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# VALUE DISTRIBUTION OF THE $q$-DIFFERENCE PRODUCT OF ENTIRE FUNCTIONS

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Abstract. For a complex value $q \neq 0, 1$, and a transcendental entire function $f(z)$ with order $0 < \sigma(f) < \infty$, we study the value distribution of $q$-difference product $f(z)f(qz)$ and $f^n(z)(f(qz) - f(z))$. Properties of entire solution of a certain $q$-difference linear equation are also considered.

## 1. Introduction and main results

A meromorphic function $f(z)$ means meromorphic in the complex plane $\mathbb{C}$. If no poles occur, then $f(z)$ reduces to an entire function. For every real number $x \geq 0$, we define $\log^+ x := \max\{0, \log x\}$. Assume that $n(r, f)$ counts the number of the poles of $f$ in $|z| \leq r$, each pole according to its multiplicity, and that $\overline{n}(r, f)$ counts the number of the distinct poles of $f$ in $|z| \leq r$, ignoring the multiplicity. The characteristic function of $f$ is defined by

$$T(r, f) := m(r, f) + N(r, f),$$

where

$$N(r, f) := \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,$$

$$m(r, f) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta.$$

The notation $\overline{N}(r, f)$ is similarly defined with $\overline{n}(r, f)$ instead of $n(r, f)$. For more notations and definitions of the Nevanlinna's value distribution theory of meromorphic functions, refer to [9, 13].

A meromorphic function $\alpha(z)$ is called a small function with respect to $f(z)$, if $T(r, \alpha) = S(r, f)$, where $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \longrightarrow \infty$ outside a possible exceptional set $E$ of logarithmic density 0. The order and the exponent of convergence of zeros of meromorphic function $f(z)$ is respectively defined as

$$\sigma(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r},$$

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$$\lambda(f)=\limsup_{r\to\infty}\frac{\log N(r,\frac{1}{f})}{\log r}.$$

The reduced deficiency of $a$ with respect to $f(z)$ is defined by

$$\Theta(a,f)=1-\limsup_{r\to\infty}\frac{\overline{N}(r,\frac{1}{f-a})}{T(r,f)}.$$

The difference operators for a meromorphic function $f$ are defined as

$$\triangle_c f(z)=f(z+c)-f(z)\quad(c\neq 0),$$
$$\nabla_q f(z)=f(qz)-f(z)\quad(q\neq 0,1).$$

A Borel exceptional value of $f(z)$ is any value $a$ satisfying $\lambda(f-a)<\sigma(f)$.

Recently, the difference variant of the Nevanlinna theory has been established independently in [2, 6, 7, 8]. Using these theories, value distributions of difference polynomials have been studied by many papers. For example, Laine and Yang [10] proved if $f(z)$ is a transcendental entire function of finite order, $c$ is a nonzero complex constant and $n\geq 2$, then $f^n(z)f(z+c)$ takes every nonzero value infinitely often. Liu and Yang [11] proved the following theorem.

**Theorem 1.1** ([11, Theorem 1.4])**.** *Let $f(z)$ be a transcendental entire function of finite order, and $c$ be a nonzero complex constant, $\triangle_c f(z)=f(z+c)-f(z)\not\equiv 0$. Then for $n\geq 2$, $f^n(z)\triangle_c f(z)-p(z)$ has infinitely many zeros, where $p(z)\not\equiv 0$ is a polynomial in $z$.*

The following theorems discussed the case $n\geq 2$. For the case $n=1$, Chen [3], Chen-Huang-Zheng [5] considered value distributions of $f(z)f(z+c), f(z)\Delta_c f(z)$.

**Theorem 1.2** ([5, Corollary 1.3])**.** *Let $f(z)$ be a transcendental entire function of finite order, and $c$ be a nonzero complex constant. If $f(z)$ has the Borel exceptional value $0$, then $H(z)=f(z)f(z+c)$ takes every nonzero value $a\in\mathbb{C}$ infinitely often.*

**Theorem 1.3** ([3, Theorem 2])**.** *Let $f(z)$ be a finite order transcendental entire function with a finite Borel exceptional value $d$, and let $c\in\mathbb{C}\setminus\{0\}$ be a constant satisfying $f(z+c)\not\equiv f(z)$. Set $H(z)=f(z)\Delta_c f(z)$ where $\triangle_c f(z)=f(z+c)-f(z)$. Then the following statements hold:*

1. *$H(z)$ takes every nonzero value $a\in\mathbb{C}$ infinitely often and satisfies $\lambda(H-a)=\sigma(f)$.*
2. *If $d\neq 0$, then $H(z)$ has no any finite Borel exceptional value.*
3. *If $d=0$, then $0$ is also the Borel exceptional value of $H(z)$. So that $H(z)$ has no nonzero finite Borel exceptional value.*

**Theorem 1.4** ([3, Theorem 3])**.** *Let $f(z)$ be a transcendental entire function of finite order and let $c\in\mathbb{C}\setminus\{0\}$ be a constant satisfying $f(z+c)\not\equiv f(z)$. Set $H(z)=f(z)\Delta_c f(z)$ where $\triangle_c f(z)=f(z+c)-f(z)$. If $f(z)$ has infinitely many multi-order zeros, then $H(z)$ takes every value $a\in\mathbb{C}$ infinitely often.*

Chen [3] also considered zeros of difference product $H_n(z)=f^n(z)\Delta_c f(z)$ and gave some conditions guarantee $H_n(z)$ has finitely many zeros or infinitely many zeros.

**Theorem 1.5** ([3, Theorem 1])**.** *Let $f(z)$ be a transcendental entire function of finite order and $c\in\mathbb{C}\setminus\{0\}$ be a constant satisfying $f(z+c)\not\equiv f(z)$. Set $H_n(z)=$*

*$f^n(z)\triangle_c f(z)$ where $\triangle_c f(z) = f(z+c) - f(z)$, $n \geq 2$ is an integer. Then the following statements hold:*

(1) *If $f(z)$ satisfies $\sigma(f) \neq 1$, or has infinitely many zeros, then $H_n(z)$ has infinitely many zeros.*
(2) *If $f(z)$ has only finitely many zeros and $\sigma(f) = 1$, then $H_n(z)$ has only finitely many zeros.*

The Nevanlinna theory for the $q$-difference operator plays an important part in considering value distributions of $q$-difference polynomials. Since $q$-difference logarithmic derivative lemma is only use for meromorphic functions of zero order, most papers only consider meromorphic functions of zero order. For example, Zhang and Korhonen [15] proved that for a transcendental entire function $f(z)$ of zero order and a nonzero complex constant $q$ and $n \geq 2$, $f^n(z)f(qz)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often. Recently, Liu-Liu-Cao [12] extended this to consider zero distributions of $q$-difference products $f^n(z)(f^m(z) - a)f(qz + c)$ and $f^n(z)(f^m(z) - a)[f(qz + c) - f(z)]$ for meromorphic function $f$ with order zero.

It is natural to ask how about value distribution of $q$-difference products for functions with positive order? The main purpose of this paper is to consider a transcendental entire function $f$ with positive and finite order, and obtain some results on the value distributions of $q$-difference products $f(z)f(qz)$ and $f^n(z)(f(qz) - f(z))$. However, in this case, we have to add the condition that $f$ has finitely many zeros, or something like that. The first main theorem is as follows.

**Theorem 1.6.** *Let $f(z)$ be a transcendental entire function of finite and positive order $\sigma(f)$, $q \in \mathbb{C} \setminus \{0\}$ be a constant satisfying $q^{\sigma(f)} \neq -1$. Set $H(z) = f(z)f(qz)$. If $f(z)$ has finitely many zeros, then $H(z) - \alpha(z)$ has infinitely many zeros, where $\alpha(z)$ is a nonzero small entire function with respect to $f(z)$.*

If replacing by the condition that $f(z)$ has infinitely many multi-order zeros and considering any value $a$ which can be zero, then we have another theorem.

**Theorem 1.7.** *Let $f(z)$ be a transcendental entire function of finite and positive order $\sigma(f)$, $q \in \mathbb{C} \setminus \{0\}$ be a constant. Set $H(z) = f(z)f(qz)$. If $f(z)$ has infinitely many multi-order zeros, then $H(z)$ takes every value $a \in \mathbb{C}$ infinitely often.*

For the $q$-difference product $f(z)(f(qz) - f(z))$ we have the following main theorem.

**Theorem 1.8.** *Let $f(z)$ be a transcendental entire function of finite and positive order $\sigma(f)$, $q \in \mathbb{C} \setminus \{0, 1\}$ be a constant satisfying $q^{\sigma(f)} \neq \pm 1$ and $f(z) \not\equiv f(qz)$, set $H(z) = f(z)\nabla_q f(z)$. If $f(z)$ has finitely many zeros, then $H(z) - \alpha(z)$ has infinitely many zeros, where $\alpha(z)$ is a small entire function with respect to $f(z)$.*

By the definition of Borel exceptional value and the proof of Theorem 1.8, the following result is immediately true.

**Corollary 1.9.** *Let $f(z)$ be a transcendental entire function of finite and positive order $\sigma(f)$, $q \in \mathbb{C} \setminus \{0, 1\}$ be a constant satisfying $q^{\sigma(f)} \neq \pm 1$ and $f(z) \not\equiv f(qz)$, set $H(z) = f(z)\nabla_q f(z)$. If $f(z)$ has finitely many zeros, then $H(z)$ has no any finite Borel exceptional value.*

If replacing by the condition that $f(z)$ has infinitely many multi-order zeros, we also have the following theorem.

**Theorem 1.10.** *Let $f(z)$ be a transcendental entire function of finite and positive order $\sigma(f)$, $q \in \mathbb{C} \setminus \{0,1\}$ be a constant satisfying $q^{\sigma(f)} \neq \pm 1$ and $f(z) \not\equiv f(qz)$, set $H(z) = f(z)\nabla_q f(z)$. If $f(z)$ has infinitely many multi-order zeros, then $H(z)$ takes every value $a \in \mathbb{C}$ infinitely often.*

If considering zero distribution of $q$-difference product $f^n(z)\nabla_q f(z)$, we have the following result whether $f$ has finitely many zeros or not.

**Theorem 1.11.** *Let $f(z)$ be a transcendental entire function of finite and positive order $\sigma(f)$, $q \in \mathbb{C} \setminus \{0,1\}$ be a constant satisfying $q^{\sigma(f)} \neq 1$ and $f(z) \not\equiv f(qz)$. Set $H_n(z) = f^n(z)\nabla_q f(z)$, $n \geq 1$ is an integer. Then $H_n(z)$ has infinitely many zeros.*

Chen [4] considered complex linear difference equations and obtained that the relation between $\lambda(g)$ and $\sigma(g)$ of entire solutions to nonhomogeneous linear difference equations is better than that of homogeneous equations. Next, we will consider a special $q$-difference linear equation and obtain the following result.

**Theorem 1.12.** *Let $F(z)$ and $h_j(z)(j = 1,\dots,n)$ be entire functions with orders all less than one, such that at least one of $h_j(z) \not\equiv 0$, and let $q_j(j = 1,\dots,n) \in \mathbb{C}\setminus\{0,1\}$ be constants satisfying $(\frac{q_s}{q_t})^{\sigma(f)} \neq 1$ for any $s \neq t$. Suppose that $f(z)$ is a finite and positive order transcendental entire solution of linear $q$-difference equation*

$$h_n(z)f(q_n z) + \dots + h_1(z)f(q_1 z) = F(z). \tag{1.1}$$

*Then $f(z)$ has infinitely many zeros.*

There exist many solutions which satisfy the functional equation (1.1). For example:

**Example 1.13.** It is known that the transcendental entire function $f(z) = z + \cos z^3$ with order three has infinitely many zeros. Let $h_1(z) = z^5 = -h_2(z)$, and let $q_2 = 2$, $q_1 = -2$. Obviously, $(\frac{q_2}{q_1})^{\sigma(f)} = (-1)^3 \neq 1$. Then the function $f(z)$ satisfies the non-homogeneous linear $q$-difference equation

$$h_2(z)f(q_2 z) + h_1(z)f(q_1 z) = -4z^6.$$

The following example shows that the condition $(\frac{q_s}{q_t})^{\sigma(f)} \neq 1$ for any $s \neq t$ in Theorem 1.12 is necessary.

**Example 1.14.** Let $h_1(z) = -h_2(z) \not\equiv 0$, $q_2 = q_1 = q \neq 0, 1$. Then the function $f(z) = e^z$ with order one satisfies the homogeneous linear $q$-difference equation

$$h_2(z)f(q_2 z) + h_1(z)f(q_1 z) = 0.$$

Here, $(\frac{q_2}{q_1})^{\sigma(f)} = 1$, and but $f(z) = e^z$ has no zeros.

## 2. Lemmas

To prove our results, we need some lemmas. The first one is the well-known Weierstrass factorization theorem and Hadamard factorization theorem.

**Lemma 2.1** ([1]). *If an entire function $f$ has a finite exponent of convergence $\lambda(f)$ for its zero-sequence, then $f$ has a representation in the form*

$$f(z) = Q(z)e^{g(z)},$$

*satisfying $\lambda(Q) = \sigma(Q) = \lambda(f)$. Further, if $f$ is of finite order, then $g$ in the above form is a polynomial of degree less or equal to the order of $f$.*

**Lemma 2.2** ([14]). *Suppose that $f_1(z), f_2(z), \dots, f_n(z), (n \geq 2)$ are meromorphic functions and $g_1(z), g_2(z), \dots, g_n(z)$ are entire functions satisfying the following conditions*

(1) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;
(2) $g_j(z) - g_k(z)$ *are not constants for* $1 \leq j < k \leq n$;
(3) *For* $1 \leq j \leq n, 1 \leq h < k \leq n$, $T(r, f_j) = o(T(r, e^{g_h - g_k}))$ $(r \to \infty, r \notin E)$.

*Then* $f_j(z) \equiv 0 (j = 1, 2, \dots, n)$.

**Lemma 2.3.** *Let $f(z)$ be a transcendental entire function of finite and positive order $\sigma(f)$, $q \in \mathbb{C} \setminus \{0\}$ be a constant satisfying $q^{\sigma(f)} \neq -1$. Set $H(z) = f(z)f(qz)$. If $f(z)$ has finitely many zeros, then $H(z)$ is a transcendental entire function and $\sigma(H) = \sigma(f)$.*

*Proof.* Since $f(z)$ is a transcendental entire function of finite order and has finitely many zeros, by Lemma 2.1, $f(z)$ can be written as

$$f(z) = g(z)e^{h(z)},$$

where $g(z)(\not\equiv 0)$, $h(z)$ are polynomials. Set

$$h(z) = a_k z^k + \dots + a_0,$$

where $a_k, \dots, a_0$ are constants, $a_k \neq 0$. Since $\sigma(f) \neq 0$, it follows that $\sigma(f) = \deg(h(z)) = k \geq 1$. So

$$H(z) = f(z)f(qz) = g(z)g(qz)e^{(a_k + a_k q^k)z^k + \dots + 2a_0}.$$

Since $g(z)(\not\equiv 0)$ is a polynomial, $q^{\sigma(f)} = q^k \neq -1$, it follows that $H(z)$ is a transcendental entire function and $\sigma(H) = \sigma(f) = k$. $\square$

## 3. Proofs of main results

3.1. **Proof of Theorem 1.6.** Since $f(z)$ is a transcendental entire function of finite order and has finitely many zeros, by Lemma 2.1, $f(z)$ can be written as

$$f(z) = g(z)e^{h(z)},$$

where $g(z)(\not\equiv 0)$, $h(z)$ are polynomials. Set

$$h(z) = a_k z^k + \dots + a_0,$$

where $a_k, \dots, a_0$ are constants, $a_k \neq 0$. Since $\sigma(f) \neq 0$, it follows that $\sigma(f) = \deg(h(z)) = k \geq 1$. So

$$H(z) = f(z)f(qz) = g(z)g(qz)p(z)e^{(1+q^k)a_k z^k}, \tag{3.1}$$

where $p(z) = e^{(1+q^{k-1})a_{k-1}z^{k-1} + \dots + 2a_0}$, $\sigma(p) \leq k - 1 < k$.

Suppose $H(z) - \alpha(z)$ has finitely many zeros, by Lemma 2.3, $\sigma(H - \alpha) = \sigma(H) = \sigma(f)$, then $H(z) - \alpha(z)$ can be written as

$$H(z) - \alpha(z) = s(z)e^{tz^k}, \tag{3.2}$$

where $s(z)$ is an entire function with $\sigma(s) < k$, $t \neq 0$ is a constant. By (3.1) and (3.2), we obtain

$$g(z)g(qz)p(z)e^{(1+q^k)a_k z^k} - s(z)e^{tz^k} - \alpha(z) = 0. \tag{3.3}$$

Case 1: $(1+q^k)a_k \neq t$. Since $\sigma(p) < k$, $\sigma(s) < k$, $g(z)$ and $g(qz)$ are polynomials, by Lemma 2.2, we obtain

$$g(z)g(qz)p(z) \equiv 0, s(z) \equiv 0, \alpha(z) \equiv 0.$$

Which is a contradiction.

Case 2: $(1+q^k)a_k = t$. Then, (3.3) can be written as

$$g(z)g(qz)p(z) - s(z) = \alpha(z)e^{-tz^k}.$$

Since $\sigma(g(z)g(qz)p(z) - s(z)) < k$, while $\sigma(\alpha(z)e^{-tz^k}) = k$, which is a contradiction.

Therefore, $H(z) - \alpha(z)$ has infinitely many zeros.

3.2. **Proof of Theorem 1.7.** If $a = 0$, then $H(z)$ has obviously infinitely many zeros since $f(z)$ has infinitely many zeros.

Now we consider $a \neq 0$. Suppose $H(z) - a$ has finitely many zeros, by Lemma 2.1, $H(z) - a$ can be rewritten as

$$H(z) - a = f(z)f(qz) - a = g(z)e^{h(z)}, \tag{3.4}$$

where $g(z)(\not\equiv 0)$, $h(z)$ are polynomials, $\deg(h(z)) \geq 1$. Differentiating (3.4) and eliminating $e^{h(z)}$, we obtain

$$\frac{(f(z)f(qz))'}{f(z)f(qz)} = \frac{g'(z) + g(z)h'(z)}{g(z)} - a\frac{g'(z) + g(z)h'(z)}{g(z)}\frac{1}{f(z)f(qz)}. \tag{3.5}$$

Since $g(z)(\not\equiv 0)$, $h(z)$ are polynomials and $\deg(h(z)) \geq 1$, it follows that $g'(z) + g(z)h'(z) \not\equiv 0$. As $f(z)$ has infinitely many multi-order zeros, there is a multi-order zero $z_0$, such that $\mid z_0 \mid$ is sufficiently large and $g(z_0) \neq 0, g'(z_0) + g(z_0)h'(z_0) \neq 0$. Thus the right side of (3.5) has a multi-order pole at $z_0$, but the left side of (3.5) has only a simple pole at $z_0$, which is a contradiction.

Hence $H(z)$ takes every value $a \in \mathbb{C}$ infinitely often.

3.3. **Proof of Theorem 1.8.** Since $f(z)$ is a transcendental entire function of finite order and has finitely many zeros, by Lemma 2.1, $f(z)$ can be written as

$$f(z) = g(z)e^{h(z)},$$

where $g(z)(\not\equiv 0)$, $h(z)$ are polynomials, set

$$h(z) = a_k z^k + \dots + a_0,$$

where $a_k, \dots, a_0$ are constants, $a_k \neq 0$. Since $\sigma(f) \neq 0$, it follows that $\sigma(f) = \deg(h(z)) = k \geq 1$.

$$H(z) = f(z)\nabla_q f(z) = g(z)g(qz)p_1(z)e^{(1+q^k)a_k z^k} - g^2(z)p_2(z)e^{2a_k z^k}, \tag{3.6}$$

where

$$p_1(z) = e^{(1+q^{k-1})a_{k-1}z^{k-1}+\dots+2a_0}, \quad \sigma(p_1) \leq k-1 < k;$$
$$p_2(z) = e^{2a_{k-1}z^{k-1}+\dots+2a_0}, \quad \sigma(p_2) \leq k-1 < k.$$

Since $g(z)(\not\equiv 0)$ is a polynomial, $q^{\sigma(f)} = q^k \neq \pm 1$, $\sigma(p_1) < k$, $\sigma(p_2) < k$, it follows $H(z)$ is a transcendental entire function and $\sigma(H) = \sigma(f) = k$.

Suppose $H(z) - \alpha(z)$ has finitely many zeros, then $\lambda(H - \alpha) < \sigma(H) = \sigma(f)$, $H(z) - \alpha(z)$ can be written as

$$H(z) - \alpha(z) = s(z)e^{tz^k}, \tag{3.7}$$

where $s(z)$ is an entire function with $\sigma(s) < k$, $t \neq 0$ is a constant. By (3.6) and (3.7), we obtain

$$g(z)g(qz)p_1(z)e^{(1+q^k)a_k z^k} - g^2(z)p_2(z)e^{2a_k z^k} - s(z)e^{tz^k} - \alpha(z) = 0. \tag{3.8}$$

Since $q^{\sigma(f)} = q^k \neq 1$, it follows that $(1+q^k)a_k \neq 2a_k$.

Case 1: $(1+q^k)a_k \neq t, 2a_k \neq t$. By Lemma 2.2, we obtain

$$g(z)g(qz)p_1(z) \equiv 0, g^2(z)p_2(z) \equiv 0, s(z) \equiv 0, \alpha(z) \equiv 0.$$

This is a contradiction.

Case 2: $(1+q^k)a_k = t$. Then (3.8) can be written as

$$(g(z)g(qz)p_1(z) - s(z))e^{(1+q^k)a_k z^k} - g^2(z)p_2(z)e^{2a_k z^k} - \alpha(z) = 0.$$

By Lemma 2.2, we obtain

$$g(z)g(qz)p_1(z) - s(z) \equiv 0, g^2(z)p_2(z) \equiv 0, \alpha(z) \equiv 0,$$

which is a contradiction.

Case 3: $2a_k = t$. Then using the same method as above, we also obtain a contradiction.

Hence $H(z) - \alpha(z)$ has infinitely many zeros.

3.4. **Proof of Theorem 1.10.** If $a = 0$, then $H(z)$ has obviously infinitely many zeros as $f(z)$ has infinitely many zeros and $f(qz) - f(z) \not\equiv 0$.

Now we consider $a \neq 0$. Suppose $H(z) - a$ has finitely many zeros, from the proof of Theorem 1.8, $H(z)$ is a transcendental entire function and $\sigma(H) = \sigma(f)$. So by Lemma 2.1, $H(z) - a$ can be rewritten as

$$H(z) - a = f(z)f(qz) - f^2(z) - a = g(z)e^{h(z)}, \tag{3.9}$$

where $g(z)(\not\equiv 0)$, $h(z)$ are polynomials, $\deg(h(z)) \geq 1$. Differentiating (3.9)and eliminating $e^{h(z)}$, we obtain

$$\frac{(f(z)f(qz))'}{f(z)f(qz)} - \frac{2f'(z)}{f(qz)} = \frac{g'(z) + g(z)h'(z)}{g(z)}[1 - \frac{f(z)}{f(qz)} - \frac{a}{f(z)f(qz)}]. \tag{3.10}$$

Since $g(z)(\not\equiv 0)$, $h(z)$ are polynomials and $\deg(h(z)) \geq 1$, it follows that $g'(z) + g(z)h'(z) \not\equiv 0$. As $f(z)$ has infinitely many multi-order zeros, there is a multi-order zero $z_0$ of multiplicity $k \geq 2$, such that $|z_0|$ is sufficiently large and $g(z_0) \neq 0, g'(z_0) + g(z_0)h'(z_0) \neq 0$.

If $f(qz)$ has zero at $z_0$ of multiplicity $s \geq 1$, then $\frac{(f(z)f(qz))'}{f(z)f(qz)}$ has a simple pole at $z_0$; $-\frac{2f'(z)}{f(qz)}$ has pole at $z_0$ of multiplicity $s-k+1$; $\frac{f(z)}{f(qz)}$ has pole at $z_0$ of multiplicity $s-k$; but $\frac{a}{f(z)f(qz)}$ has pole at $z_0$ of multiplicity $s+k$. This is a contradiction.

If $f(qz_0) \neq 0$, then $\frac{(f(z)f(qz))'}{f(z)f(qz)}$ has a simple pole at $z_0$; $\frac{f(z_0)}{f(qz_0)} = 0$; $-\frac{2f'(z_0)}{f(qz_0)} = 0$; but $\frac{a}{f(z)f(qz)}$ has pole at $z_0$ of multiplicity $k \geq 2$. We also have a contradiction.

Hence $H(z)$ takes every value $a \in \mathbb{C}$ infinitely often.

3.5. **Proof of Theorem 1.11.** If $f(z)$ has infinitely many zeros, then $H_n(z)$ has infinitely many zeros since $f(qz) - f(z) \not\equiv 0$.

Now we consider $f(z)$ has finitely many zeros, suppose $H_n(z)$ has only finitely many zeros. Since $f(z)$ is a transcendental entire function of finite order and has finitely many zeros, by Lemma 2.1, $f(z)$ can be written as

$$f(z) = g(z)e^{h(z)},$$

where $g(z)(\not\equiv 0)$, $h(z)$ are polynomials. Set

$$h(z) = a_k z^k + \dots + a_0,$$

where $a_k, \dots, a_0$ are constants, $a_k \neq 0$. Since $\sigma(f) \neq 0$, it follows that $\sigma(f) = \deg(h(z)) = k \geq 1$. So

$$H_n(z) = f^n(z)\nabla_q f(z) = g^n(z)g(qz)e^{nh(z)+h(qz)} - g^{n+1}(z)e^{(n+1)h(z)} \tag{3.11}$$

and

$$(n+1)h(z)-(nh(z)+h(qz)) = (1-q^k)a_k z^k + (1-q^{k-1})a_{k-1}z^{k-1} + \dots + (1-q)a_1 z.$$

Since $q^{\sigma(f)} = q^k \neq 1$, it follows that $(n+1)h(z)-(nh(z)+h(qz))$ is not a constant. So $H_n(z)$ is a transcendental entire function, by Lemma 2.1, $H_n(z)$ can be written as

$$H_n(z) = g_1(z)e^{h_1(z)}, \tag{3.12}$$

where $g_1(z)(\not\equiv 0)$, $h_1(z)$ are polynomials. By (3.11) and (3.12), we obtain

$$g^n(z)g(qz)e^{nh(z)+h(qz)} - g^{n+1}(z)e^{(n+1)h(z)} - g_1(z)e^{h_1(z)} = 0. \tag{3.13}$$

Note that $(n+1)h(z)-(nh(z)+h(qz))$ is not a constant and $g(z)$ and $g_1(z)$ are polynomials. If $(nh(z)+h(qz)) - h_1(z)$ and $(n+1)h(z) - h_1(z)$ are not constants, then by (3.13) and Lemma 2.2, we obtain

$$g^n(z)g(qz) \equiv 0, g^{n+1}(z) \equiv 0, g_1(z) \equiv 0.$$

This is a contradiction.

If $(nh(z)+h(qz)) - h_1(z) = c$, where $c$ is a constant, then (3.13) can be written as

$$(g^n(z)g(qz) - e^{-c}g_1(z))e^{nh(z)+h(qz)} - g^{n+1}(z)e^{(n+1)h(z)} = 0. \tag{3.14}$$

By (3.14) and Lemma 2.2, we obtain

$$g^n(z)g(qz) - e^{-c}g_1(z) \equiv 0, g^{n+1}(z) \equiv 0,$$

Which is a contradiction.

If $(n+1)h(z) - h_1(z) = c$, where $c$ is a constant, then using the same method as above, we also obtain a contradiction.

Hence $H_n(z)$ has infinitely many zeros.

3.6. **Proof of Theorem 1.12.** Suppose $f(z)$ has finitely many zeros, since $f(z)$ is a transcendental entire function of finite and positive order, it follows by Lemma 2.1, that $f(z)$ can be written as

$$f(z) = g(z)e^{h(z)}, \tag{3.15}$$

where $g(z)(\not\equiv 0)$, $h(z)$ are polynomials. Set

$$h(z) = a_k z^k + \dots + a_0,$$

where $a_k, \dots, a_0$ are constants, $a_k \neq 0$. Since $\sigma(f) \neq 0$, it follows that $\sigma(f) = \deg(h(z)) = k \geq 1$. Substituting (3.15)into (1.1), we obtain

$$h_n(z)g(q_n z)e^{h(q_n z)} + \dots + h_1(z)g(q_1 z)e^{h(q_1 z)} = F(z) \tag{3.16}$$

and

$$h(q_s z) - h(q_t z) = a_k(q_s^k - q_t^k)z^k + a_{k-1}(q_s^{k-1} - q_t^{k-1})z^{k-1} + \dots + a_1(q_s - q_t)z,$$

where $s \neq t$. Since $q_j (j = 1, \dots, n) \in \mathbb{C} \setminus \{0, 1\}$ and $(\frac{q_s}{q_t})^{\sigma(f)} \neq 1$ for $s \neq t$, it follows that

$$q_s^k \neq q_t^k, \deg(h(q_s z) - h(q_t z)) = k, \sigma(e^{h(q_s z) - h(q_t z)}) = k.$$

Since $\sigma(h_j(z)g(q_j z)) < 1 < k$ for $j = 1, \dots, n$, $\sigma(F) < 1 < k$, by (3.16) and Lemma 2.2, we obtain

$$h_n(z)g(q_n z) \equiv 0, \dots, h_1(z)g(q_1 z) \equiv 0, F(z) \equiv 0.$$

We get a contradiction. Hence $f(z)$ has infinitely many zeros.

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