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COMPOSITION OF S^p-WEIGHTED PSEUDO ALMOST AUTOMORPHIC FUNCTIONS AND APPLICATIONS

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Abstract. Since the background space of S^p-weighted pseudo almost automorphic functions (abbr. S^p-wpaa functions) is endowed with a norm coming from L^p norm, it is natural to consider the composition of S^p-wpaa functions under conditions of L^p norm. However, the known resutls for composition of S^p-wpaa functions were always given under conditions of supremum norm. Motivated by this, we establish some new results for the composition of S^p-almost automorphic functions and S^p-wpaa functions under a "uniform continuity condition" with respect to L^p norm. As an application, we prove the existence of mild weighted pseudo almost automorphic solutions for some semilinear differential equations with an S^p-wpaa force term. An example of the heat equation illustrates our results.

1. Introduction

The concept of almost automorphy, which was first introduced by Bochner [2] in the earlier sixties, is a natural generalization of almost periodicity. Afterwards, this kind of property was extended to many interesting cases such as pseudo almost automorphy, weighted pseudo almost automorphy, S^p-almost automorphy, S^p-pseudo almost automorphy, etc. (see e.g., [1, 4, 17, 19, 20, 22, 24]). Meanwhile, applications to differential equations, partial differential equations and functional differential equations of these properties have been widely investigated (see e.g., [6, 8, 9, 10, 11, 13, 25] and the references therein).

Recently, a new class of property called S^p-weighted pseudo almost automorphy was introduced by Zhang et al. [25] and Xia and Fan [22], which is a natural generalization of both weighted pseudo almost automorphy (introduced by Blot et al. [1]) and S^p-pseudo almost automorphy (introduced by Diagana [4]), and the properties of this new class of functions were discussed, including the key property composition theorem. Meanwhile, as applications, some existence theorems of weighted pseudo almost automorphic solutions to some differential equations with S^p-weighted pseudo almost automorphic coefficients were obtained.

The main purpose of this work is to make a further study on the composition for S^p-almost automorphic functions and S^p-weighted pseudo almost automorphic functions. We notice that some more results were given in this line (see

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[6, 7, 8, 10, 11, 22, 23, 25]). In these works, a "Lipschitz condition" or a "uniform continuity condition" with respect to sup norm was needed in the these theorems (see Remark 3.5 and 3.9 for details). It is natural to consider the same problem under the "uniform continuity condition" with respect to L^p norm instead of the "Lipschitz condition" or "uniform continuity condition" with respect to sup norm. This is reasonable and necessary since the spaces of S^p-almost automorphic functions and S^p-weighted pseudo almost automorphic functions are endowed with an integral norm coming from L^p norm. Moreover, the "uniform continuity condition" is the main condition needed for the known theorems for composition of almost automorphic functions [18], pseudo almost automorphic functions [15] and weighted pseudo almost automorphic functions [1].

For this purpose, we present some new results on the composition of S^p-almost automorphic functions and S^p-weighted pseudo almost automorphic functions under the "uniform continuity condition" with respect to the L^p norm (see Theorems 3.3 and 3.7). As an application, we give an existence of mild weighted pseudo almost automorphic solutions to the semilinear differential equation

$$u'(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R},$$

where f is an S^p-weighted pseudo almost automorphic function (see Theorem 4.3). Moreover, a concrete example of heat equation is given to illustrate our abstract theorems at the end of this paper, where the "uniform continuity condition" with respect to the L^p norm is satisfied while the "Lipschitz condition" is not.

2. Preliminaries

We first introduce some classical notation. Let $(\mathbb{X}, \|\cdot\|), (\mathbb{Y}, \|\cdot\|)$ be two Banach spaces, and $BC(\mathbb{R}, \mathbb{X})$ (resp. $BC(\mathbb{R}\times\mathbb{Y}, \mathbb{X})$) be the space of bounded continuous functions $u : \mathbb{R} \to \mathbb{X}$ (resp. $u : \mathbb{R}\times\mathbb{Y} \to \mathbb{X}$). Then endowed with the sup norm $\|u\| = \sup_{t\in\mathbb{R}} \|u(t)\|$, $BC(\mathbb{R}, \mathbb{X})$ is a Banach space. $C(\mathbb{R}, \mathbb{X})$ (resp. $C(\mathbb{R}\times\mathbb{Y}, \mathbb{X})$) stands for the space of continuous functions from $\mathbb{R}$ to $\mathbb{X}$ (resp. from $\mathbb{R}\times\mathbb{Y}$ to $\mathbb{X}$). We note that even though the notation $\|\cdot\|$ is used for norms in different spaces, no confusion should arise.

2.1. **Weighted pseudo almost automorphic functions.**

Definition 2.1 ([2]). (i) $f \in C(\mathbb{R}, \mathbb{X})$ is said to be almost automorphic if for every sequence of real numbers $\{s'_n\}$, there exists a subsequence $\{s_n\}$ such that $g(t) = \lim_{n\to\infty} f(t + s_n)$ is well-defined for $t \in \mathbb{R}$, and $\lim_{n\to\infty} g(t - s_n) = f(t)$ for $t \in \mathbb{R}$. Denote by $AA(\mathbb{X})$ the set of all such functions.

(ii) $f \in C(\mathbb{R}\times\mathbb{Y}, \mathbb{X})$ is said to be almost automorphic if $f(t, u)$ is almost automorphic in $t \in \mathbb{R}$ uniformly for all $u \in K$, where K is any bounded subset of $\mathbb{Y}$. Denote by $AA(\mathbb{R}\times\mathbb{Y}, \mathbb{X})$ the set of all such functions.

Denote by $AA_u(\mathbb{X})$ the closed subspace of all functions $f \in AA(\mathbb{X})$ with $g \in C(\mathbb{R}, \mathbb{X})$. We note that $f \in AA_u(\mathbb{X})$ if and only if $f \in AA(\mathbb{X})$ and all convergences in Definition 2.1 are uniform on compact intervals (i.e. in the Fréchet space), and f is uniformly continuous if $g \in C(\mathbb{R}, \mathbb{X})$. Moreover, the range $f(\mathbb{R})$ of $f \in AA(\mathbb{X})$ is precompact, and $AA(\mathbb{X})$ is a closed subspace of $BC(\mathbb{R}, \mathbb{X})$ (cf. [17, 19, 20]).

Let U be the set of all functions $\rho : \mathbb{R} \to (0, \infty)$ which are locally integrable over $\mathbb{R}$. For $T > 0$ and $\rho \in U$, set

$$\mu(T, \rho) = \int_{-T}^{T} \rho(t)dt.$$

Define

$$U_\infty = \{\rho \in U : \lim_{T\to\infty} \mu(T, \rho) = \infty\}.$$

For $\rho \in U_\infty, T > 0$ and $f \in BC(\mathbb{R}, \mathbb{X})$, denote

$$\mathcal{W}(T, f, \rho) = \frac{1}{\mu(T, \rho)} \int_{-T}^{T} \|f(t)\|\rho(t)dt.$$

Then the weighted ergodic spaces $PAA_0(\mathbb{R}, \mathbb{X}, \rho)$ and $PAA_0(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \rho)$ are defined by

$$PAA_0(\mathbb{X}, \rho) = \{f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{T\to\infty} \mathcal{W}(T, f, \rho) = 0\},$$

$$PAA_0(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \rho) = \Big\{f \in BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X}) : \lim_{T\to\infty} \mathcal{W}(T, f(\cdot, u), \rho) = 0 \text{ uniformly in } u \in \mathbb{Y}\Big\}.$$

For $\rho \in U_\infty$, the spaces $WPAA(\mathbb{X}, \rho)$ and $WPAA(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \rho)$ of weighted pseudo almost automorphic functions were introduced in [1]:

$$WPAA(\mathbb{X}, \rho) = \{f = g + \phi \in BC(\mathbb{R}, \mathbb{X}) : g \in AA(\mathbb{X}), \phi \in PAA_0(\mathbb{X}, \rho)\},$$

$$\begin{aligned} &WPAA(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \rho) \\ &= \{f = g + \phi \in BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X}) : g \in AA(\mathbb{R} \times \mathbb{Y}, \mathbb{X}), \phi \in PAA_0(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \rho)\}. \end{aligned}$$

Lemma 2.2 ([3, 14, 16])**.** *If $PAA_0(\mathbb{X}, \rho)$ is translation invariant, then the space $WPAA(\mathbb{X}, \rho)$ endowed with the supremum norm is a Banach space and the decomposition of the functions in $WPAA(\mathbb{X}, \rho)$ is unique.*

2.2. **S^p-weighted pseudo almost automorphic functions.** In this subsection, the definitions and basic results on S^p-weighted pseudo almost automorphic functions can be found (or simply deduced from the results) in [3, 5, 12, 14, 20, 21, 22, 26, 25]. We always denote by $\|\cdot\|_p$ the norm of space $L^p(0, 1; \mathbb{X})$ for $p \in [1, \infty)$.

Definition 2.3. (i) The Bochner transform $f^b(t, s), (t, s) \in \mathbb{R} \times [0, 1]$, of a function $f : \mathbb{R} \to \mathbb{X}$ is defined by $f^b(t, s) = f(t + s)$.

(ii) The Bochner transform $f^b(t, s, u)$, $(t, s, u) \in \mathbb{R} \times [0, 1] \times \mathbb{Y}$, of a function $f : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$ is defined by $f^b(t, s, u) = f(t + s, u)$.

Obviously, if $f = g + \phi$, then $f^b = g^b + \phi^b$, and $(\lambda f)^b = \lambda f^b$ for each scalar λ.

Definition 2.4. (i) The space $BS^p(\mathbb{X})$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R} \to \mathbb{X}$ such that $f^b \in L^\infty(\mathbb{R}, L^p(0, 1; \mathbb{X}))$. It is a Banach space with the norm

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \|f(\tau)\|^p d\tau\Big)^{1/p} = \sup_{t\in\mathbb{R}} \|f(t + \cdot)\|_p.$$

(ii) The space $BS^p(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ of all Stepanov bounded functions consists of all measurable functions $f : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$ such that

$$f^b(\cdot, \cdot, y) \in L^\infty(\mathbb{R}, L^p(0, 1; \mathbb{X})), \quad t \mapsto f^b(t, \cdot, y) \in L^p(0, 1; \mathbb{X}), \quad t \in \mathbb{R}$$

(abbr.

$$f^b(\cdot, y) \in L^\infty(\mathbb{R}, L^p(0,1;\mathbb{X})), \quad t \mapsto f^b(t,y) \in L^p(0,1;\mathbb{X}), \quad t \in \mathbb{R})$$

for each $y \in \mathbb{Y}$.

Definition 2.5. (i) The space $AS^p(\mathbb{X})$ of Stepanov-like almost automorphic functions (abbr. S^p-almost automorphic functions) consists of all $f \in BS^p(\mathbb{X})$ such that $f^b \in AA(L^p(0,1;\mathbb{X}))$. That is, a function $f \in L^p_{\rm loc}(\mathbb{R},\mathbb{X})$ is said to be S^p-almost automorphic if its Bochner transform $f^b : \mathbb{R} \to L^p(0,1;\mathbb{X})$ is almost automorphic in the sense that for every sequence of real numbers $\{s'_n\}$, there exist a subsequence $\{s_n\}$ and a function $g \in L^p_{Loc}(\mathbb{R},\mathbb{X})$ such that

$$\lim_{n\to\infty} \|f(t+s_n+\cdot) - g(t+\cdot)\|_p = \lim_{n\to\infty} \|g(t-s_n+\cdot) - f(t+\cdot)\|_p = 0$$

pointwisely for $t \in \mathbb{R}$.

(ii) A function $f \in BS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ is said to be S^p-almost automorphic in $t \in \mathbb{R}$ for $u \in \mathbb{Y}$, if $f(\cdot,u) \in AS^p(\mathbb{X})$ for $u \in \mathbb{Y}$. Denote by $AS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ the set of all such functions.

Lemma 2.6 ([20])**.** (i) $(AS^p(\mathbb{X}), \|\cdot\|_{S^p})$ is a Banach space.
(ii) $f \in AS^p(\mathbb{X})$ if and only if $f^b \in AA_u(L^p(0,1;\mathbb{X}))$.
(iii) $AA(\mathbb{X})$ is continuously embedded in $AS^p(\mathbb{X})$.

For $\rho \in U_\infty$, the spaces $S^pWPAA(\mathbb{X},\rho)$ and $S^pWPAA(\mathbb{R}\times\mathbb{Y},\mathbb{X},\rho)$ of Stepanov-like weighted pseudo almost automorphic functions (abbr. S^p-weighted pseudo almost automorphic functions) are defined by:

$$S^pWPAA(\mathbb{X},\rho) = \{f = g+\phi \in BS^p(\mathbb{X}) : g \in AS^p(\mathbb{X}), \phi^b \in PAA_0(L^p(0,1;\mathbb{X}),\rho)\},$$

$$\begin{aligned} S^pWPAA(\mathbb{R}\times\mathbb{Y},\mathbb{X},\rho) = \Big\{ & f = g+\phi \in BS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X}) : g \in AS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X}), \\ & \phi^b \in PAA_0(\mathbb{R}\times\mathbb{Y}, L^p(0,1;\mathbb{X}),\rho)\Big\}. \end{aligned}$$

Lemma 2.7. (i) *Assume that $PAA_0(L^p(0,1;\mathbb{X}),\rho)$ is translation invariant. Then the decomposition of an S^p-weighted pseudo almost automorphic function is unique.*
(ii) *The space $S^pWPAA(\mathbb{X},\rho)$ equipped with $\|\cdot\|_{S^p}$ is a Banach space.*
(iii) *$WPAA(\mathbb{X})$ is continuously embedded in $S^pWPAA(\mathbb{X},\rho)$.*

Remark 2.8. Note that Lemma 2.7 (i) can be proved by the same argument as that in [14, Theorem 3.3], and Lemma 2.7 (i) does not hold in general without the assumption "$PAA_0(L^p(0,1;\mathbb{X}),\rho)$ is translation-invariant"(see [14, Remark 3.3]). Similar result can be found also in [16, Theorem 2.10, Remark 2.11]. Lemma 2.7 (ii) comes from [22, Theorem 3.3], and Lemma 2.7(iii) comes from [22, Theorem 3.1].

In the sequel, we write $u = x+y \in S^pWPAA(\mathbb{X},\rho)$ implies that $x \in AS^p(\mathbb{X})$ and $y^b \in PAA_0(L^p(0,1;\mathbb{X}),\rho)$, and similarly $f = g+\phi \in S^pWPAA(\mathbb{R}\times\mathbb{Y},\mathbb{X},\rho)$ implies that $g \in AS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ and $\phi^b \in PAA_0(\mathbb{R}\times\mathbb{Y}, L^p(0,1;\mathbb{X}),\rho)$.

3. Results on compositions of functions

For any bounded set $K \subset \mathbb{X}$, denote by $AS^p_K(\mathbb{R}\times\mathbb{X},\mathbb{X}) \subset AS^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$, the set of all functions such that the convergence in Definition 2.5 (ii) are uniform for $u \in K$. In addition, we always assume that $p \in (1,\infty)$.

We introduce the following "uniform continuity condition" with respect to the L^p norm for a function $h : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ with $h(\cdot, u) \in L^p_{\rm loc}(\mathbb{R}, \mathbb{X})$ for each $u \in \mathbb{X}$:

(A1) For $\varepsilon > 0$, there exists $\sigma > 0$ such that $x, y \in L^p(0, 1; \mathbb{X})$ and $\|x - y\|_p < \sigma$ imply that
$$\|h(t + \cdot, x(\cdot)) - h(t + \cdot, y(\cdot))\|_p < \varepsilon \quad \text{for } t \in \mathbb{R}.$$

In the sequel, we say that a function ψ satisfies (A1) means that (A1) holds for ψ in the place of h.

Let $f \in AS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$. Then for a sequence $\{s_n\} \subset \mathbb{R}$, there exist a subsequence $\{\tau_n\}$ and a function $g : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ with $g(\cdot, x) \in L^p_{\rm loc}(\mathbb{R}, \mathbb{X})$, $x \in \mathbb{X}$ such that for each $t \in \mathbb{R}$,
$$\lim_{n\to\infty} \|f(t+\tau_n+\cdot, x)-g(t+\cdot, x)\|_p = \lim_{n\to\infty} \|g(t-\tau_n+\cdot, x)-f(t+\cdot, x)\|_p = 0. \tag{3.1}$$

We will use the following "uniform continuity condition" with respect to the L^p norm:

(A2) $f \in AS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ satisfies (A1), and for a sequence $\{s_n\} \subset \mathbb{R}$, there exist a subsequence $\{\tau_n\}$ and a function g given above such that g satisfies (A1).

Remark 3.1. (i) It is clear that assumption (A2) implies that $f(t, \cdot) : \mathbb{X} \to AS^p(\mathbb{X})$ is uniformly continuous and $g^b(t, \cdot) : \mathbb{X} \to L^p(0, 1; \mathbb{X})$ is uniformly continuous uniformly in $t \in \mathbb{R}$. As a result, we can check easily that (3.1) holds uniformly on each compact $K \subset \mathbb{X}$, that is $f \in AS^p_K(\mathbb{R} \times \mathbb{X}, \mathbb{X})$.

(ii) If f satisfies (A1), for a sequence $\{s_n\} \subset \mathbb{R}$, it is not necessary that each function g associated with some subsequence $\{\tau_n\}$ of $\{s_n\}$ satisfies (A1) even in the periodic case. For example, let $f(t, x) = \sin 2\pi t$, $(t, x) \in \mathbb{R}^2$. For sequence $\{1/n\}$, $g(t, x)$ can be chosen as
$$g(t, x) = \begin{cases} 2, & t = n + x,\ n \in \mathbb{Z},\ x \in (0, 1), \\ \sin 2\pi t, & \text{otherwise}. \end{cases}$$

It is clear that f satisfies (A1), while g does not. In fact, for $\sigma \in (0, 1)$, let $x(s) = s, y(s) = s + \sigma/2, s \in [0, 1]$. Then $x, y \in L^p(0, 1; \mathbb{X})$, $\|x - y\|_p < \sigma$, and for $t = 0$,
$$\begin{aligned} \|g(t + \cdot, x(\cdot)) - g(t + \cdot, y(\cdot))\|_p &= \Big(\int_0^1 \|g(s, s) - g(s, s + \sigma/2)\|^p ds\Big)^{1/p} \\ &= \Big(\int_0^1 \|2 - \sin 2\pi s\|^p ds\Big)^{1/p} > 1. \end{aligned}$$

However, if we choose $g = f$, g satisfies (A1).

Lemma 3.2. *Let h be the function in* (A1)*, and $x : \mathbb{R} \to \mathbb{X}$ with $\overline{x(\mathbb{R})}$ compact. For $\varepsilon > 0$, there exists a finite set $\{x_k\}_{k=1}^m \subset \overline{x(\mathbb{R})}$ such that*
$$\|h(t + \cdot, x(t + \cdot))\|_p < \varepsilon + m \sup_{1\le k\le m} \|h(t + \cdot, x_k)\|_p, \quad t \in \mathbb{R}.$$

Proof. For $\varepsilon > 0$, let $\sigma > 0$ be given by (A1). Since $\overline{x(\mathbb{R})}$ is compact, we can find finite open balls $O_k (k = 1, 2, \dots, m)$ with center $x_k \in \overline{x(\mathbb{R})}$ and radius σ such that $\overline{x(\mathbb{R})} \subset \cup_{k=1}^m O_k$. Set $B_k = \{s \in \mathbb{R} : x(s) \in O_k\}$. Then $\mathbb{R} = \cup_{k=1}^m B_k$. Let $E_1 = B_1$, $E_k = B_k \setminus (\cup_{j=1}^{k-1} B_j) (2 \le k \le m)$, then $E_i \cap E_j = \emptyset$ for $i \ne j$ and $\mathbb{R} = \cup_{k=1}^m E_k$.

Define a step function $\hat{x} : \mathbb{R} \to \mathbb{X}$ by $\hat{x}(s) = x_k$, $s \in E_k$, $k = 1, 2, \dots, m$. It is clear that $\|x(s) - \hat{x}(s)\| < \sigma$ for $s \in \mathbb{R}$. So $\|x(s + \cdot) - \hat{x}(s + \cdot)\|_p < \sigma, s \in \mathbb{R}$. By (A1),

$$\|h(t + \cdot, x(s + \cdot)) - h(t + \cdot, \hat{x}(s + \cdot))\|_p < \varepsilon, \quad t, s \in \mathbb{R}.$$

Thus for $t \in \mathbb{R}$,

$$\begin{aligned}
\|h(t + \cdot, x(t + \cdot))\|_p &\le \|h(t + \cdot, x(t + \cdot)) - h(t + \cdot, \hat{x}(t + \cdot))\|_p + \|h(t + \cdot, \hat{x}(t + \cdot))\|_p \\
&< \varepsilon + \Big(\int_t^{t+1} \|h(s, \hat{x}(s))\|^p ds \Big)^{1/p} \\
&= \varepsilon + \Big(\sum_{k=1}^m \int_{E_k \cap [t, t+1]} \|h(s, x_k)\|^p ds \Big)^{1/p} \\
&\le \varepsilon + \Big(\sum_{k=1}^m \int_t^{t+1} \|h(s, x_k)\|^p ds \Big)^{1/p} \\
&\le \varepsilon + m \sup_{1 \le k \le m} \|h(t + \cdot, x_k)\|_p.
\end{aligned}$$

This completes the proof. □

Theorem 3.3. *Assume that f satisfies* (A2), *and $x \in AS^p(\mathbb{X})$ with $\overline{x(\mathbb{R})}$ compact. Then $f(\cdot, x(\cdot)) \in AS^p(\mathbb{X})$.*

Proof. Let $K = \overline{x(\mathbb{R})}$. Since $f(\cdot, x) \in L^p_{\rm loc}(\mathbb{R}, \mathbb{X})$ for each $x \in \mathbb{X}$ and K is compact, by (A2) and a standard method, it is easy to verify that $f(\cdot, x(\cdot)) \in L^p_{\rm loc}(\mathbb{R}, \mathbb{X})$.

Since $x \in AS^p(\mathbb{X})$ and $f \in AS^p_K(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ by Remark 3.1 (i), for every sequence $\{s_n\} \subset \mathbb{R}$, there exist a subsequence $\{\tau_n\}$ and functions $y : \mathbb{R} \to \mathbb{X}$ and $g : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ with $y, g(\cdot, z) \in L^p_{\rm loc}(\mathbb{R}, \mathbb{X})$, $z \in \mathbb{X}$ such that for each $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \|x(t + \tau_n + \cdot) - y(t + \cdot)\|_p = \lim_{n\to\infty} \|y(t - \tau_n + \cdot) - x(t + \cdot)\|_p = 0, \tag{3.2}$$

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{z\in K} \|f(t + \tau_n + \cdot, z) - g(t + \cdot, z)\|_p \\
&= \lim_{n\to\infty} \sup_{z\in K} \|g(t - \tau_n + \cdot, z) - f(t + \cdot, z)\|_p = 0.
\end{aligned} \tag{3.3}$$

Clearly, (3.2) implies that $y(t) \in K$ for a.e. $t \in \mathbb{R}$. Let $R_0 = \{t \in \mathbb{R} : y(t) \in K\}$. Then the measure $m(\mathbb{R} \setminus R_0) = 0$. Fix $y_0 \in K$, define $\bar{y}(t) = y(t)$ if $t \in R_0$ and $\bar{y}(t) = y_0$ if $t \in \mathbb{R} \setminus R_0$. Then $\bar{y}(\mathbb{R}) \subset K$. Notice that $f(\tau_n + \cdot, \cdot) - g$ satisfies (A1) for all n (for the same σ). Then by Lemma 3.2, for $\varepsilon > 0$, there exists a finite set $\{y_k\}_{k=1}^m \subset K$ such that for $t \in \mathbb{R}$,

$$\|f(t+\tau_n+\cdot, \bar{y}(t+\cdot)) - g(t+\cdot, \bar{y}(t+\cdot))\|_p < \varepsilon + m \sup_{1\le k\le m} \|f(t+\tau_n+\cdot, y_k)) - g(t+\cdot, y_k)\|_p.$$

Fix $t \in \mathbb{R}$, by (3.3), there exists $N_1 > 0$ such that for $n > N_1$,

$$\sup_{z\in K} \|f(t + \tau_n + \cdot, z) - g(t + \cdot, z)\|_p < \frac{\varepsilon}{m}.$$

Thus for $n > N_1$,

$$\begin{aligned}
&\|f(t + \tau_n + \cdot, y(t + \cdot)) - g(t + \cdot, y(t + \cdot))\|_p \\
&= \|f(t + \tau_n + \cdot, \bar{y}(t + \cdot)) - g(t + \cdot, \bar{y}(t + \cdot))\|_p < \varepsilon + m\frac{\varepsilon}{m} = 2\varepsilon.
\end{aligned} \tag{3.4}$$

By (3.2), there is $N_2 > 0$ such that for $n > N_2$, $\|x(t+\tau_n+\cdot) - y(t+\cdot)\|_p < \sigma$, where σ is given by (A1) with f in the place of h. Then for $n > N_2$,

$$\begin{aligned} &\|f(t+\tau_n+\cdot, x(t+\tau_n+\cdot)) - f(t+\tau_n+\cdot, y(t+\cdot))\|_p \\ &\le \sup_{r\in\mathbb{R}} \|f(r+\cdot, x(t+\tau_n+\cdot)) - f(r+\cdot, y(t+\cdot))\|_p \le \varepsilon. \end{aligned} \tag{3.5}$$

Now, by (3.4) and (3.5), for $n > \max\{N_1, N_2\}$,

$$\begin{aligned} &\|f(t+\tau_n+\cdot, x(t+\tau_n+\cdot)) - g(t+\cdot, y(t+\cdot))\|_p \\ &\le \|f(t+\tau_n+\cdot, x(t+\tau_n+\cdot)) - f(t+\tau_n+\cdot, y(t+\cdot))\|_p \\ &\quad + \|f(t+\tau_n+\cdot, y(t+\cdot)) - g(t+\cdot, y(t+\cdot))\|_p < 3\varepsilon. \end{aligned}$$

This implies that for $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \|f(t+\tau_n+\cdot, x(t+\tau_n+\cdot)) - g(t+\cdot, y(t+\cdot))\|_p = 0.$$

Similarly, we can prove that for $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \|g(t-\tau_n+\cdot, y(t-\tau_n+\cdot)) - f(t+\cdot, x(t+\cdot))\|_p = 0.$$

Therefore, $f(\cdot, x(\cdot)) \in AS^p(\mathbb{X})$. The proof is complete. □

If $x \in AA(\mathbb{X})$, $x \in AS^p(\mathbb{X})$ and $\overline{x(\mathbb{R})}$ is compact. Then we obtain the following result by Theorem 3.3.

Corollary 3.4. *Assume that f satisfies* (A2) *and $x \in AA(\mathbb{X})$. Then $f(\cdot, x(\cdot)) \in AS^p(\mathbb{X})$.*

Remark 3.5. There are lots of works devoted to the composition of S^p-almost automorphic functions, in which the function f is assumed to satisfy a "Lipschitz condition" of the form

$$\|f(t,u) - f(t,v)\| \le L(t)\|u - v\|, \quad u, v \in \mathbb{X}, \tag{3.6}$$

where $L(t)$ is a positive constant in [6, 8], $L \in AS^r(\mathbb{R})$ with $r \ge \max\{p, \frac{p}{p-1}\}$ in [7], and $L \in BS^r(\mathbb{R})$ with $r \ge p$ in [10]; or in the form of L^p norm in [11]:

$$\|f(t+\cdot, x(\cdot)) - f(t+\cdot, y(\cdot))\|_p \le L\|x - y\|_p, \quad t \in \mathbb{R}.$$

where $L > 0$ and $x, y \in L^p_{\rm loc}(\mathbb{R}, \mathbb{X})$. In addition, $f(t,x)$ is also assumed to be uniformly continuous on each bounded subset $K' \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$ in [25]; i.e., f satisfies the "uniform continuity condition" with respect to the sup norm.

The following lemma will be useful later.

Lemma 3.6 ([25])**.** *Let $\rho \in U_\infty$ and $f \in BS^p(\mathbb{X})$. Then $f^b \in PAA_0(L^p(0,1;\mathbb{X}), \rho)$ if and only if for any $\varepsilon > 0$,*

$$\lim_{T\to\infty} \frac{\mu(M_{T,\varepsilon}(f), \rho)}{\mu(T, \rho)} = 0,$$

where

$$M_{T,\varepsilon}(f) = \{t \in [-T, T] : \|f(t+\cdot)\|_p \ge \varepsilon\}, \quad \mu(M_{T,\varepsilon}(f), \rho) = \int_{M_{T,\varepsilon}(f)} \rho(t)dt\,.$$

Next, we give a result in the composition of S^p-weighted pseudo almost automorphic functions.

Theorem 3.7. *Let $f = g + \phi$ belong to $S^pWPAA(\mathbb{R}\times\mathbb{X}, \mathbb{X}, \rho)$ and let $u = x + y$ belong to $S^pWPAA(\mathbb{X}, \rho)$ with $\overline{x(\mathbb{R})}$ compact. Assume that g satisfies* (A2), *ϕ satisfies* (A1) *and $\{f(\cdot, z) : z \in J\}$ is bounded in $S^pWPAA(\mathbb{X}, \rho)$ for any bounded $J \subset \mathbb{X}$. Then $f(\cdot, u(\cdot)) \in S^pWPAA(\mathbb{X}, \rho)$.*

Proof. Let $I_1(t) = g(t, x(t))$, $I_2(t) = f(t, u(t)) - f(t, x(t))$ and $I_3(t) = \phi(t, x(t)), t \in \mathbb{R}$. Then

$$f(t, u(t)) = I_1(t) + I_2(t) + I_3(t), \quad t \in \mathbb{R}.$$

We have $I_1 \in AS^p(\mathbb{X})$ by Theorem 3.3. So we need only to prove that $I_2^b, I_3^b \in PAA_0(L^p(0, 1; \mathbb{X}), \rho)$.

It is easy to see that $I_2 \in BS^p(\mathbb{X})$ since u and x are bounded and $\{f(\cdot, z) : z \in J\}$ is bounded in $S^pWPAA(\mathbb{X}, \rho)$ for any bounded $J \subset \mathbb{X}$. Noticing that f satisfies (A1) since g and ϕ satisfy (A1), for $\varepsilon > 0$, let $\sigma > 0$ be given by (A1) with f in the place of h. Then

$$\|I_2(t + \cdot)\|_p = \|f(t + \cdot, u(t + \cdot)) - f(t + \cdot, x(t + \cdot))\|_p < \varepsilon$$

for $\|y(t + \cdot)\|_p < \sigma, t \in \mathbb{R}$. This implies that $M_{T,\varepsilon}(I_2) \subset M_{T,\sigma}(y)$. Here we use the notation in Lemma 3.6. Meanwhile, since $y^b \in PAA_0(L^p(0, 1; \mathbb{X}), \rho)$, by Lemma 3.6,

$$\lim_{T\to\infty} \frac{\mu(M_{T,\sigma}(y), \rho)}{\mu(T, \rho)} = 0.$$

Thus

$$\lim_{T\to\infty} \frac{\mu(M_{T,\varepsilon}(I_2), \rho)}{\mu(T, \rho)} = 0,$$

which shows that $I_2^b \in PAA_0(L^p(0, 1; \mathbb{X}), \rho)$.

For $\varepsilon > 0$, let σ be given by (A1) with ϕ in the place of h. By Lemma 3.2, there is a finite set $\{x_k\}_{k=1}^m \subset \overline{x(\mathbb{R})}$ such that for $t \in \mathbb{R}$,

$$\|\phi(t + \cdot, x(t + \cdot))\|_p < \varepsilon + m \sup_{1\le k\le m} \|\phi(t + \cdot, x_k)\|_p.$$

Since $\phi^b(\cdot, x) \in PAA_0(L^p(0, 1; \mathbb{X}), \rho)$ for each $x \in \mathbb{X}$, there is $T_0 > 0$ such that for $T > T_0, 1 \le k \le m$,

$$\mathcal{W}(T, \phi^b(\cdot, x_k), \rho) = \frac{1}{\mu(T, \rho)} \int_{-T}^{T} \rho(t)\|\phi(t + \cdot, x_k)\|_p dt < \frac{\varepsilon}{m}.$$

Then for $T > T_0$,

$$\begin{aligned}
\mathcal{W}(T, I_3^b, \rho) &= \mathcal{W}(T, \phi^b(\cdot, x(\cdot)), \rho) \\
&= \frac{1}{\mu(T, \rho)} \int_{-T}^{T} \rho(t)\|\phi(t + \cdot, x(t + \cdot))\|_p dt \\
&< \varepsilon + m \sup_{1\le k\le m} \frac{1}{\mu(T, \rho)} \int_{-T}^{T} \rho(t)\|\phi(t + \cdot, x_k))\|_p dt \\
&= \varepsilon + m \sup_{1\le k\le m} \mathcal{W}(T, \phi^b(\cdot, x_k), \rho) \\
&< \varepsilon + m\frac{\varepsilon}{m} = 2\varepsilon.
\end{aligned}$$

This yields that $\lim_{T\to\infty} \mathcal{W}(T, I_3^b, \rho) = 0$. That is $I_3^b \in PAA_0(L^p(0, 1; \mathbb{X}), \rho)$. The proof is complete. $\square$

By Lemma 2.7 (iii) and Theorem 3.7, we have the following corollary.

Corollary 3.8. *Let $f = g + \phi \in S^pWPAA(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \rho)$ and $u \in WPAA(\mathbb{X}, \rho)$. Assume that g satisfies* (A2)*, ϕ satisfies* (A1) *and $\{f(\cdot, z) : z \in J\}$ is bounded in $S^pWPAA(\mathbb{X}, \rho)$ for any bounded $J \subset \mathbb{X}$. Then $f(\cdot, u(\cdot)) \in S^pWPAA(\mathbb{X}, \rho)$.*

Remark 3.9. The composition of S^p-weighted pseudo almost automorphic functions was also studied in some recent papers, where the functions $f = g + \phi \in S^pWPAA(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \rho)$ are assumed to satisfy a "Lipschitz condition": Xia an Fan [22] investigated the case when f and g satisfies (3.6) with constant $L > 0$ or $L \in AS^r(\mathbb{R})$ with $r \geq \max\{p, \frac{p}{p-1}\}$; Zhang et al. [25] studied the case when f satisfies

$$\|f(t, u) - f(t, v)\|_{S^p} \leq L(t)\|u - v\|,$$

where $L \in BS^p(\mathbb{R})$ and $u, v \in L^p_{\rm loc}(\mathbb{R}, \mathbb{X})$, and g satisfies the "uniform continuity condition" with respect to the sup norm. Moreover, the case when f and g satisfy the "uniform continuity condition" with respect to the sup norm is also studied.

4. Mild weighted pseudo almost automorphic solutions

Applying the theorems obtained in the last section, we study the existence of mild weighted pseudo almost automorphic solutions of the semilinear differential equation

$$u'(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R}, \tag{4.1}$$

where A is the infintesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ and $f = g + \phi \in S^pWPAA(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \rho)$ satisfies the conditions of Corollary 3.8. We recall that u is said to be a mild weighted pseudo almost automorphic solution of (4.1) if $u \in WPAA(\mathbb{X}, \rho)$ and

$$u(t) = \int_{-\infty}^{t} T(t - s)f(s, u(s))ds.$$

In the sequel, we assume that $PAA_0(L^p([0, 1], \mathbb{X}), \rho)$ is translation invariant. Then $WPAA(\mathbb{X}, \rho)$ and $S^pWPAA(\mathbb{X}, \rho)$ are Banach spaces by Lemma 2.2 and 2.7. Moreover, we will use the following assumption:

- (A3) $T(t)$ is compact for $t > 0$ and there exist constants $M, c > 0$ such that $\|T(t)\| \leq Me^{-ct}$ for $t \geq 0$.

Let $q > 1$ such that $1/p + 1/q = 1$. Denote

$$\alpha_0 = M\Big(\frac{e^{qc} - 1}{qc}\Big)^{1/q}, \quad \alpha = \alpha_0 \sum_{k=1}^{\infty} e^{-ck}.$$

For $u \in WPAA(\mathbb{X}, \rho)$, by Corollary 3.8, we have $f(\cdot, u(\cdot)) \in S^pWPAA(\mathbb{X}, \rho)$. So we can give the following assumptions which will be used later:

- (A4) There exists $r > 0$ such that $\|f(\cdot, u(\cdot))\|_{S^p} \leq r/\alpha$ for $u \in WPAA(\mathbb{X}, \rho)$ with $\|u\| \leq r$.
- (A5) Let $\{u_n\}$ be a bounded sequence in $WPAA(\mathbb{X}, \rho)$ and uniformly convergent in each compact subset of $\mathbb{R}$. Then $\{f(\cdot, u_n(\cdot))\}$ is relatively compact in $S^pWPAA(\mathbb{X}, \rho)$.

For $u \in WPAA(\mathbb{X}, \rho)$, define

$$Vu = \int_{-\infty}^{t} T(t-s)f(s,u(s))ds = \int_0^\infty T(s)f(t-s,u(t-s))ds.$$

Then we have the following lemma.

Lemma 4.1. *Assume that* (A3) *holds. Then* $V : WPAA(\mathbb{X}, \rho) \to WPAA(\mathbb{X}, \rho)$ *is continuous.*

Proof. For any $\chi \in BS^p(\mathbb{X})$, by (A3), for $t \in \mathbb{R}$,

$$\begin{aligned} \Big\| \int_0^\infty T(\tau)\chi(t-\tau)d\tau \Big\| &\le M \sum_{k=1}^\infty \int_{k-1}^k e^{-c\tau} \|\chi(t-\tau)\| d\tau \\ &\le M \sum_{k=1}^\infty \Big(\int_{k-1}^k e^{-cq\tau} d\tau \Big)^{1/q} \Big(\int_{k-1}^k \|\chi(t-\tau)\|^p d\tau \Big)^{1/p} \\ &= \alpha_0 \sum_{k=1}^\infty e^{-ck} \|\chi(t+k-2+\cdot)\|_p \le \alpha \|\chi\|_{S^p}. \end{aligned} \tag{4.2}$$

Let $u \in WPAA(\mathbb{X}, \rho)$, and denote $\psi(t) = f(t, u(t))$. Then $\psi \in S^pWPAA(\mathbb{X}, \rho)$ by Corollary 3.8. Let $\psi = \psi_1 + \psi_2$ with $\psi_1 \in AS^p(\mathbb{X})$ and $\psi_2^b \in PAA_0(L^p([0,1], \mathbb{X}), \rho)$. Denote

$$\Psi_i(t) = \int_0^\infty T(\tau)\psi_i(t-\tau)d\tau, \quad t \in \mathbb{R}, i = 1, 2.$$

Then by (4.2), for $t, s \in \mathbb{R}$, $i = 1, 2$,

$$\|\Psi_i(t)\| \le \alpha \|\psi_i\|_{S^p},$$

$$\|\Psi_i(t) - \Psi_i(s)\| \le \alpha_0 \sum_{k=1}^\infty e^{-ck} \|\psi_i(t+k-2+\cdot) - \psi_i(s+k-2+\cdot)\|_p.$$

Notice that $\sum_{k=1}^\infty e^{-ck}\|\psi_i(t+k-2+\cdot) - \psi_i(s+k-2+\cdot)\|_p$ is convergent uniformly in $t, s \in \mathbb{R}$. Therefore $\Psi_i \in BC(\mathbb{R}, \mathbb{X}), i = 1, 2$. Now the proof is completed by the following 3 steps.

Step 1. We prove that $\Psi_1 \in AA(\mathbb{X})$. For a sequence $\{s_n'\} \subset \mathbb{R}$, since $\psi_1^b \in AA(L^p(0,1;\mathbb{X}))$, there exist a subsequence $\{s_n\}$ and a function $\hat{\psi}_1 \in L^p_{\rm loc}(\mathbb{R}, \mathbb{X})$ such that, for $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \|\psi_1(t+s_n+\cdot) - \hat{\psi}_1(t+\cdot)\|_p = \lim_{n\to\infty} \|\hat{\psi}_1(t-s_n+\cdot) - \psi_1(t+\cdot)\|_p = 0. \tag{4.3}$$

Let

$$\hat{\Psi}_1(t) = \int_0^\infty T(\tau)\hat{\psi}_1(t-\tau)d\tau, \quad t \in \mathbb{R}.$$

It is easy to see that

$$\sum_{k=1}^\infty e^{-ck} \|\psi_1(t+s_n+k-2+\cdot) - \hat{\psi}_1(t+k-2+\cdot)\|_p$$

is convergent uniformly in $t \in \mathbb{R}$. Then by (4.2) and (4.3), for $t \in \mathbb{R}$,

$$\|\Psi_1(t+s_n) - \hat{\Psi}_1(t)\| = \Big\| \int_0^\infty T(\tau)(\psi_1(t+s_n-\tau) - \psi_1(t-\tau))d\tau \Big\|$$

$$\begin{aligned}&\le \alpha_0 \sum_{k=1}^{\infty} e^{-ck} \|\psi_1(t+s_n+k-2+\cdot) - \hat{\psi}_1(t+k-2+\cdot)\|_p \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Similarly, we can prove that

$$\lim_{n\to\infty} \|\hat{\Psi}_1(t-s_n) - \Psi_1(t)\| = 0 \quad \text{for } t \in \mathbb{R}.$$

That is, $\Psi_1 \in AA(\mathbb{X})$.

Step 2. We prove that $\Psi_2 \in PAA_0(\mathbb{X}, \rho)$. Noticing that $PAA_0(L^p([0,1],\mathbb{X}), \rho)$ is translation invariant, we have

$$\lim_{T\to\infty} \mathcal{W}(T, \psi_2^b(\cdot + k - 2), \rho) = \lim_{T\to\infty} \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t) \|\psi_2(t+k-2+\cdot)\|_p dt = 0,$$

$k = 1, 2, \dots$. It is clear that $\sum_{k=1}^{\infty} e^{-ck} \|\psi_2(t+k-2+\cdot)\|_p$ is convergent uniformly in $t \in \mathbb{R}$ and $\sum_{k=1}^{\infty} e^{-ck} \mathcal{W}(T, \psi_2^b(\cdot + k-2), \rho)$ is convergent uniformly in $T \in (0, \infty)$. Then by (4.2),

$$\begin{aligned}\mathcal{W}(T, \Psi_2, \rho) &= \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t) \Big\| \int_0^{\infty} T(\tau) \psi_2(t-\tau) d\tau \Big\| dt \\ &\le \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t) \big[\alpha_0 \sum_{k=1}^{\infty} e^{-ck} \|\psi_2(t+k-2+\cdot)\|_p\big] dt \\ &= \alpha_0 \sum_{k=1}^{\infty} e^{-ck} \mathcal{W}(T, \psi_2^b(\cdot + k-2), \rho) \to 0 \quad \text{as } T \to \infty.\end{aligned}$$

This means that $\Psi_2 \in PAA_0(\mathbb{X}, \rho)$.

Step 3. We prove the continuity of V. For $\varepsilon > 0$, let $\sigma > 0$ be given by (A1) with f in the place of h, and $u, v \in WPAA(\mathbb{X}, \rho)$ such that $\|u - v\| < \sigma$. Then $\|u(t+\cdot) - v(t+\cdot)\|_p < \sigma$ for $t \in \mathbb{R}$. Denote $\omega(t) = f(t, u(t)) - f(t, v(t)), t \in \mathbb{R}$. We have $\|\omega(t+\cdot)\|_p < \varepsilon$ for $t \in \mathbb{R}$ by (A1), which yields that $\|\omega\|_{S^p} \le \varepsilon$. Thus by (4.2),

$$\|Vu - Vv\| = \sup_{t\in\mathbb{R}} \Big\| \int_0^{\infty} T(\tau) \omega(t-\tau) ds \Big\| \le \alpha \|\omega\|_{S^p} \le \alpha\varepsilon.$$

This implies that $V : WPAA(\mathbb{X}, \rho) \to WPAA(\mathbb{X}, \rho)$ is uniformly continuous. The proof is complete. □

Assume that (A4) holds, and let

$$\mathcal{B} = \{u \in WPAA(\mathbb{X}, \rho) : \|u\| \le r\},$$

where r is given by (A4). Then $\mathcal{B}$ is a bounded closed convex subset of $WPAA(\mathbb{X}, \rho)$, and we have the following result.

Lemma 4.2. *Assume that* (A3)-(A5) *hold. Then* $V : \mathcal{B} \to \mathcal{B}$ *is continuous and* $V(\mathcal{B})$ *satisfies the following conditions:*

(a) $V(\mathcal{B})(t) = \{Vu(t) : u \in \mathcal{B}\} \subset \mathbb{X}$ *is relatively compact for each* $t \in \mathbb{R}$*;*
(b) *As a set of functions from* $\mathbb{R}$ *to* $\mathbb{X}$*,* $V(\mathcal{B})$ *is equicontinuous.*

Proof. For $u \in \mathcal{B}$, by (A4) and (4.2),

$$\|Vu\| = \sup_{t\in\mathbb{R}} \Big\| \int_0^{\infty} T(\tau) f(t-\tau, u(t-\tau)) d\tau \Big\| \le \alpha \|f(\cdot, u(\cdot))\|_{S^p} \le r.$$

Then $V:\mathcal{B}\to\mathcal{B}$ is continuous by Lemma 4.1.

For $t\in\mathbb{R}$, $0<\varepsilon<1$, $u\in\mathcal{B}$, let

$$\begin{aligned}V_\varepsilon u(t)&=\int_{-\infty}^{t-\varepsilon}T(t-s)f(s,u(s))ds\\&=T(\varepsilon)\int_{-\infty}^{t-\varepsilon}T(t-\varepsilon-s)f(s,u(s))ds=T(\varepsilon)Vu(t-\varepsilon).\end{aligned}$$

This implies that $\{V_\varepsilon u(t):u\in\mathcal{B}\}$ is relatively compact in $\mathbb{X}$ since $T(\varepsilon)$ is compact. By (A3) and (A4),

$$\begin{aligned}\|Vu(t)-V_\varepsilon u(t)\|&\le\int_{t-\varepsilon}^{t}\|T(t-\tau)f(\tau,u(\tau))\|d\tau\\&\le M\int_{t-\varepsilon}^{t}e^{-c(t-\tau)}\|f(\tau,u(\tau))\|d\tau\\&\le M\Big(\int_{t-\varepsilon}^{t}e^{-cq(t-\tau)}d\tau\Big)^{1/q}\Big(\int_{t-\varepsilon}^{t}\|f(\tau,u(\tau))\|^pd\tau\Big)^{1/p}\\&\le M\varepsilon^{1/q}\|f(\cdot,u(\cdot))\|_{S^p}\le\frac{Mr}{\alpha}\varepsilon^{1/q}.\end{aligned}$$

This implies that $\{Vu(t):u\in\mathcal{B}\}$ is relatively compact in $\mathbb{X}$ for each $t\in\mathbb{R}$, and (a) holds.

Let $u\in\mathcal{B}$, $t_1,t_2\in\mathbb{R}$ with $t_1<t_2$ and $0<\varepsilon<1$ such that $\eta=(\alpha\varepsilon/(6Mr))^q\le1$. We can decompose $Vu(t_2)-Vu(t_1)=J_1+J_2+J_3$, where

$$\begin{gathered}J_1=\int_{-\infty}^{t_1-\eta}(T(t_2-\tau)-T(t_1-\tau))f(\tau,u(\tau))d\tau,\\J_2=\int_{t_1-\eta}^{t_1}(T(t_2-\tau)-T(t_1-\tau))f(\tau,u(\tau))d\tau,\\J_3=\int_{t_1}^{t_2}T(t_2-\tau)f(\tau,u(\tau))d\tau.\end{gathered}$$

Since $(T(t))_{t\ge0}$ is a C_0-semigroup and $T(t)$ is compact for $t>0$, there exists $\delta\in(0,\eta)$ such that $\|T(t_2-t_1+\eta)-T(\eta)\|\le\varepsilon/(3r)$ for $t_2-t_1<\delta$. Then by (A3), (A4) and (4.2),

$$\begin{aligned}\|J_1\|&=\Big\|\int_{-\infty}^{t_1-\eta}(T(t_2-\tau)-T(t_1-\tau))f(\tau,u(\tau))d\tau\Big\|\\&=\Big\|(T(t_2-t_1+\eta)-T(\eta))\int_{-\infty}^{t_1-\eta}T(t_1-\eta-\tau)f(\tau,u(\tau))d\tau\Big\|\\&\le\frac{\varepsilon}{3r}\Big\|\int_0^\infty T(\tau)f(t_1-\eta-\tau,u(t_1-\eta-\tau))d\tau\Big\|\\&\le\frac{\varepsilon}{3r}\alpha\|f(\cdot,u(\cdot))\|_{S^p}\\&\le\frac{\varepsilon}{3r}r=\frac{\varepsilon}{3},\end{aligned}\tag{4.4}$$

$$\begin{aligned}
\|J_2\| &\le \int_{t_1-\eta}^{t_1} (Me^{-c(t_2-\tau)} + Me^{-c(t_1-\tau)})\|f(\tau,u(\tau))\|d\tau \\
&\le M\Big(\int_{t_1-\eta}^{t_1} \Big(e^{-c(t_2-\tau)} + e^{-c(t_1-\tau)}\Big)^q d\tau\Big)^{1/q}\Big(\int_{t_1-\eta}^{t_1} \|f(\tau,u(\tau))\|^p d\tau\Big)^{1/p} \\
&\le 2M\eta^{1/q}\|f(\cdot,u(\cdot))\|_{S^p} \\
&\le 2M\eta^{1/q}\frac{r}{\alpha} = \frac{\varepsilon}{3}
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
\|J_3\| &\le \int_{t_1}^{t_2} Me^{-c(t_2-\tau)}\|f(\tau,u(\tau))\|d\tau \\
&\le M\Big(\int_{t_1}^{t_2} e^{-cq(t_2-\tau)}d\tau\Big)^{1/q}\Big(\int_{t_1}^{t_2} \|f(\tau,u(\tau))\|^p d\tau\Big)^{1/p} \\
&\le M\delta^{1/q}\|f(\cdot,u(\cdot))\|_{S^p} \\
&\le M\eta^{1/q}\frac{r}{\alpha} = \frac{\varepsilon}{6}.
\end{aligned} \tag{4.6}$$

Combining (4.4)–(4.6), we obtain

$$\|Vu(t_2) - Vu(t_1)\| < \varepsilon.$$

This implies (b), and completes the proof. □

Now we are in the position to give the main result of this section.

Theorem 4.3. *Assume that* (A3)–(A5) *hold. Then* (4.1) *has a mild weighted pseudo almost automorphic solution u such that $\|u\| \le r$.*

Proof. Denote by $\overline{\mathrm{co}}V(\mathcal{B})$ the closed convex hull of $V(\mathcal{B})$. Since $V(\mathcal{B}) \subset \mathcal{B}$ and $\mathcal{B}$ is closed convex, $\overline{\mathrm{co}}V(\mathcal{B}) \subset \mathcal{B}$. Thus $V(\overline{\mathrm{co}}V(\mathcal{B})) \subset V(\mathcal{B}) \subset \overline{\mathrm{co}}V(\mathcal{B})$. It is easy to see that $\overline{\mathrm{co}}V(\mathcal{B})$ also satisfies conditions (a) and (b) in Lemma 4.2 since $V(\mathcal{B})$ does. By the Arzela-Ascoli theorem, the restriction of $\overline{\mathrm{co}}V(\mathcal{B})$ to every compact subset $I \subset \mathbb{R}$, namely $\{u(t) : u \in \overline{\mathrm{co}}V(\mathcal{B})\}_{t\in I}$, is relatively compact in $C(I,\mathbb{X})$.

Let $\{u_n\} \subset \overline{\mathrm{co}}V(\mathcal{B})$. Then $\{u_n(t)\}_{t\in I}$ is relatively compact in $C(I,\mathbb{X})$, and there is a subsequence of $\{u_n(t)\}_{t\in I}$, denoted by $\{u_n(t)\}_{t\in I}$ again, which is convergent in $C(I,\mathbb{X})$. By (A5), $\{f(\cdot,u_n(\cdot))\}$ is relatively compact in $S^pWPAA(\mathbb{X},\rho)$. Thus there exists a subsequence of $\{f(\cdot,u_n(\cdot))\}$, denoted again by $\{f(\cdot,u_n(\cdot))\}$, which is convergent in $S^pWPAA(\mathbb{X},\rho)$, that is for $\varepsilon > 0$, there is $N > 0$ such that

$$\|f(\cdot,u_n(\cdot)) - f(\cdot,u_m(\cdot))\|_{S^p} < \frac{\varepsilon}{\alpha} \quad \text{for } m,n > N.$$

By (4.2), for $m,n > N$,

$$\|Vu_n - Vu_m\| = \sup_{t\in\mathbb{R}}\|Vu_n(t) - Vu_m(t)\| \le \alpha\|f(\cdot,u_n(\cdot)) - f(\cdot,u_m(\cdot))\|_{S^p} < \varepsilon,$$

which implies that $\{Vu_n\}$ is convergent in $WPAA(\mathbb{X},\rho)$. Therefore $V : \overline{\mathrm{co}}V(\mathcal{B}) \to \overline{\mathrm{co}}V(\mathcal{B})$ is a compact operator. Now it follows from Schauder's fixed point theorem that V has a fixed point $u \in \overline{\mathrm{co}}V(\mathcal{B})$. This u is a mild weighted pseudo almost automorphic solution of (4.1) such that $\|u\| \le r$. The proof is complete. □

5. An example

To close this work, we consider the following heat equation with Dirichlet boundary conditions:

$$\begin{gathered}\frac{\partial}{\partial t}u(t,x)=\frac{\partial^2}{\partial x^2}u(t,x)+\sin\frac{1}{2+\cos t+\cos\pi t}+a(t)h(u(t,x)),\quad t\in\mathbb{R},\ x\in[0,1],\\ u(t,0)=u(t,1)=0,\quad t\in\mathbb{R},\end{gathered}\tag{5.1}$$

where $a(t)=\frac{1}{1+t^2}$ and

$$h(u)=\begin{cases}u\sin\frac{1}{u}, & u\neq 0,\\ 0, & u=0.\end{cases}$$

Let $\mathbb{X}=L^2(0,1)$, A be defined by $Au=u''$ with domain $D(A)=\{u\in\mathbb{X}:u''\in\mathbb{X},u(0)=u(1)=0\}$, and $f=g+\phi$ with

$$g(t,u(t))=\sin\frac{1}{2+\cos t+\cos\pi t},\quad \phi(t,u(t))=a(t)h(u(t)),\quad (t,u(t))\in\mathbb{R}\times\mathbb{X}.$$

Then (5.1) can be formulated in the abstract equation (4.1) where $u(t)=u(t,\cdot)\in\mathbb{X},t\in\mathbb{R}$.

It is well known that A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ satisfying that $T(t)$ is compact for $t>0$ and $\|T(t)\|\leq e^{-t}$ for $t\geq 0$. So (A3) holds with $M=c=1$. Clearly, $g\in AS^p(\mathbb{X})$, and it is easy to verify that $PAA_0(\mathbb{R}\times\mathbb{X},L^p(0,1;\mathbb{X}),|t|)$ is translation invariant and

$$\phi^b\in PAA_0(\mathbb{R}\times\mathbb{X},L^p(0,1;\mathbb{X}),|t|).$$

Then $f\in S^pWPAA(\mathbb{R}\times\mathbb{X},\mathbb{X},|t|)$. Moreover, by a long winded but fundamental calculation (the details are omitted), we can check that f satisfies all the conditions of Corollary 3.8 and (A4), (A5), and we can choose $r=2\alpha<2e$. As a result, (5.1) has a mild weighted pseudo almost automorphic solution u such that $\|u\|\leq 2\alpha$ by Theorem 4.3. That is the solution $u\in WPAA(\mathbb{X},|t|)$ satisfies

$$\|u\|=\sup_{t\in\mathbb{R}}\|u(t,\cdot)\|_{\mathbb{X}}=\sup_{t\in\mathbb{R}}\Big(\int_0^1 u^2(t,x)dx\Big)^{1/2}\leq 2\alpha<2e.$$

It is obvious that f does not satisfy any sort of “Lipschitz condition”. Therefore, the results in the literature under some “Lipschitz condition” are not applicable.

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