0$, $\|w_k\|_\infty \le c_k$ and

$$I(\lambda, w_k) < I(\lambda, u_{k-1}) \text{ for all } \lambda > \lambda_k, \tag{4.3}$$

to complete the proof. We first show that for all $x \in \Omega$,

$$F(x, u_{k-1}(x)) \le \max\{F(x,s) : 0 \le s \le a_{k-1}(x),\ x \in \overline{\Omega}\}.$$

The assertion is obvious if $u_{k-1}(x) \le a_{k-1}(x)$. For the case $a_{k-1}(x) \le u_{k-1}(x) \le c_{k-1}$, we obtain that $f(x, u_{k-1}(x)) \le 0$ and

$$\begin{aligned} F(x, u_{k-1}(x)) &= \int_0^{a_{k-1}(x)} f(x,s) ds + \int_{a_{k-1}(x)}^{u_{k-1}(x)} f(x,s) ds \\ &\le \int_0^{a_{k-1}(x)} f(x,s) ds \\ &= F(x, a_{k-1}(x)) \\ &\le \max\{F(x,s) : 0 \le s \le a_{k-1}(x),\ x \in \overline{\Omega}\}. \end{aligned}$$

From this inequality and (F4k) it follows that

$$F(x, a_k(x)) \ge F(x, u_{k-1}(x)) + \alpha_k(x), \forall x \in \Omega,$$

and hence,

$$\int_\Omega F(x,a_k(x))\,dx \geq \int_\Omega F(x,u_{k-1}(x))\,dx + \int_\Omega \alpha_k(x)\,dx. \tag{4.4}$$

For $\delta > 0$, let $\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) < \delta\}$. Then $|\Omega_\delta| \to 0$ as $\delta \to 0$. For each small $\delta > 0$, there exists $w_\delta \in W_0^{1,p(x)}(\Omega)$ such that $w_\delta(x) = a_k(x)$ for $x \in \Omega\backslash\Omega_\delta$ and $0 \leq w_\delta(x) \leq a_k(x)$ for $x \in \Omega$. Thus

$$\begin{aligned}\int_\Omega F(x,w_\delta(x))\,dx &= \int_{\Omega\backslash\Omega_\delta} F(x,a_k(x))\,dx + \int_{\Omega_\delta} F(x,w_\delta(x))\,dx\\ &= \int_\Omega F(x,a_k(x))\,dx - \int_{\Omega_\delta} [F(x,a_k(x)) - F(x,w_\delta(x))]\,dx\\ &\geq \int_\Omega F(x,a_k(x))\,dx - C_k|\Omega_\delta|,\end{aligned}$$

where $C_k := 2\max\{|F(x,s)| : 0 \leq s \leq a_k(x),\ x \in \overline{\Omega}\}$. By (4.4),

$$\int_\Omega F(x,w_\delta(x))\,dx \geq \int_\Omega F(x,u_{k-1}(x))\,dx + \int_\Omega \alpha_k(x)\,dx - C_k|\Omega_\delta|.$$

Fixing $\delta > 0$ such that

$$\eta := \int_\Omega \alpha_k(x)\,dx - C_k|\Omega_\delta| > 0,$$

and setting $w_k := w_\delta$, we obtain

$$\begin{aligned}&I(\lambda,w_k) - I(\lambda,u_{k-1})\\ &= \int_\Omega \frac{1}{p(x)}\left(|\nabla w_k|^{p(x)} - |\nabla u|^{p(x)}\right)\,dx - \lambda\int_\Omega (F(x,w_k(x)) - F(x,u_{k-1}(x)))\,dx\\ &\leq \int_\Omega \frac{1}{p(x)}|\nabla w_k|^{p(x)}\,dx - \lambda\eta,\end{aligned}$$

which implies that there exists $\lambda_k > 0$ such that (4.3) is satisfied. □

Next we shall give some results by using the degree theory for (S_+) type maps in the Banach space. For the basic properties of the degree of (S_+) type maps, we refer to [2, 14]. For each $k \in \{1,2,\dots,m\}$ and $\epsilon > 0$, let $\mathcal{U}_\epsilon(\mathcal{C}_k(\lambda))$ be the ϵ-neighborhood of $\mathcal{C}_k(\lambda)$ in $W_0^{1,p(x)}(\Omega)$. For $m \geq 2$, $\mathcal{C}_{k-1}(\lambda) \subsetneq \mathcal{C}_k(\lambda)$ for each $k \in \{2,\dots,m\}$. By Proposition 2.2, $\mathcal{C}_k(\lambda)$ is a compact set in $W_0^{1,p(x)}(\Omega)$.

Let $B_R(0)$ denote the open ball in $W_0^{1,p(x)}(\Omega)$ with radius $R > 0$ and center at the origin. By the boundedness of f_k, for sufficiently large $R = R(\lambda) > 0$, $I_k'(\lambda,u)u > 0$ for any $u \in \partial B_R(0)$. Thus, by the property for the degree of (S_+) type operator, we have

$$\deg(I_k'(\lambda,\cdot), B_R(0), 0) = 1. \tag{4.5}$$

By the modified arguments which were used in [10, Lemma 3] for the Hilbert space, we have the following lemma.

Lemma 4.5. *Fix $k \in \{2,\dots,m\}$ and assume that* (F1), (F2), (F6), (F4k) *hold. Then there exists $\epsilon_k = \epsilon_k(\lambda) > 0$ such that for any $\epsilon \in (0,\epsilon_k)$,*

$$\deg(I_k'(\lambda,\cdot), \mathcal{U}_\epsilon(\mathcal{C}_{k-1}(\lambda)), 0) = 1. \tag{4.6}$$

Proof. By (4.5) and the excision property of the degree, for any $\epsilon > 0$,

$$\deg(I'_{k-1}(\lambda, \cdot), \mathcal{U}_\epsilon(\mathcal{C}_{k-1}(\lambda)), 0) = 1.$$

We claim that there exists $\epsilon_{k-1} > 0$ such that, for all $\epsilon \in (0, \epsilon_{k-1})$ and all $\mu \in [0, 1]$,

$$\mu I'_{k-1}(\lambda, v) + (1-\mu) I'_k(\lambda, v) \neq 0 \quad \text{for } v \in \partial\mathcal{U}_\epsilon(\mathcal{C}_{k-1}(\lambda)).$$

Indeed, if the assertion were false then there are a sequences of positive numbers δ_n approaching 0, and sequences $\{\mu_n\}_{n=1}^\infty \subset [0, 1]$ and $\{v_n\}_{n=1}^\infty \subset W_0^{1,p(x)}(\Omega)$ such that

$$\operatorname{dist}(v_n, \mathcal{C}_{k-1}(\lambda)) = \delta_n, \tag{4.7}$$

and

$$\mu_n I'_{k-1}(\lambda, v_n) + (1-\mu_n) I'_k(\lambda, v_n) = 0.$$

Thus v_n satisfies

$$\begin{gathered} -\Delta_{p(x)} v_n = \lambda(\mu_n f_{k-1}(x, v_n) + (1-\mu_n) f_k(x, v_n)), \quad x \in \Omega, \\ v_n(x) = 0, \quad x \in \partial\Omega. \end{gathered}$$

Since

$$\begin{aligned} &\mu_n f_{k-1}(x, s) + (1-\mu_n) f_k(x, s) \\ &= \begin{cases} f(x, 0), & (x, s) \in \overline{\Omega} \times (-\infty, 0], \\ f(x, s), & (x, s) \in \overline{\Omega} \times (0, c_{k-1}], \\ \mu_n f(x, c_{k-1}) + (1-\mu_n) f(x, c_k), & (x, s) \in \overline{\Omega} \times (c_k, \infty), \end{cases} \end{aligned}$$

by Lemma 2.4, $0 \le v_n(x) \le c_k$ for a.e $x \in \Omega$ and all $n \in \mathbb{N}$, and thus by Proposition 2.2, $\{v_n\}_{n=1}^\infty$ is relatively compact in $C^1(\overline{\Omega})$. Then, there exist a subsequence of $\{v_n\}_{n=1}^\infty$, still denote by $\{v_n\}_{n=1}^\infty$, and $v \in C^1(\overline{\Omega})$ such that $v_n \to v$ in $C^1(\overline{\Omega})$. It follows from (4.7) that $v \in \mathcal{C}_{k-1}(\lambda)$. Hence, by Lemma 4.2, $0 \le v(x) \le c_{k-1}$ for all $x \in \Omega$.

Next, we show that $\|v\|_\infty < c_{k-1}$. Indeed, by (F6),

$$-\Delta_{p(x)}(c_{k-1}) + Lc_{k-1} \ge f(x, c_{k-1}) + Lc_{k-1} \ge f(x, v) + Lv = -\Delta_{p(x)} v + Lv,$$

and

$$-\Delta_{p(x)}(c_{k-1} - v) + L(c_{k-1} - v) \ge 0. \tag{4.8}$$

Since $v = 0$ on $\partial\Omega$, $c_{k-1} - v \not\equiv 0$ in Ω. Applying Proposition 2.3 with $q(x) \equiv 2$, it follows from (4.8) that $v(x) < c_{k-1}$ for all $x \in \overline{\Omega}$ and hence $\|v\|_\infty < c_{k-1}$. Since $v_n \notin \mathcal{C}_{k-1}(\lambda)$ and $\|v_n\|_\infty > c_{k-1}$, letting $n \to \infty$, we get a contradiction. Thus (4.6) holds by the homotopy invariance property of the degree. ☐

5. Proofs of main results and an example

Now we give the proofs of Theorems 3.1, 3.2, 3.4, 3.5 and 3.7.

Proof of Theorem 3.1. Fix $\lambda > \max\{\lambda_k : k = 2, \dots, m\}$, where λ_k are taken as in Lemma 4.4. Also as in Lemma 4.4, denote by $u_k(\lambda)$ the global minimizer of $I_k(\lambda, \cdot)$. Then, by Lemma 4.2 and Lemma 4.4, we have $0 \le u_k(\lambda) \le c_k$ and

$$\begin{gathered} 0 \le \|u_1(\lambda)\|_\infty \le c_1 < \|u_2(\lambda)\| \le c_2 < \dots < c_{m-1} < \|u_m(\lambda)\|_\infty \le c_m, \\ I(\lambda, u_m(\lambda)) < \dots < I(\lambda, u_2(\lambda)) < I(\lambda, u_1(\lambda)) \le 0 = I(\lambda, 0). \end{gathered}$$

By Proposition 2.3, we deduce $u_2(\lambda), \dots, u_m(\lambda)$ are $m-1$ positive solutions of problem (1.1). Once again, by Proposition 2.3, if $f(x,0) \not\equiv 0$, then u_1 is also a positive solution. □

Proof of Theorem 3.2. First, by Lemma 4.4, $u_k \notin \mathcal{C}_{k-1}(\lambda)$. If u_k is not an isolated critical point of $I_k(\lambda,\cdot)$, then there are infinitely many positive solutions in $\mathcal{C}_k(\lambda) \setminus \mathcal{C}_{k-1}(\lambda)$, the proof is complete. Otherwise, u_k is an isolated critical point of $I_k(\lambda,\cdot)$ and it follows from [2, Theorem 1.8] that

$$\deg(I_k'(\lambda,\cdot), B_\epsilon(u_k), 0) = 1, \tag{5.1}$$

where ϵ is so small that

$$\mathcal{U}_\epsilon(\mathcal{C}_{k-1}(\lambda)) \cap B_\epsilon(u_k) = \emptyset.$$

By the additivity property of the degree, (4.5), (4.6) and (5.1),

$$\deg(I_k'(\lambda,\cdot), B_R(0) \setminus (\overline{\mathcal{U}_\epsilon(\mathcal{C}_{k-1}(\lambda))} \cup \overline{B_\epsilon(u_k)}), 0) = -1.$$

Consequently, there exists $\hat{u}_k \in \mathcal{C}_k(\lambda) \setminus \mathcal{C}_{k-1}(\lambda)$ such that $\hat{u}_k \neq u_k$. By (F6), using the same argument as in the proof of Lemma 4.5, we conclude that $\|u_k\|_\infty$, $\|\hat{u}_k\|_\infty \in (c_{k-1}, c_k)$. □

Proof of Theorem 3.4. In the case $m = 2$, by Lemma 4.3, $I_2(\lambda, u_2(\lambda)) < 0$ for $\lambda > \lambda_2$, and $u_2(\lambda) \not\equiv 0$. Hence, $u_2(\lambda)$ is positive by Proposition 2.3. In the case $m \geq 3$, fix $\lambda > \max\{\lambda_k : k = 2, \dots, m\}$, where λ_2 is taken as in Lemma 4.3 whereas λ_k $(k = 3, \dots, m)$ are taken as in Lemma 4.4. Using the same argument as in the proof of Theorem 3.1 with noting that $I_2(\lambda, u_2(\lambda)) < 0$, it follows that problem (1.1) has $m-1$ positive solutions $u_2(\lambda), \dots, u_m(\lambda)$ such that $\|u_k(\lambda)\|_\infty \in (c_{k-1}, c_k]$ and $I(\lambda, u_k(\lambda)) < 0$ for $k \in \{2, \dots, m\}$. If we assume in addition that (F6) holds, then by the same argument as in the proof of Theorem 3.2, there exists other $m-2$ positive solutions $\hat{u}_3(\lambda), \dots, \hat{u}_m(\lambda)$ such that $\|\hat{u}_k(\lambda)\|_\infty \in (c_{k-1}, c_k)$ and $\hat{u}_k(\lambda) \neq u_k(\lambda)$ for $k \in \{3, \dots, m\}$. □

Proof of Theorem 3.5. Since $p^+ < p^*(x)$ for all $x \in \overline{\Omega}$, we can choose a constant q such that $q \in (p^+, p^*(x))$ for all $x \in \overline{\Omega}$. From the fact that $a_1(x) = 0$ for all $x \in \Omega$, there exists a constant $C(q) > 0$ such that

$$\begin{gathered} f_2(x,s) \leq C(q)|s|^{q-1}, \quad (x,s) \in \Omega \times \mathbb{R}, \\ F_2(x,s) \leq C(q)\frac{|s|^q}{q}, \quad (x,s) \in \Omega \times \mathbb{R}. \end{gathered}$$

Let $0 < \delta < \min\{1, 1/C_q\}$, where C_q is the imbedding constant such that $\|u\|_q \leq C_q\|u\|$ for $u \in W_0^{1,p(x)}(\Omega)$. For $\|u\| < \delta$, we estimate

$$\begin{aligned} I_2(\lambda,u) &\geq \int_\Omega \frac{1}{p(x)}|\nabla u(x)|^{p(x)}\,dx - \lambda\frac{C(q)}{q}\int_\Omega |u(x)|^q\,dx \\ &\geq \Big[\frac{1}{p^+} - \lambda\frac{C(q)C_q^q}{q}\|u\|^{q-p^+}\Big]\|u\|^{p^+}. \end{aligned}$$

Thus, for each $\lambda > 0$, there exists $\rho \in (0, \delta)$ such that $I_2(\lambda, u) > 0 = I_2(\lambda, 0)$ if $0 < \|u\| \leq \rho$. Fix $\lambda > 0$ such that $I_2(\lambda, u_2(\lambda)) < 0$. It follows from Mountain pass Theorem that $I_2(\lambda,\cdot)$ has another critical point $\hat{u}_1$ such that

$$I_2(\lambda, \hat{u}_1(\lambda)) > 0 > I_2(\lambda, u_2(\lambda)),$$

and thus, for sufficiently large λ, problem (1.1) has other positive solution $\hat{u}_1(\lambda)$, which is different from $2m-3$ positive solutions $u_2, \dots, u_m, \hat{u}_3, \dots, \hat{u}_m$ obtained in Theorem 3.4, satisfying $\|\hat{u}_1(\lambda)\|_\infty < c_2$ and $I(\lambda, \hat{u}_1(\lambda)) > 0$. □

Remark 5.1. If we replace (F3) by (F1) as in Loc-Schmitt's work [13], the conclusions of Theorems 3.1, 3.2, 3.4, 3.5, and Corollary 3.9 remain valid with the non-negativity of solutions not the positivity.

Proof of Theorem 3.7. By contradiction, assume that $\{(\lambda_n, u_n)\}_{n=1}^\infty$ is a sequence such that u_n is a positive solution of (1.1) with $\lambda = \lambda_n$ for each $n \in \mathbb{N}$, and $\lambda_n \to 0$ as $n \to \infty$. Then $\|u_n\|_\infty > C_1$ for all $n \in \mathbb{N}$, since $f(x,s) \le 0$ for all $x \in \overline{\Omega}$ and $0 \le s \le C_1$. Indeed, assume on the contrary that $\|u_n\|_\infty \le C_1$ for some $n \in \mathbb{N}$. It follows from the comparison principle [9, Proposition 2.3] that $u_n \le 0$, which contradicts the fact that u_n is a positive solution of problem (1.1) with $\lambda = \lambda_n$. By Lemma 2.4, $\|u_n\|_\infty \le C_2$ for all $n \in \mathbb{N}$. Let $h_n = \lambda_n f(\cdot, u_n)$, then $h_n \to 0$ as $n \to \infty$ in $L^\infty(\Omega)$. By Proposition 2.2, $u_n := K(h_n) \to 0$ as $n \to \infty$ in $C^1(\Omega)$ which contradicts the fact that $\|u_n\|_\infty > C_1$ for all $n \in \mathbb{N}$. □

Example 5.2. To illustrate Corollary 3.9 in the case $m = 2$, let us consider the nonautonomous cubic nonlinearity

$$f(x,s) = s^{p(x)-1}(s - b(x))(c(x) - s),$$

where $p \in C^1(\overline{\Omega})$ with $p^+ < p^*(x)$ for all $x \in \overline{\Omega}$, and $b, c \in C(\overline{\Omega})$ such that $0 < b(x) < c(x) < 1$ for any $x \in \overline{\Omega}$. If we assume that there exists an open ball $B_1 \subseteq \Omega$ such that $c(x) \in C^1(\overline{B}_1)$ and

$$0 < \left(1 + \frac{2}{p^+}\right) b(x) < c(x) \quad \text{in } B_1,$$

it is easy to verify that all assumptions of Corollary 3.9 are satisfied. Thus, problem (1.1) has at least two positive solutions for large $\lambda > 0$, and it has no positive solutions for small $\lambda > 0$.

Acknowledgements. The third author was supported by the National Research Foundation of Korea, Grant funded by the Korea Government (MEST) (NRF-2012R1A1A2000739).

References

[1] G. Dinca, P. Jebelean, J. Mawhin; *Variational and topological methods for Dirichlet problems with p-Laplacian*, Port. Math. (N.S.), 58 (3) (2001), 339–378.
[2] P. Drábek, A. Kufner, F. Nicolosi; *Quasilinear elliptic equations with degenerations and singularities*. de Gruyter Series in Nonlinear Analysis and Applications, 5. Walter de Gruyter and Co., Berlin, 1997.
[3] D. Edmunds, J. Răkosnik; *Sobolev embeddings with variable exponent*, Studia Math. 143 (2000), 267–293.
[4] D. Edmunds, J. Răkosnik; *Sobolev embeddings with variable exponent II*, Math. Nachr., no. 246-247 (2002), 53–67.
[5] X. L. Fan, D. Zhao; *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. 263 (2001), 424–446.
[6] X. L. Fan, Q. H. Zhang; *Existence of solutions for p(x)-Laplacian Dirichlet problems*, Nonlinear Anal. 52 (2003), 1843–1852.
[7] X. L. Fan, Y. Z. Zhao, Q. H. Zhang; *A strong maximum principle for p(x)-Laplace equations*, Chinese J. Contemp. Math. 24 (3) (2003), 277–282.
[8] X. L. Fan; *Global $C^{1,\alpha}$ regularity for variable exponent elliptic equations in divergence form*, J. Differential Equations, 235 (2007), 397–417.

[9] X. L. Fan; *On the sub-supersolution method for p(x)-Laplacian equations*, J. Math. Anal. Appl., 330 (2007), 665–682.

[10] P. Hess; *On multiple positive solutions of nonlinear elliptic eigenvalue problems*, Comm. Partial Differential Equations, 6 (1981), 951–961.

[11] C.-G. Kim, J. Shi; *Existence and multiplicity of positive solutions to a quasilinear elliptic equation with strong Allee effect growth rate*, Results Math. 64 (2013), 165–173.

[12] O. Kováčik, J. Răkosnik; *On spaces* $L^{p(x)}$ *and* $W^{k,p(x)}$, Czechoslovak Math. J. 41(116) (1991), 592–618.

[13] N. H. Loc, K. Schmitt; *On positive solutions of quasilinear elliptic equations*, Differential Integral Equations 22 (2009), 829–842.

[14] D. Motreanu, V. V. Motreanu, N. Papageorgiou; *Topological and Variational Methods with Applications to Nonlinear Boundary Value Problems*, Springer, New York Heidelberg Dordrecht London, 2014.

[15] M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Lecture Notes in Mathematics, 1748, Springer-Verlag, Berlin, 2000.

Ky Ho
Department of Mathematics, University of Ulsan, Ulsan 680-749, Korea
E-mail address: hnky81@gmail.com

Chan-Gyun Kim
Department of Mathematics Education, Pusan National University, Busan 609-735, Korea
E-mail address: cgkim75@gmail.com

Inbo Sim
Department of Mathematics, University of Ulsan, Ulsan 680-749, Korea
E-mail address: ibsim@ulsan.ac.kr