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ENTIRE SOLUTIONS FOR A MONO-STABLE DELAY POPULATION MODEL IN A 2D LATTICE STRIP

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Abstract. This article concerns the entire solutions of a mono-stable age-structured population model in a 2D lattice strip. In a previous publication, we established the existence of entire solutions related to traveling wave solutions with speeds larger than the minimal wave speed $c_{\min}$. However, the existence of entire solutions related to the minimal wave fronts remains open open question. In this article, we first establish a new comparison theorem. Then, applying the theorem we obtain the existence of entire solutions by mixing any finite number of traveling wave fronts with speeds $c \geq c_{\min}$, and a solution without the j variable. In particular, we show the relationship between the entire solution and the traveling wave fronts that they originate.

1. Introduction

In this article, which may be regarded as a sequel to [13], we consider the entire solutions of the following age-structured population model in a 2-dimensional (2D) lattice strip with Neumann boundary conditions [8, 13],

$$\begin{aligned} \frac{du_{i,j}(t)}{dt} &= D_m \Delta u_{i,j}(t) - d_m u_{i,j}(t) \\ &\quad + \mu \sum_{i_1=1}^{N} \sum_{j_1=-\infty}^{+\infty} G(i, i_1, j, j_1, \alpha) b\big(u_{i_1,j_1}(t-\tau)\big), \\ u_{0,j}(t) &= u_{1,j}(t), \quad u_{N,j}(t) = u_{N+1,j}(t), \end{aligned} \tag{1.1}$$

where $i \in [1, N]_{\mathbb{Z}} := \{1, \dots, N\}$, $j \in \mathbb{Z}$, $t \in \mathbb{R}$, N is a positive integer,

$$\Delta u_{i,j}(t) = u_{i+1,j}(t) + u_{i-1,j}(t) + u_{i,j+1}(t) + u_{i,j-1}(t) - 4u_{i,j}(t); \tag{1.2}$$

$u_{i,j}(t)$ is the density of the mature population of the species at position (i, j) and time t; $\tau > 0$ is the maturation time; $D_m, d_m > 0$ are the diffusion and death rates of mature individuals, respectively; $b(\cdot)$ is the birth function which satisfies the following assumption:

- (A1) $b \in C^2([0, K], \mathbb{R})$, $b(0) = \mu b(K) - d_m K = 0$, $\mu b(u) > d_m u$ and $b'(u) \leq b'(0)$ for $u \in (0, K)$, where $K > 0$ is a constant,
- (A2) $b'(u) \geq 0$ for all $u \in [0, K]$.

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Assume that there is a single species divided into juveniles and adults, which is distributed on the patches in a 2D lattice strip domain $\Omega := [1,N]_{\mathbb{Z}} \times \mathbb{Z}$ with the patches located at the integer nodes $(i,j) \in \Omega$. The above model is derived to express the dynamics for the mature population of the single species by Weng [8] with the following coefficients:

$$\mu = \exp\Big\{-\int_0^\tau d(z)dz\Big\}, \quad \alpha = \int_0^\tau D(z)dz,$$
$$G(i,i_1,j,j_1,t) = G_1(i,i_1,t)\beta_t(j-j_1), \quad \beta_t(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{k\omega i - 4t\sin^2(\omega/2)}\,d\omega,$$

where i is the imaginary unit; $D(a)$ and $d(a)$ are the diffusion and death rates of the juvenile population at age a, $0 < a < \tau$, respectively, and $G_1(i,i_1,t)$ is the Green function of the boundary-value problem

$$\begin{gathered}\frac{dU_i(t)}{dt} = U_{i+1}(t) + U_{i-1}(t) - 2U_i(t), \quad i \in [1,N]_{\mathbb{Z}},\ t > 0,\\ U_0(t) = U_1(t), \quad U_N(t) = U_{N+1}(t), \quad t \geq 0.\end{gathered} \tag{1.3}$$

Assuming mono-stable and quasi-monotone conditions, Weng [8] obtained the spreading speed and its coincidence with the minimal speed of monotone traveling waves by employing the theory of spreading speed and traveling waves for monotone semiflows developed by Liang and Zhao [3]. The study of the traveling wave solutions and spreading speed are important in population dynamics. They can describe certain dynamical behavior of the studied problem such as (1.1). However, the dynamics of delayed lattice differential equations is so rich that there might be other interesting patterns. Recently, quite a few entire solutions have been found in many problems, see e.g. [1, 2, 4, 5, 7, 11, 10, 12, 9]. Here an entire solution is meant by a classical solution defined for all space and time. It is obvious that traveling wave solutions are special examples of the entire solutions.

Recently, in [13], we constructed some new types of entire solutions which are different from traveling wave fronts for (1.1) by considering a combination of traveling wave fronts coming from opposite sides of the j-axis with speeds $c > c_{\min}$ and a solution of (1.1) without j variable. The basic idea in [13], similar to [2], is to use traveling wave fronts and their exponential decay at $-\infty$ to build subsolutions and upper estimates, respectively, and then prove the existence results by employing comparison principle. However, the issue of the existence of entire solution for (1.1) connecting traveling wave fronts with minimal wave speed $c_{\min}$ is still open. Resolving this issue represents a main contribution of our current study.

More precisely, in this paper, we continue to consider the entire solutions of (1.1). Since the decay of the minimal wave front at $-\infty$ is not exponential, we can not apply directly the method in [2, 13] to construct appropriate upper estimates. To overcome this difficulty, we first establish a new comparison theorem (see Lemma 3.1) based on a concavity assumption of the birth function b. Then, applying the comparison theorem, we establish an appropriate upper estimate (supersolution) (see Lemma 3.2) and construct some new types of entire solutions by mixing any number of traveling wave fronts coming from opposite sides of the j-axis with speeds $c \geq c_{\min}$ and a solution of (1.1) without j variable (see Theorem 3.3). Various qualitative features of the entire solutions are also investigated (see Theorem 3.4). In particular, we show the relationship between the entire solution and the traveling wave fronts which they originated.

It should be mention that, in [13], we also established the existence of entire solutions of (1.1) connecting the traveling wave solutions with speeds $c > c_{\min}$ when the quasi-monotone condition does not hold. The main idea is to introduce two auxiliary quasi-monotone equations and establish a comparison argument for the Cauchy problems of the three systems. For the case where the quasi-monotone condition does not hold, we can apply the similar argument as in the proof of Theorem 3.3 to obtain the existence of entire solutions of (1.1) connecting traveling wave solutions with speeds $c \geq c_{\min}$. We leave the details to the readers.

The rest of the paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we establish the existence of entire solutions of (1.1). Various qualitative features of the entire solutions are also investigated.

2. Preliminaries

We first recall some known results on traveling wave fronts and solutions of (1.1) without j variable. Then, we state the well-posedness of initial value problem of (1.1), and establish some comparison theorems.

A traveling wave solution of (1.1) refers to a solution with the form $u_{i,j}(t) = \Phi_c(i, j + ct)$, where $c > 0$ is the wave speed. Letting $\xi = j + ct$, then the profile function of traveling wave solution satisfies the equation

$$\begin{aligned} c\frac{d}{d\xi}\Phi_c(i,\xi) &= D_m[\Phi_c(i+1,\xi) + \Phi_c(i-1,\xi) - 2\Phi_c(i,\xi)] \\ &\quad + D_m[\Phi_c(i,\xi+1) + \Phi_c(i,\xi-1) - 2\Phi_c(i,\xi)] - d_m\Phi_c(i,\xi) \\ &\quad + \mu\sum_{i_1=1}^{N}\sum_{j_1=-\infty}^{+\infty} G_1(i,i_1,\alpha)\beta_\alpha(j_1)b\big(\Phi_c(i_1,\xi-j_1-c\tau)\big), \\ \Phi_c(0,\xi) &= \Phi_c(1,\xi), \quad \Phi_c(N,\xi) = \Phi_c(N+1,\xi), \end{aligned} \tag{2.1}$$

where $i \in [1, N]_{\mathbb{Z}}$ and $\xi \in \mathbb{R}$. The characteristic problem for (2.1) with respect to the trivial equilibrium is

$$\begin{gathered} M(\lambda)v_i = D_m[v_{i+1} + v_{i-1} - 2v_i] + [2D_m(\cosh\lambda - 1) - d_m]v_i \\ + \mu b'(0)e^{-M(\lambda)\tau}e^{2\alpha(\cosh\lambda-1)}\sum_{i_1=1}^{N} G_1(i,i_1,\alpha)v_{i_1}, \\ i \in [1,N]_{\mathbb{Z}},\ \lambda \in \mathbb{R}, \\ v_0 = v_1, \quad v_N = v_{N+1}. \end{gathered} \tag{2.2}$$

From Weng [8], we see that: (i) (2.2) has a positive principal eigenvalue $M(\lambda)$ with strictly positive eigenfunction $v(\lambda) = \{v_i(\lambda)\}_{i\in[1,N]_{\mathbb{Z}}}$; (ii) there exist $c_{\min} > 0$ and $\lambda_* > 0$ such that

$$c_{\min} = \frac{M(\lambda_*)}{\lambda_*} = \inf_{\lambda>0}\frac{M(\lambda)}{\lambda},$$

and for any $c > c_{\min}$, there exists a unique $\lambda_1 := \lambda_1(c) \in (0, \lambda_*)$ such that $M(\lambda_1) = c\lambda_1$, and $M(\lambda) < c\lambda$ for any $\lambda \in (\lambda_1, \lambda_*)$. Moreover, the following result holds, see [13, Proposition 3.1].

Proposition 2.1. *Assume* (A1)–(A2) *hold. For each $c \geq c_{\min}$, system* (1.1) *has a non-decreasing traveling wave solution $\Phi_c(i, j + ct)$ which satisfies $\Phi_c(i, -\infty) = 0$*

and $\Phi_c(i,+\infty)=K$. *Moreover, if* $c>c_{\min}$, *then*

$$\Phi_c'(i,\xi)>0,\quad \lim_{\xi\to-\infty}\Phi_c(i,\xi)e^{-\lambda_1(c)\xi}=v_i(\lambda_1(c)),\quad \Phi_c(i,\xi)\le e^{\lambda_1(c)\xi}v_i(\lambda_1(c))$$

for all $i\in[1,N]_{\mathbb{Z}}$ *and* $\xi\in\mathbb{R}$.

Next, we consider the existence and asymptotic behavior of solutions of (1.1) without j variable; that is, solutions of the problem

$$\begin{aligned} \frac{d\Gamma_i(t)}{dt}&=D_m[\Gamma_{i+1}(t)+\Gamma_{i-1}(t)-2\Gamma_i(t)]-d_m\Gamma_i(t)\\ &\quad+\mu\sum_{i_1=1}^{N}G_1(i,i_1,\alpha)b\big(\Gamma_{i_1}(t-\tau)\big),\quad i\in[1,N]_{\mathbb{Z}},t\in\mathbb{R},\\ &\Gamma_0(t)=\Gamma_1(t),\quad \Gamma_N(t)=\Gamma_{N+1}(t),\quad t\in\mathbb{R}. \end{aligned}\tag{2.3}$$

The characteristic problem for (2.3) with respect to the trivial equilibrium is

$$\begin{aligned} \varsigma v_i&=D_m[v_{i+1}+v_{i-1}-2v_i]-d_mv_i\\ &\quad+\mu b'(0)e^{-\varsigma\tau}\sum_{i_1=1}^{N}G_1(i,i_1,\alpha)v_{i_1},\quad i\in[1,N]_{\mathbb{Z}},\\ &\qquad v_0=v_1,\quad v_N=v_{N+1}. \end{aligned}\tag{2.4}$$

Following [8, 13], Equation (2.4) has a positive principal eigenvalue λ^* with strictly positive eigenfunction $v^*=\{v_i^*\}_{i\in[1,N]_{\mathbb{Z}}}$ and the following result holds.

Proposition 2.2. *Assume* (A1), (A2) *hold. Then there exists a solution* $\Gamma(t)=\{\Gamma_i(t)\}_{i\in[1,N]_{\mathbb{Z}}}$ *of* (2.3) *such that* $\Gamma_i(-\infty)=0$ *and* $\Gamma_i(+\infty)=K$ *for* $i\in[1,N]_{\mathbb{Z}}$. *Moreover*

$$\lim_{t\to-\infty}\Gamma_i(t)e^{-\lambda^*t}=v_i^*,\quad \Gamma_i'(t)>0,\quad \Gamma_i(t)\le e^{\lambda^*t}v_i^*,\quad \text{for } i\in[1,N]_{\mathbb{Z}},\ t\in\mathbb{R}.$$

We now consider the initial value problem of (1.1) with initial condition

$$u_{i,j}(s)=\varphi_{i,j}(s),\quad (i,j)\in\Omega, s\in[r-\tau,r],\tag{2.5}$$

where $r\in\mathbb{R}$ is an any given constant. For convenience, we introduce some notation.

(1) Let $X:=\big\{\phi:\Omega\to\mathbb{R}:\{\phi_{i,j}\}_{(i,j)\in\Omega}\text{ is bounded}\big\}$, $X^+:=\big\{\phi\in X:\phi_{i,j}\ge 0\text{ for }(i,j)\in\Omega\big\}$ and $X_{[0,K]}:=\big\{\phi\in X:\phi_{i,j}\in[0,K]\text{ for }(i,j)\in\Omega\big\}$. It is obvious that X^+ is a closed cone of X under the partial ordering induced by X^+. Moreover, we denote

$$T(t)[\phi](i,j):=e^{-d_mt}\sum_{i_1=1}^{N}\sum_{j_1=-\infty}^{+\infty}G(i,i_1,j,j_1,D_mt)\phi_{i_1,j_1},\quad \forall\phi\in X,\ t>0.$$

We equip X^+ with a compact open topology and define the norm

$$\|\phi\|_X=\sum_{k=0}^{\infty}\frac{\max_{i\in[1,N]_{\mathbb{Z}},|j|\le k}|\phi_{i,j}|}{2^k}.$$

It is clear that $(X,\|\cdot\|_X)$ is a normed space. Let $d(\cdot,\cdot)$ be the metric on X induced by the norm $\|\cdot\|_X$. Then X is a Banach lattice, and $T(t):X\to X$ is a linear C_0-semigroup with $T(t)X^+\subseteq X^+$ for $t>0$.

(2) Let $\mathcal{C} := C([-\tau, 0], X)$ be the Banach space of continuous functions from $[-\tau, 0]$ into X with the supremum norm and $\mathcal{C}^+ := \{\phi \in \mathcal{C} : \phi(s) \in X^+, s \in [-\tau, 0]\}$. Then $\mathcal{C}^+$ is a closed (positive) cone of $\mathcal{C}$. Moreover, we denote

$$\mathcal{C}_{[0,K]} := \{\varphi \in \mathcal{C} : \varphi_{i,j}(s) \in [0, K], \forall (i, j) \in \Omega, s \in [-\tau, 0]\}.$$

As usual, we identify an element $\varphi \in \mathcal{C}$ as a function from $\Omega \times [-\tau, 0]$ into $\mathbb{R}$ defined by $\varphi(i, j, s) = \varphi_{i,j}(s)$. For any continuous function $w : [-\tau, b) \to X$, $b > 0$, we define $w_t \in \mathcal{C}$, $t \in [0, b)$ by $w_t(s) = w(t + s)$, $s \in [-\tau, 0]$. Then $t \to w_t$ is a continuous function from $[0, b)$ to $\mathcal{C}$. For any $\varphi \in \mathcal{C}_{[0,K]}$, define

$$F(\varphi)(i, j) := \mu \sum_{i_1=1}^{N} \sum_{j_1=-\infty}^{+\infty} G(i, i_1, j, j_1, \alpha) b\big(\varphi_{i_1,j_1}(-\tau)\big).$$

Then $F(\varphi) \in X$ and $F : \mathcal{C}_{[0,K]} \to X$ is globally Lipschitz continuous.

The definitions of supersolution and subsolution are given as follows.

Definition 2.3. A continuous function $v : [-\tau, b) \to X$, $b > 0$, is called a supersolution (or subsolution) of (1.1) on $[0, b)$ if for all $0 \le s < t < b$,

$$v(t) \ge (or \le) T(t - s)[v(s)] + \int_s^t T(t - \theta)[F(v_\theta)]d\theta. \tag{2.6}$$

The following results follow from [8, Lemmas 3.1 and 3.3] and [13, Lemma 3.5].

Proposition 2.4. *Assume* (A1)–(A2) *hold. Then the following statements hold.*

(1) For any $\varphi \in \mathcal{C}_{[0,K]}$, there exists a unique solution $u(t; \varphi) = \{u_{i,j}(t; \varphi)\}_{(i,j)\in\Omega}$ of (1.1) *on $[r, +\infty)$ such that $u_{i,j}(s; \varphi) = \varphi_{i,j}(s)$ and $0 \le u_{i,j}(t; \varphi) \le K$ for $(i, j) \in \Omega$, $s \in [r - \tau, r]$ and $t \ge r$. Moreover, there exists a positive constant M, independent of φ and r, such that*

$$\big|u'_{i,j}(t; \varphi)\big| \le M, \quad \big|u''_{i,j}(t; \varphi)\big| \le M \quad \text{for any } (i, j) \in \Omega,\ t > r + \tau.$$

(2) Let $\{u^+_{i,j}(t)\}_{(i,j)\in\Omega}$ and $\{u^-_{i,j}(t)\}_{(i,j)\in\Omega}$ be a supersolution and subsolution of (1.1) *on $[r, +\infty)$ respectively. If $u^+_{i,j}(s) \ge u^-_{i,j}(s)$ for $(i, j) \in \Omega$ and $s \in [r - \tau, r]$, then $u^+_{i,j}(t) \ge u^-_{i,j}(t)$ for $(i, j) \in \Omega$ and $t \ge r$. If, in addition, $u^+_{i,j}(0) \not\equiv u^-_{i,j}(0)$, then $u^+_{i,j}(t) > u^-_{i,j}(t)$ for $(i, j) \in \Omega$ and $t > r$.*

3. Existence of entire solutions

In this section, we establish the existence of entire solutions by mixing any finite number of traveling wave fronts with speeds $c \ge c_{\min}$ and a solution without j variable. In particular, we show the relationship between the entire solution and the traveling wave fronts which they originated.

We first establish a comparison theorem. For this, we need the concavity assumption of the function b:

(A3) $b''(u) \le 0$ for $u \in [0, K]$.

Lemma 3.1. *Assume* (A1)–(A3)*. Let $\varphi^{(k)}, \varphi \in \mathcal{C}_{[0,K]}$, $k = 1, \dots, m$, be $m + 1$ given functions with*

$$\varphi_{i,j}(s) \le \sum_{k=1}^{m} \varphi^{(k)}_{i,j}(s) \quad \text{for } (i, j) \in \Omega, s \in [-\tau, 0].$$

Let $u^{(k)}$ and u be the solutions of Cauchy problems of (1.1) *with the initial values:*

$$u_{i,j}^{(k)}(s)=\varphi_{i,j}^{(k)}(s) \quad u_{i,j}(s)=\varphi_{i,j}(s), \quad (i,j)\in\Omega,\ s\in[-\tau,0], \tag{3.1}$$

respectively. Then

$$0\le u_{i,j}(t)\le \min\big\{K,\sum_{k=1}^{m}u_{i,j}^{(k)}(t)\big\}$$

for all $(i,j)\in\Omega$ and $t\ge 0$.

Proof. Set $\Pi(t)=\{\Pi_{i,j}(t)\}_{(i,j)\in\Omega}$ and $Z(t)=\{Z_{i,j}(t)\}_{(i,j)\in\Omega}$, where

$$\Pi_{i,j}(t)=\sum_{k=1}^{m}u_{i,j}^{(k)}(t), \quad Z_{i,j}(t):=\min\big\{K,\Pi_{i,j}(t)\big\}$$

for $(i,j)\in\Omega$, $t\ge-\tau$. Then $u_{i,j}(s)\le Z_{i,j}(s)$ for $(i,j)\in\Omega$ and $s\in[-\tau,0]$. By Proposition 2.4, it suffices to show that $Z(t)$ is a supersolution of (1.1), i.e.

$$Z(t)\ge T(t-s)[Z(s)]+\int_s^t T(t-r)[F(Z_r)]dr \quad \text{for any } 0\le s<t<+\infty. \tag{3.2}$$

Since $b'(u)\ge 0$ for $u\in[0,K]$, it is easy to see that

$$T(t-s)[Z(s)]+\int_s^t T(t-r)[F(Z_r)]dr\le K \quad \text{for } 0\le s<t<+\infty. \tag{3.3}$$

Now, we show that

$$T(t-s)[Z(s)]+\int_s^t T(t-r)[F(Z_r)]dr\le \Pi(t) \quad \text{for } 0\le s<t<+\infty. \tag{3.4}$$

First, we show that for any $u_k\in(0,K]$, $k=1,\dots,m$,

$$b(\min\{K,u_1+\dots+u_m\})\le b(u_1)+\dots+b(u_m). \tag{3.5}$$

For $m=1$, (3.5) holds obviously. For $m=2$, we consider the following two cases: (i) $u_1+u_2>K$ and (ii) $u_1+u_2\le K$.

For case (i), using the concavity of the function b again, we obtain

$$\frac{b(K)-b(u_1)}{K-u_1}\le\frac{b(u_1)}{u_1}, \quad \frac{b(K)-b(u_2)}{K-u_2}\le\frac{b(u_2)}{u_2},$$

which implies that $u_1b(K)\le Kb(u_1)$ and $u_2b(K)\le Kb(u_2)$. Thus, we have

$$(u_1+u_2)b(K)\le K(b(u_1)+b(u_2))\le(u_1+u_2)(b(u_1)+b(u_2))$$

and hence,

$$b(\min\{K,u_1+u_2\})=b(K)\le b(u_1)+b(u_2).$$

The case (ii) can be considered similarly. Using mathematical induction, we can show that (3.5) holds. By (3.5), it is easy to verify that (3.4) holds. Therefore, $Z(t)$ is a supersolution of (1.1) and the assertion of this lemma follows from Proposition 2.4. $\square$

For any $m,n\in\mathbb{N}\cup\{0\}$, $\theta_1,\dots,\theta_m,\theta_1',\dots,\theta_n',\theta\in\mathbb{R}$, $c_1,\dots,c_m,c_1',\dots,c_n'\ge c_{\min}$ and $\chi\in\{0,1\}$ with $m+n+\chi\ge 2$, we denote

$$\begin{aligned}\varphi_{i,j}^{n}(s):=\max\Big\{&\max_{1\le l\le m}\Phi_{c_l}(i,j+c_ls+\theta_l),\max_{1\le k\le n}\Phi_{c_k'}(i,-j+c_k's+\theta_k'),\\ &\chi\Gamma_i(s+\theta)\Big\},\end{aligned}$$

$$
\begin{aligned}
\underline{u}_{i,j}(t) := \max\Big\{ &\max_{1\le l\le m} \Phi_{c_l}(i, j + c_l t + \theta_l),\ \max_{1\le k\le n} \Phi_{c'_k}(i, -j + c'_k t + \theta'_k), \\
&\chi\Gamma_i(t+\theta)\Big\},
\end{aligned}
$$

where $(i,j)\in\Omega$, $s\in[-n-\tau,-n]$ and $t>-n$. Let $U^n(t)=\{U^n_{i,j}(t)\}_{(i,j)\in\Omega}$ be the unique solution of (1.1) with the initial data

$$
U^n_{i,j}(s)=\varphi^n_{i,j}(s),\quad (i,j)\in\Omega,\ s\in[-n-\tau,-n]. \tag{3.6}
$$

By Proposition 2.4, we have

$$
\underline{u}_{i,j}(t)\le U^n_{i,j}(t)\le K \quad \text{for all } (i,j)\in\Omega,\ t\ge -n.
$$

Applying the comparison lemma 3.1, we obtain the following result which provides the appropriate upper estimate of $U^n(t)$.

Lemma 3.2. *Assume* (A1)–(A3). *The function $U^n(t)=\{U^n_{i,j}(t)\}_{(i,j)\in\Omega}$ satisfies*

$$
U^n_{i,j}(t)\le \overline{U}_{i,j}(t) := \min\big\{K, \Pi(i,j,t)\big\}
$$

for any $(i,j)\in\Omega$ and $t\ge -n$, where

$$
\Pi(i,j,t)=\sum_{l=1}^{m}\Phi_{c_l}(i,j+c_l t+\theta_l)+\sum_{k=1}^{n}\Phi_{c'_k}(i,-j+c'_k t+\theta'_k)+\chi\Gamma_i(t+\theta).
$$

Proof. It is clear that $U^n_{i,j}(s)=\varphi^n_{i,j}(s)\le \Pi(i,j,s)$ for $(i,j)\in\Omega, s\in[-n-\tau,-n]$, and the assertion of this lemma follows directly from Lemma 3.1. $\square$

Following the priori estimate of Proposition 2.4 and upper estimates of Lemma 3.2, we can obtain the following existence result. In the next theorems, we say that a sequence of functions $\Psi_p(t)=\{\Psi_{i,j;p}(t)\}_{(i,j)\in\Omega}$ converges to a function $\Psi_{p_0}(t)=\{\Psi_{i,j;p_0}(t)\}_{(i,j)\in\Omega}$ in the sense of topology $\mathcal{T}$ if, for any compact set $S\subset\Omega\times\mathbb{R}$, the functions $\Psi_{i,j;p}(t)$ and $\Psi'_{i,j;p}(t)$ converge uniformly in S to $\Psi_{i,j;p_0}(t)$ and $\Psi'_{i,j;p_0}(t)$ respectively as p tends to p_0.

Theorem 3.3. *Assume* (A1), (A2) *hold. For any $m,n\in\mathbb{N}\cup\{0\}$, $\theta_1,\dots,\theta_m$, $\theta'_1,\dots,\theta'_n$, $\theta\in\mathbb{R}$, $c_1,\dots,c_m,c'_1,\dots,c'_n\ge c_{\min}$ and $\chi\in\{0,1\}$ with $m+n+\chi\ge 2$, there exists an entire solution $U_p(t)=\big\{U_{i,j;p}(t)\big\}_{(i,j)\in\Omega}$ of* (1.1) *such that*

$$
\underline{u}_{i,j}(t)\le U_{i,j;p}(t)\le K \quad \text{for all } (i,j,t)\in\Omega\times\mathbb{R}, \tag{3.7}
$$

where $p:=p_{m,n,\chi}=\big(c_1,\theta_1,\dots,c_m,\theta_m,c'_1,\theta'_1,\dots,c'_n,\theta'_n,\chi\theta\big)$. Furthermore, the following properties hold.

- (i) *$0<U_{i,j;p}(t)<K$ and $\frac{d}{dt}U_{i,j;p}(t)>0$ for any $(i,j,t)\in\Omega\times\mathbb{R}$.*
- (ii) *If* (A3) *holds, then $U_{i,j;p}(x,t)\le\overline{U}_{i,j}(t)$ for any $(i,j,t)\in\Omega\times\mathbb{R}$.*
- (iii) *For any $\gamma\in\mathbb{R}$, $U_{i,j;p_{m,n,1}}(t)$ converges to $U_{i,j;p_{m,n,0}}(t)$ as $\theta\to-\infty$ in $\mathcal{T}$, and uniformly on $(i,j,t)\in T_\gamma=[1,N]_{\mathbb{Z}}\times\mathbb{Z}\times(-\infty,\gamma]$.*

Proof. By Proposition 2.4, we have

$$
\underline{u}_{i,j}(t)\le U^n_{i,j}(t)\le U^{n+1}_{i,j}(t)\le K \quad \text{for all } (i,j)\in\Omega \text{ and } t\ge -n. \tag{3.8}
$$

Thus, from the priori estimate of Proposition 2.4, there exists a function $U_p(t)=\big\{U_{i,j;p}(t)\big\}_{(i,j)\in\Omega}$ such that $\lim_{n\to+\infty}U^n_{i,j}(t)=U_{i,j;p}(t)$. It is clear that $U_p(t)$ is an entire solution of (1.1). Also, (3.7) follows from (3.8). Moreover, by Lemma 3.2, the assertion of part (ii) holds. The proof of assertion of part (i) is similar to that of [13, Theorem 3.9] and is omitted. We only prove the assertion of part (iii).

(iii) For $\chi = 0$, we denote

$$\varphi^n(s) = \{\varphi^n_{i,j}(s)\}_{(i,j)\in\Omega} \text{ by } \varphi^n_{p_{m,n,0}}(s) = \{\varphi^n_{i,j;p_{m,n,0}}(s)\}_{(i,j)\in\Omega},$$
$$U^n(t) = \{U^n_{i,j}(t)\}_{n\in\mathbb{Z}} \text{ by } U^n_{p_{m,n,0}}(t) = \{U^n_{i,j;p_{m,n,0}}(t)\}_{(i,j)\in\Omega}.$$

Similarly, for $\chi = 1$, we denote $\varphi^n(s)$ by $\varphi^n_{p_{m,n,1}}(s)$ and $U^n(t)$ by $U^n_{p_{m,n,1}}(t)$. Let

$$W^n(t) = \{W^n_{i,j}(t)\}_{n\in\mathbb{Z}} := U^n_{p_{m,n,1}}(t) - U^n_{p_{m,n,0}}(t), \quad (i,j)\in\Omega,\ t \geq -n-\tau.$$

Then $0 \leq W^n_{i,j}(t) \leq K$ for all $(i,j,t) \in \Omega\times[-n,+\infty)$. Moreover, by the assumption $b'(u) \leq b'(0)$ for $u \in [0,K]$, we have

$$\begin{aligned}
\frac{dW^n_{i,j}(t)}{dt} &\leq D_m\Delta W^n_{i,j}(t) - d_m W^n_{i,j}(t)\\
&\quad + \mu b'(0)\sum_{i_1=1}^{N}\sum_{j_1=-\infty}^{+\infty} G(i,i_1,j,j_1,\alpha)W^n_{i_1,j_1}(t-\tau), \quad (i,j)\in\Omega, t>-n,\\
W^n_{0,j}(t) &= W^n_{1,j}(t), \quad W^n_{N,j}(t) = W^n_{N+1,j}(t), \quad j\in\mathbb{Z},\ t\geq -n.
\end{aligned}$$

Let us define the function

$$\widehat{W}(t) = \big\{\widehat{W}_{i,j}(t)\big\}_{(i,j)\in\Omega} = \big\{e^{\lambda^*(t+\theta)}v_i^*\big\}_{(i,j)\in\Omega}.$$

By Proposition 2.2, we have

$$W^n_{i,j}(s) = \varphi^n_{i,j;p_{m,n,1}}(s) - \varphi^n_{i,j;p_{m,n,0}}(s) \leq \Gamma_i(s+\theta) \leq e^{\lambda^*(s+\theta)}v_i^* = \widehat{W}_{i,j}(s)$$

for $(i,j)\in\Omega$, $s \in [-n-\tau,-n]$. Moreover, it is easy to verify that $\widehat{W}(t)$ satisfies the linear system

$$\begin{aligned}
\frac{d\widehat{W}_{i,j}(t)}{dt} &= D_m\Delta \widehat{W}_{i,j}(t) - d_m \widehat{W}_{i,j}(t)\\
&\quad + \mu b'(0)\sum_{i_1=1}^{N}\sum_{j_1=-\infty}^{+\infty} G(i,i_1,j,j_1,\alpha)\widehat{W}_{i_1,j_1}(t-\tau), \quad (i,j)\in\Omega, t>-n,\\
\widehat{W}_{0,j}(t) &= \widehat{W}_{1,j}(t), \quad \widehat{W}_{N,j}(t) = \widehat{W}_{N+1,j}(t), \quad j\in\mathbb{Z},\ t\geq -n.
\end{aligned}$$

It then follows from Proposition 2.4 that

$$0 \leq W^n_{i,j}(t) \leq e^{\lambda^*(t+\theta)}v_i^* \quad \text{for all } (i,j,t)\in\Omega\times[-n,+\infty).$$

Since $\lim_{n\to+\infty} U^n_{i,j;p_{m,n,k}}(t) = U_{i,j;p_{m,n,k}}(t)$, $k = 0,1$, we obtain

$$0 \leq U_{i,j;p_{m,n,1}}(t) - U_{i,j;p_{m,n,0}}(t) \leq e^{\lambda^*(t+\theta)}v_i^* \leq e^{\lambda^*(t+\theta)}\max_{i\in[1,N]_{\mathbb{Z}}} v_i^*$$

for all $(i,j,t) \in \Omega\times\mathbb{R}$, which implies that $U_{p_{m,n,1}}(t)$ converges to $U_{p_{m,n,0}}(t)$ as $\theta\to-\infty$ uniformly on $(i,j,t)\in T_\gamma$ for any $\gamma\in\mathbb{R}$.

For any sequence θ^ℓ with $\theta^\ell\to-\infty$ as $\ell\to+\infty$, the functions $U_{p^\ell_{m,n,1}}(t)$ (where $p^\ell_{m,n,1} := (c_1,\theta_1,\dots,c_m,\theta_m,c_1',\theta_1',\dots,c_n',\theta_n',\theta^\ell)$) converge to a solution of (1.1) (up to extraction of some subsequence) in the sense of topology $\mathcal{T}$, which turns out to be $U_{p_{m,n,0}}(t)$. The limit does not depend on the sequence θ^ℓ, whence all of the functions $U_{p_{m,n,1}}(t)$ converge to $U_{p_{m,n,0}}(t)$ in the sense of topology $\mathcal{T}$ as $\theta\to-\infty$. The proof is complete. □

In the following theorem, we show the relationship between the entire solution $U_p(t)$ and the traveling wave fronts which they originate.

Theorem 3.4. *Let* (A1), (A2) *hold and* $U_p(t)$ *be the entire solution of* (1.1) *stated in Theorem 3.3. Then for any* $c \geq c_{\min}$*, the following properties hold:*

(i) (a) *if* (A3) *holds and there exists* $l_0 \in \{1, \dots, m\}$ *such that* $c_{l_0} = c$ *and* $c_l > c$ *for any* $l \neq l_0$*, then* $U_{i,j-ct;p}(t) \to \Phi_{c_{l_0}}\big(i, j + \theta_{l_0}\big)$ *as* $t \to -\infty$ *with* $j - ct \in \mathbb{Z}$*;*

(b) *if* (A3) *holds and there exists* $k_0 \in \{1, \dots, n\}$ *such that* $c'_{k_0} = c$ *and* $c'_k > c$ *for any* $k \neq k_0$*, then* $U_{i,j+ct;p}(t) \to \Phi_{c'_{k_0}}\big(i, j + \theta'_{k_0}\big)$ *as* $t \to -\infty$ *with* $j + ct \in \mathbb{Z}$*;*

(c) *if* (A3) *holds and* $c_l > c$ *for all* $l \in \{1, \dots, m\}$*, then* $U_{i,j-ct;p}(t) \to 0$ *as* $t \to -\infty$ *with* $j - ct \in \mathbb{Z}$*; and if* $c'_k > c$ *for all* $k \in \{1, \dots, n\}$*, then* $U_{i,j+ct;p}(t) \to 0$ *as* $t \to -\infty$ *with* $j + ct \in \mathbb{Z}$*;*

(d) *if there exists* $l_0 \in \{1, \dots, m\}$ *such that* $c_{l_0} < c$*, then* $U_{i,j-ct;p}(t) \to K$ *as* $t \to -\infty$ *with* $j - ct \in \mathbb{Z}$*; and if there exists* $k_0 \in \{1, \dots, n\}$ *such that* $c'_{k_0} < c$*, then* $U_{i,j+ct;p}(t) \to K$ *as* $t \to -\infty$ *with* $j + ct \in \mathbb{Z}$*.*

(ii) *if there exists* $l_0 \in \{1, \dots, m\}$ *such that* $c_{l_0} > c$*, then* $U_{i,j-ct;p}(t) \to K$ *as* $t \to +\infty$ *with* $j - ct \in \mathbb{Z}$*; and if there exists* $k_0 \in \{1, \dots, n\}$ *such that* $c'_{k_0} > c$*, then* $U_{i,j+ct;p}(t) \to K$ *as* $t \to +\infty$ *with* $j + ct \in \mathbb{Z}$*.*

All the above convergence hold in $\mathcal{T}$*.*

Proof. (i) We only prove the statements (a) and (d), since the others can be proved similarly. From (3.7) and assertion (ii) of Theorem 3.3, we have

$$\begin{aligned} 0 &\leq U_{i,j-c_{l_0}t;p}(t) - \Phi_{c_{l_0}}\big(i, j + \theta_{l_0}\big) \\ &\leq \sum_{1 \leq l \leq m, l \neq l_0} \Phi_{c_l}\big(i, j + (c_l - c_{l_0})t + \theta_l\big) \\ &\quad + \sum_{k=1}^{n} \Phi_{c_k}\big(i, -j + (c'_k + c_{l_0})t + \theta'_k\big) + \chi\Gamma_i(t + \theta), \end{aligned}$$

for all $(i, j, t) \in \Omega \times \mathbb{R}$ with $j - c_{l_0}t \in \mathbb{Z}$. By our assumption, we conclude that $U_{i,j-c_{l_0}t;p}(t) \to \Phi_{c_{l_0}}\big(i, j + \theta_{l_0}\big)$ locally in j as $t \to -\infty$ with $j - c_{l_0}t \in \mathbb{Z}$. Moreover, by Proposition 2.4, the convergence also takes place in $\mathcal{T}$.

Now, we prove the statement (d). Suppose that there exists $l_0 \in \{1, \dots, m\}$ such that $c_{l_0} < c$. Using (3.7), we obtain

$$\Phi_{c_{l_0}}\big(i, j + (c_{l_0} - c)t + \theta_{l_0}\big) \leq U_{i,j-ct;p}(t) \leq K. \tag{3.9}$$

Noting that $\Phi_c(i, +\infty) = K$, we conclude that $U_{i,j-ct;p}(t) \to K$ as $t \to -\infty$ with $j - ct \in \mathbb{Z}$. By Proposition 2.4, the convergence also takes place in $\mathcal{T}$. Similarly, we can show that if there exists $k_0 \in \{1, \dots, n\}$ such that $c'_{k_0} < c$, then $U_{i,j+ct;p}(t) \to K$ as $t \to -\infty$ with $j + ct \in \mathbb{Z}$.

(ii) Suppose that there exists $l_0 \in \{1, \dots, m\}$ such that $c_{l_0} > c$. By (3.9), it is easy to see that $U_{i,j-ct;p}(t) \to K$ as $t \to +\infty$ with $j - ct \in \mathbb{Z}$. Similarly, we can prove the second conclusion of this statement. This completes the proof. □

Remark 3.5. Roughly speaking, the statement (a) of part (i) of Theorem 3.4 mean that only some fronts, those with small speeds, can be "viewed" as $t \to -\infty$, the other ones being "hidden". However, it seems impossible to view any fronts as $t \to +\infty$.

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