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EXISTENCE OF INFINITELY MANY RADIAL SOLUTIONS FOR QUASILINEAR SCHRÖDINGER EQUATIONS

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Abstract. In this article we prove the existence of radial solutions with arbitrarily many sign changes for quasilinear Schrödinger equation

$$-\sum_{i,j=1}^{N} \partial_j(a_{ij}(u)\partial_i u) + \frac{1}{2}\sum_{i,j=1}^{N} a'_{ij}(u)\partial_i u\partial_j u + V(x)u = |u|^{p-1}u,\ x \in \mathbb{R}^N,$$

where $N \geq 3$, $p \in (1, \frac{3N+2}{N-2})$. The proof is accomplished by using minimization under a constraint.

1. Introduction

We consider the quasilinear elliptic problem

$$-\sum_{i,j=1}^{N} \partial_j(a_{ij}(u)\partial_i u) + \frac{1}{2}\sum_{i,j=1}^{N} a'_{ij}(u)\partial_i u\partial_j u + V(x)u = |u|^{p-1}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $1 < p < 2(2^*) - 1 = \frac{3N+2}{N-2}$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev constant, $a_{ij} \in C^{1,\alpha}(\mathbb{R})$ is a symmetric matrix function, $\alpha \in (0,1)$ and $a'_{ij}(u) = \frac{d}{du}a_{ij}(u)$.

For $a_{ij}(u) = (1+u^2)\delta_{ij}$, Equation (1.1) is reduced to the well known Modified Nonlinear Schrödinger Equation

$$-\Delta u + V(x)u - \frac{1}{2}u\Delta(u^2) = |u|^{p-1}u, \quad x \in \mathbb{R}^N. \tag{1.2}$$

This type of equations arise from the study of steady states and standing wave solutions of time-dependent nonlinear Schrödinger equations, and are derived as models in various branches of mathematical physics; see [3, 5, 6, 8, 13, 16, 17, 19, 22].

In the literature several papers have considered problem (1.2). For example, the existence of positive ground state solution of (1.2) was proved by Poppenberg, Schmitt and Wang [20] by using a constrained minimization argument. Liu et al [15], by a change of variables, transformed the quasilinear problem into a semilinear one, and used an Orlicz space was the working space. The authors proved the existence of soliton solutions of (1.2) for a Lagrange multiplier $\lambda > 0$. Colin and Jeanjean [10] also used the change variables but work in the Sobolev space $H^1(\mathbb{R}^N)$, they proved the existence of positive solution for (1.2) with a Lagrange multiplier

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appears in the equation. The same method of changing variables was also used recently to obtain the existence of infinitely many solutions of problem (1.2) in [12]. See also [4] for the existence of positive solutions of problem (1.2) for the case of critical growth.

The main mathematical difficulties with problem (1.2) are caused by the term $\int_{\mathbb{R}^N} u^2|\nabla u|^2\,\mathrm{d}x$ which is not convex. A further problem is caused by usual lack of compactness since these problems are dealt with in the whole $\mathbb{R}^N$.

In this article, we consider a general problem (1.1). Under a certain constraint, we prove that (1.1) possess infinitely many sign-changing solutions for $p \in (1, \frac{3N+2}{N-2})$. As far as we know, besides [14], there are very few results for the existence of sign-changing solutions for (1.1). However, we point out that in [14], solutions are founded in the case $p \geq 3$.

Throughout this article, we denote the positive constants (possibly different) by $C, C_1, C_2, \dots$. First we state the following assumptions.

(V1) $V(x) \in C^\alpha(\mathbb{R}^{\mathbb{N}})$ is a radially symmetric function and satisfies

$$0 < V_0 \leq V(x) \leq \lim_{|x|\to+\infty} V(x) = V_\infty < +\infty, \quad \forall x \in \mathbb{R}^N.$$

(V2) The function $x \longmapsto x \cdot \nabla V(x)$ belongs to $L^\infty(\mathbb{R}^N)$ and $\|x \cdot \nabla V(x)\|_\infty \leq C_0 < (p-1)V_0$.

(V3) The map $s \longmapsto s^{N+2}V(sx)$ is concave for any $x \in \mathbb{R}^N$, $s \in \mathbb{R}$.

(A1) There exist constants $C_1 > 0, C_2 > 0$, such that for all $\xi \in \mathbb{R}^N$ and $s \in \mathbb{R}$,

$$C_1(1+s^2)|\xi|^2 \leq \sum_{i,j=1}^N a_{ij}(s)\xi_i\xi_j \leq C_2(1+s^2)|\xi|^2.$$

(A2) There exists constant $b > 0$ such that for all $\xi \in \mathbb{R}^N$ and $s \in \mathbb{R}$ such that

$$(b-2)\sum_{i,j=1}^N a_{ij}(s)\xi_i\xi_j \leq s\sum_{i,j=1}^N a'_{ij}(s)\xi_i\xi_j \leq (p-1)\sum_{i,j=1}^N a_{ij}(s)\xi_i\xi_j - b|\xi|^2.$$

(A3) $|s|^{N-1}\sum_{i,j=1}^N (a_{ij}(s) + \frac{1}{N}sa'_{ij}(s))\xi_i\xi_j$ is decreasing in $s \in (0,+\infty)$ and increasing in $s \in (-\infty, 0)$.

Here is our main result.

Theorem 1.1. *Assume* (V1)–(V3), (A1)–(A3). *Then for any* $k \in \{0,1,2,\dots\}$, *there exists a pair of radial solutions* $u_k^\pm$ *of* (1.1) *with the following properties:*

(i) $u_k^-(0) < 0 < u_k^+(0)$;
(ii) $u_k^\pm$ *possess exactly* k *nodes* r_l *with* $0 < r_1 < r_2 < \dots < r_k < +\infty$, *and* $u_k^\pm(x)|_{|x|=r_l} = 0$, $l = 1,2,\dots,k$.

We shall prove Theorem 1.1 under a convenient constraint, which is not of Nehari-type; instead, we use a Pohozaev identity. This kind of argument can be found in [23], see also [1, 24, 25] for different applications. Moreover, the main idea to prove Theorem 1.1 can be found in [11], see also [2, 9]. However, since we deal with a more general case and $p \in (1, \frac{3N+2}{N-2})$, there are more difficulties.

This article is organized as follows: Section 2 is devoted to establish some preliminary results and useful lemmas. Theorem 1.1 will be proved in Section 3.

2. Preliminary lemmas

Set $H_r^1(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) : u(x) = u(|x|)\}$, and $X = \{u \in H_r^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla u|^2|u|^2\,\mathrm{d}x < +\infty\}$, where $H^1(\mathbb{R}^N)$ is the usual Sobolev space and $\|u\|_{H^1}^2 = \int_{\mathbb{R}^N}(|\nabla u|^2 + V|u|^2)\,\mathrm{d}x$. X is a complete metric space with distance:

$$d_X(u,v) = \|u-v\|_{H^1} + \|\nabla u^2 - \nabla v^2\|_{L^2}.$$

Then, $u \in X$ is a weak solution of (1.1) if for all $\phi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N}\Big(\sum_{i,j=1}^N a_{ij}(u)\partial_i u\partial_j\phi + \frac{1}{2}\sum_{i,j=1}^N a'_{ij}(u)\partial_i u\partial_j u\phi + V(x)u\phi - |u|^{p-1}u\phi\Big)\,\mathrm{d}x = 0. \tag{2.1}$$

The corresponding functional is

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N}\Big(\sum_{i,j=1}^N a_{ij}(u)\partial_i u\partial_j u + V(x)u^2\Big)\,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}\,\mathrm{d}x.$$

Given $u \in X$ and $\phi \in C_0^\infty(\mathbb{R}^N)$, the Gâteaux derivative of I in the direction ϕ at u, denoted by $\langle I'(u),\phi\rangle$ is defined as $\lim_{t\to 0^+}\frac{I(u+t\phi)-I(u)}{t}$. It is easy to check that

$$\begin{aligned}&\langle I'(u),\phi\rangle\\ &= \int_{\mathbb{R}^N}\Big(\sum_{i,j=1}^N a_{ij}(u)\partial_i u\partial_j\phi + \frac{1}{2}\sum_{i,j=1}^N a'_{ij}(u)\partial_i u\partial_j u\phi + V(x)u\phi - |u|^{p-1}u\phi\Big)\,\mathrm{d}x.\end{aligned}$$

Hence, u is a weak solution of problem (1.1) if this derivative is zero in every direction $\phi \in C_0^\infty(\mathbb{R}^N)$.

From [20], we have the following two lemmas.

Lemma 2.1. *For $N \geq 2$, there is a constant $C = C(N) > 0$ such that*

$$|u(x)| \leq C|x|^{\frac{1-N}{2}}\|u\|_{H^1},$$

for any $|x| \geq 1$ and $u \in H_r^1(\mathbb{R}^N)$.

Lemma 2.2. *Let $\{u_n\} \subset H_r^1(\mathbb{R}^N)$ satisfy $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$. Then*

$$\liminf_n \int_{\mathbb{R}^N}|\nabla u_n|^2|u_n|^2\,\mathrm{d}x \geq \int_{\mathbb{R}^N}|\nabla u|^2|u|^2\,\mathrm{d}x.$$

Lemma 2.3 ([26])**.** *Let $N \geq 2$ and $2 < q < 2^*$. Then the imbedding*

$$H_r^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$$

is compact.

Lemma 2.4. *(Brézis-Lieb lemma [7]) Let $\{u_n\} \subset L^q(\mathbb{R}^N)$ be a bounded sequence, where $1 \leq q < +\infty$, such that $u_n \to u$ almost everywhere in $\mathbb{R}^N$. Then*

$$\lim_{n\to+\infty}(|u_n|_q^q - |u_n - u|_q^q) = |u|_q^q.$$

Lemma 2.5 ([14])**.** *Let u be a weak solution of (1.1). Then u and ∇u are bounded. Moreover, u satisfies the following exponential decay at infinity*

$$|u(x)| \leq Ce^{-\delta R}, \quad |x| = R, \quad \int_{\mathbb{R}^N\setminus B_R}(|\nabla u|^2 + |u|^2)\,\mathrm{d}x \leq Ce^{-\delta R},$$

for some positive constants C, δ.

Let Ω be one of the following three types of domains:

$$\begin{gathered}\{x\in\mathbb{R}^N||x|<R_1\},\\ \{x\in\mathbb{R}^N|0<R_2\le|x|<R_3<+\infty\},\\ \{x\in\mathbb{R}^N||x|\ge R_4>0\}.\end{gathered}\tag{2.2}$$

Set

$$\begin{gathered}H^1_{0,r}(\Omega)=\{u\in H^1_0(\Omega)|u(x)=u(|x|)\},\\ X(\Omega)=\{u\in H^1_{0,r}(\Omega)|\int_\Omega|\nabla u|^2u^2\,\mathrm{d}x<+\infty\}.\end{gathered}$$

Now we consider the following equation on Ω:

$$\begin{gathered}-\sum_{i,j=1}^N\partial_j(a_{ij}(u)\partial_iu)+\frac12\sum_{i,j=1}^N a'_{ij}(u)\partial_iu\partial_ju+V(x)u=|u|^{p-1}u,\quad x\in\Omega,\\ u|_{\partial\Omega}=0.\end{gathered}\tag{2.3}$$

The corresponding functional is

$$I_\Omega(u)=\frac12\int_\Omega\Big(\sum_{i,j=1}^N a_{ij}(u)\partial_iu\partial_ju+V(x)u^2\Big)\,\mathrm{d}x-\frac{1}{p+1}\int_\Omega|u|^{p+1}\,\mathrm{d}x.$$

Similarly we can define the Gâteaux derivative of I_Ω at $u\in X(\Omega)$ and weak solution of problem (2.3).

We extend any $u\in X(\Omega)$ to X by setting $u\equiv0$ on $x\in\mathbb{R}^N\backslash\Omega$. Hereafter denote by u_t the map:

$$\mathbb{R}^+\ni t\mapsto u_t\in X,\ \ u_t(x)=tu(t^{-1}x),$$

and consider

$$\begin{aligned}f_u(t):=I(u_t)=\frac{t^N}{2}\int_{\mathbb{R}^N}\sum_{i,j=1}^N a_{ij}(tu)\partial_iu\partial_ju\,\mathrm{d}x\\ +\frac{t^{N+2}}{2}\int_{\mathbb{R}^N}V(tx)u^2\,\mathrm{d}x-\frac{t^{N+p+1}}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}\,\mathrm{d}x.\end{aligned}$$

By conditions (V1) and (A1), and the fact that $p+1>2$, it is easy to see that $f_u(t)$ is positive for small t and tends to $-\infty$ if $t\to+\infty$. This implies that $f_u(t)$ attains its maximum. Moreover, thanks to (V2), $f_u:\mathbb{R}^+\to\mathbb{R}$ is C^1, and

$$\begin{aligned}f'_u(t)&=\frac N2t^{N-1}\int_{\mathbb{R}^N}\sum_{i,j=1}^N a_{ij}(tu)\partial_iu\partial_ju\,\mathrm{d}x+\frac{t^N}{2}\int_{\mathbb{R}^N}u\sum_{i,j=1}^N a'_{ij}(tu)\partial_iu\partial_ju\,\mathrm{d}x\\ &\quad+\frac{N+2}{2}t^{N+1}\int_{\mathbb{R}^N}V(tx)u^2\,\mathrm{d}x+\frac{t^{N+2}}{2}\int_{\mathbb{R}^N}\nabla V(tx)\cdot xu^2\,\mathrm{d}x\\ &\quad-\frac{N+p+1}{p+1}t^{N+p}\int_{\mathbb{R}^N}|u|^{p+1}\,\mathrm{d}x.\end{aligned}$$

Let

$$M(\Omega)=\{u\in X(\Omega)\setminus\{0\}:J_\Omega(u)=0\},$$

where $J_\Omega : X(\Omega) \to \mathbb{R}$ is defined as

$$\begin{aligned} J_\Omega(u) &= \frac{N}{2}\int_\Omega \sum_{i,j=1}^N a_{ij}(u)\partial_i u \partial_j u \,\mathrm{d}x + \frac{1}{2}\int_\Omega u \sum_{i,j=1}^N a'_{ij}(u)\partial_i u\partial_j u \,\mathrm{d}x \\ &\quad + \frac{N+2}{2}\int_\Omega V(x)u^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega \nabla V(x)\cdot x u^2\,\mathrm{d}x - \frac{N+p+1}{p+1}\int_\Omega |u|^{p+1}\,\mathrm{d}x. \end{aligned}$$

In other words, $M(\Omega)$ is the set of functions $u \in X(\Omega)$ such that $f'_u(1) = 0$. Moreover, $M(\Omega) \neq \emptyset$ (actually, given any $u \neq 0$, there exists $t > 0$ such that $u_t \in M(\Omega)$ (cf. [23])).

In the appendix of [14], by using Moser and De Giorgi iterations, the authors proved that weak solutions of (1.1) are bounded in $L^\infty(\mathbb{R}^N)$. Their arguments work also for $p \in (1,3)$. A density argument show that weak formulation (2.1) holds also for test functions in $H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. By [18, theorems 5.2 and 6.2 in chapter 4] it follows that $u \in C^{1,\alpha}$. From Schauder theory we conclude that $u \in C^{2,\alpha}$ is a classical solution of (1.1). Moreover, if $u \in X$ is a solution, u, Du, D^2u have an exponential decay as $|x| \to +\infty$ (see [14]). By [21], assume that $u \in X$ is a C^2 solution of (1.1). Then, for all $a \in \mathbb{R}$, we have the identity

$$\begin{aligned} &(\frac{N-2}{2} - a)\int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij}(u)\partial_i u\partial_j u\,\mathrm{d}x - \frac{a}{2}\int_{\mathbb{R}^N} u\sum_{i,j=1}^N a'_{ij}(u)\partial_i u\partial_j u\,\mathrm{d}x \\ &+ (\frac{N}{2} - a)\int_{\mathbb{R}^N} V(x)u^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} \nabla V(x)\cdot x u^2\,\mathrm{d}x \\ &+ (a - \frac{N}{p+1})\int_{\mathbb{R}^N} |u|^{p+1}\,\mathrm{d}x = 0. \end{aligned} \tag{2.4}$$

Observe also that $M(\Omega)$ is nothing but the set of functions $u \in X(\Omega)$ such that the identity (2.4) holds for $a = -1$. Then, all solutions belong to $M(\Omega)$.

Lemma 2.6. *For any $u \in X(\Omega)$, the map f_u attains its maximum at exactly one point t^u. Moreover, f_u is positive and increasing for $t \in [0, t^u]$ and decreasing for $t > t^u$. Also,*

$$c := \inf_{M(\Omega)} I_\Omega = \inf_{u\in X(\Omega), u\neq 0} \max_{t>0} I(u_t).$$

Proof. We employ a similar argument as in [23, Lemma 3.1]. Set

$$g(t) = \frac{t^N}{2}\int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij}(tu)\partial_i u\partial_j u\,\mathrm{d}x - \frac{t^{N+p+1}}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,\mathrm{d}x.$$

Let $t_1 \in \mathbb{R}^+$, $t_2 \in \mathbb{R}^+$, $t_1 \neq t_2$, then we have

$$\begin{aligned} g'(t_1) - g'(t_2) &= \frac{N}{2}\int_{\mathbb{R}^N} t_1^{N-1}\Big(\sum_{i,j=1}^N a_{ij}(t_1 u) + \frac{1}{N} t_1 u \sum_{i,j=1}^N a'_{ij}(t_1 u)\Big)\partial_i u\partial_j u\,\mathrm{d}x \\ &\quad - \frac{N}{2}\int_{\mathbb{R}^N} t_2^{N-1}\Big(\sum_{i,j=1}^N a_{ij}(t_2 u) + \frac{1}{N} t_2 u \sum_{i,j=1}^N a'_{ij}(t_2 u)\Big)\partial_i u\partial_j u\,\mathrm{d}x \\ &\quad - \frac{N+p+1}{p+1}(t_1^{N+p} - t_2^{N+p})\int_{\mathbb{R}^N} |u|^{p+1}\,\mathrm{d}x. \end{aligned}$$

By using (A3) we obtain

$$(g'(t_1) - g'(t_2))(t_1 - t_2) \le 0.$$

This implies that $g(t)$ is a concave function. Then by assumption (V3),

$$f_u(t) = g(t) + \frac{t^{N+2}}{2} \int_{\mathbb{R}^N} V(tx) u^2 \, \mathrm{d}x$$

is a concave function. We already know that it attains its maximum. Let t^u be the unique point at which this maximum is achieved. Then t^u is the unique critical point of f_u and f_u is positive and increasing for $0 < t < t^u$ and decreasing for $t > t^u$.

In particular, for any $u \in X(\Omega) \setminus \{0\}$, $t^u \in \mathbb{R}$ is the unique value such that u_{t^u} belongs to $M(\Omega)$, and $I(u_t)$ reaches a global maximum for $t = t^u$. □

Similar to [23, Proposition 3.3], we can prove the coercivity of $I_\Omega |_{M(\Omega)}$.

Proposition 2.7. *There exists $C > 0$ such that for any $u \in M(\Omega)$,*

$$I_\Omega(u) \ge C \int_\Omega (u^2 + |\nabla u|^2 + u^2 |\nabla u|^2) \, \mathrm{d}x.$$

Proof. Take $u \in M(\Omega)$ and extend u to X by setting $u \equiv 0$ on $\mathbb{R}^N \setminus \Omega$. Choose $t \in (0,1)$, then

$$\begin{aligned} I(u_t) - t^{N+p+1} I(u) &= \int_{\mathbb{R}^N} \sum_{i,j=1}^N \Big(\frac{t^N}{2} a_{ij}(tu) \partial_i u \partial_j u - \frac{t^{N+p+1}}{2} a_{ij}(u) \partial_i u \partial_j u \Big) \, \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} \Big(\frac{t^{N+2}}{2} V(tx) - \frac{t^{N+p+1}}{2} V(x) \Big) u^2 \, \mathrm{d}x. \end{aligned}$$

Observe that $V(tx) \ge V_0 \ge \delta V_\infty \ge \delta V(x)$, for some positive $\delta \in (0,1)$ depending only on V_0 and V_∞. By choosing a smaller t, if necessary, we obtain

$$\frac{t^{N+2}}{2} V(tx) - \frac{t^{N+p+1}}{2} V(x) \ge \Big(\delta \frac{t^{N+2}}{2} - \frac{t^{N+p+1}}{2} \Big) V(x) \ge \gamma_0,$$

for a fixed constant $\gamma_0 > 0$. Since $u \in M(\Omega)$, from Lemma 2.6 we obtain that $I(u_t) \le I(u)$. By choosing $t \in \Big(0, (\frac{C_1}{C_2})^{\frac{1}{p-1}}\Big)$ small enough, from (A1) we have

$$\begin{aligned} &(1 - t^{N+p+1}) I(u) \\ &\ge I(u_t) - t^{N+p+1} I(u) \\ &\ge \int_{\mathbb{R}^N} \Big(\frac{t^N}{2} C_1 (1 + (tu)^2) |\nabla u|^2 - \frac{t^{N+p+1}}{2} C_2 (1 + u^2) |\nabla u|^2 \Big) \, \mathrm{d}x + \gamma_0 \int_{\mathbb{R}^N} u^2 \, \mathrm{d}x \\ &= \frac{t^N}{2} \int_{\mathbb{R}^N} \Big((C_1 - t^{p+1} C_2) + \big(C_1 - C_2 t^{p-1}\big)(tu)^2 \Big) |\nabla u|^2 \, \mathrm{d}x + \gamma_0 \int_{\mathbb{R}^N} u^2 \, \mathrm{d}x \\ &\ge \frac{t^N}{2} (C_1 - C_2 t^{p-1}) \int_{\mathbb{R}^N} (1 + t^2 u^2) |\nabla u|^2 \, \mathrm{d}x + \gamma_0 \int_{\mathbb{R}^N} u^2 \, \mathrm{d}x \\ &\ge \frac{t^{N+2}}{2} (C_1 - C_2 t^{p-1}) \int_{\mathbb{R}^N} (1 + u^2) |\nabla u|^2 \, \mathrm{d}x + \gamma_0 \int_{\mathbb{R}^N} u^2 \, \mathrm{d}x. \end{aligned}$$

Note that $u \equiv 0$ on $\mathbb{R}^N \setminus \Omega$, we conclude by defining

$$C = \min \Big\{ \frac{C_1 t^{N+2} - C_2 t^{N+p+1}}{2(1 - t^{N+p+1})} \frac{\gamma_0}{1 - t^{N+p+1}} \Big\}.$$

□

Lemma 2.8. *Suppose that the domain* Ω *is one of the forms of* (2.2). *Then* $c = \inf_{M(\Omega)} I_\Omega(u)$ *can be achieved by some positive function* u *which is a solution of problem* (2.3). *Moreover,* $\int_\Omega u^2|\nabla\phi|^2\,\mathrm{d}x < +\infty$, $\int_\Omega \phi^2|\nabla u|^2\,\mathrm{d}x < +\infty$.

Proof. We divide the proof into three steps.

Step 1. c is attained. By the definition of c, there exists a sequence $\{u_n\} \subset M(\Omega)$ such that

$$I_\Omega(u_n) = c + o(1), \quad J_\Omega(u_n) = 0.$$

By Proposition 2.7, $\{u_n\}$ is bounded in $X(\Omega)$. Hence, by Lemma 2.3, we can extract a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$), such that

$$\begin{gathered} u_n \rightharpoonup u \quad \text{in } X(\Omega), \\ u_n \to u \quad \text{in } L^q(\Omega),\ 2 < q < 2^*. \end{gathered}$$

Since $\nabla(u_n)^2$ is uniformly bounded in $L^2(\Omega)$, by Sobolev's inequality we have $|u_n^2|_{2^*} \le C$, which gives $|u_n|_{22^*} \le C$. By Hölder's inequality we have

$$u_n \to u \quad \text{in } L^q(\Omega),\ 2 < q < 22^*.$$

Taking the limit in n, it follows from $J_\Omega(u_n) = 0$ that

$$\begin{aligned} J_\Omega(u) &= \frac{N}{2}\int_\Omega \sum_{i,j=1}^N a_{ij}(u)\partial_i u\partial_j u\,\mathrm{d}x + \frac{1}{2}\int_\Omega u\sum_{i,j=1}^N a'_{ij}(u)\partial_i u\partial_j u\,\mathrm{d}x \\ &\quad + \frac{N+2}{2}\int_\Omega V(x)u^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega \nabla V(x)\cdot xu^2\,\mathrm{d}x - \frac{N+p+1}{p+1}\int_\Omega |u|^{p+1}\,\mathrm{d}x \\ &\le 0. \end{aligned}$$

By Lemma 2.6, there exists $t > 0$ such that $J_\Omega(u_t) = 0$. Extend u_n and u to X by setting $u_n \equiv 0$ and $u \equiv 0$ on $\mathbb{R}^N \setminus \Omega$. In the following we just need to recall the expression of $I((u_n)_t)$,

$$c = \lim_{n\to+\infty} I_\Omega(u_n) = \lim_{n\to+\infty} I(u_n) \ge \lim_{n\to+\infty} \inf I((u_n)_t) \ge I(u_t), \quad \forall t > 0.$$

So $\max_t I(u_t) = c$. Then, by Lemma 2.6, there exists $t_0 > 0$ such that $u_{t_0} \in M(\Omega)$, which implies that c is attained.

Step 2. u is a radial solution of (2.3). We use an indirect argument which is based on a general idea used in [14]. Suppose that $u \in M(\Omega)$, $I_\Omega(u) = c$ but $I'_\Omega(u) \ne 0$. In such a case, we can find a function $\phi \in X(\Omega)$ with the property that $\int_\Omega u^2|\nabla\phi|^2\,\mathrm{d}x < +\infty$, $\int_\Omega \phi^2|\nabla u|^2\,\mathrm{d}x < +\infty$ but

$$\langle I'_\Omega(u), \phi\rangle \le -1.$$

Extend $u \in X(\Omega)$ to X as above and choose $\varepsilon > 0$ small enough such that

$$\langle I'(u_t + \sigma\phi), \phi\rangle \le -\frac{1}{2}, \quad \forall |t-1| + |\sigma| \le \varepsilon.$$

Let η be a cut-off function,

$$\eta(t) = \begin{cases} 1, & |t-1| \le \frac{1}{2}\varepsilon, \\ 0, & |t-1| \ge \varepsilon. \end{cases}$$

Define

$$\gamma(t)=\begin{cases}u_t, & |t-1|\ge\varepsilon,\\ u_t+\varepsilon\eta(t)\phi, & |t-1|<\varepsilon.\end{cases}$$

Next we estimate $\sup_t I(\gamma(t))$. If $|t-1|\le\varepsilon$, then

$$\begin{aligned} I(\gamma(t))&=I(u_t+\varepsilon\eta(t)\phi)\\ &=I(u_t)+\int_0^1\langle I'(u_t+\sigma\varepsilon\eta(t)\phi),\varepsilon\eta(t)\phi\rangle\,\mathrm{d}\sigma\\ &\le I(u_t)-\frac{1}{2}\varepsilon\eta(t). \end{aligned}\tag{2.5}$$

If $|t-1|\ge\varepsilon$, then $\eta(t)=0$, and the above estimate is trivial. Now since $u\in M(\Omega)$, for $t\neq1$ we get $I(u_t)<I(u)$. Hence it follows from (2.5) that

$$I(u_t+\varepsilon\eta(t)\phi)\le\begin{cases}I(u_t)<I(u), & t\neq1,\\ I(u)-\frac{1}{2}\varepsilon\eta(1)=I(u)-\frac{1}{2}\varepsilon, & t=1.\end{cases}\tag{2.6}$$

In any case we have $I(\gamma(t))<I(u)=c$.

To conclude observe that $J(\gamma(1-\varepsilon))>0$ and $J(\gamma(1+\varepsilon))<0$. As a result, we can find $t_0\in(1-\varepsilon,1+\varepsilon)$ such that $J(\gamma(t_0))=0$, which implies that $\gamma(t_0)=u_{t_0}+\varepsilon\eta(t_0)\phi\in M(\Omega)$. However, it follows from (2.6) that $I_\Omega(\gamma(t_0))<c$. This is a contradiction.

Step 3. $u>0$. Consider $u\in M(\Omega)$ a minimizer of $I_\Omega|_{M(\Omega)}$. Then the absolute value $|u|\in M(\Omega)$ is also a minimizer. By the classical maximum principle and the fact that solutions are C^2, $|u|>0$. □

3. Proof of Theorem 1.1

For given $k+2$ numbers r_l $(l=0,1,\dots,k+1)$ such that $0=r_0<r_1<\dots<r_k<r_{k+1}=+\infty$, denote

$$\Omega^1=\{x\in\mathbb{R}^N:|x|<r_1\},\quad \Omega^l=\{x\in\mathbb{R}^N:r_{l-1}<|x|<r_l\}.$$

We will always extend $u_l\in X(\Omega^l)$ to X by setting $u\equiv0$ on $x\in\mathbb{R}^N\backslash\Omega^l$ for every $u_l\in X(\Omega^l), l=1,2,\dots,k+1$. In this sense, we use $I(u_l)$ to replace $I_{\Omega^l}(u_l)$ and $J(u_l)$ to replace $J_{\Omega^l}(u_l)$. Define

$$\begin{aligned} Y_k^\pm(r_1,r_2,\dots,r_{k+1})=\Big\{u\in X:u=\pm\sum_{l=1}^{k+1}(-1)^{l-1}u_l,\ u_l\ge0,\\ u_l\not\equiv0,u_l\in X(\Omega^l),\ l=1,2,\dots,k+1\Big\}, \end{aligned}$$

$$\begin{aligned} M_k^\pm=\Big\{u\in X:\exists 0<r_1<r_2<\dots<r_k<r_{k+1}=+\infty,\ \text{such that}\\ u\in Y_k^\pm(r_1,r_2,\dots,r_k,r_{k+1})\text{ and }u_l\in M(\Omega^l),l=1,2,\dots,k+1\Big\}. \end{aligned}$$

Note that $M_k^\pm\neq\emptyset$, $k=1,2,\dots$. In the following we always refer to M_k and we drop the " + ". For M_k^-, everything could be done exactly in the same way. By Lemma 2.6, it is easy to verify that for all u,

$$u=\sum_{l=1}^{k+1}(-1)^{l-1}u_l\in M_k\Leftrightarrow I(u)=\max_{\substack{\alpha_l>0\\1\le l\le k+1}}I\Big(\sum_{l=1}^{k+1}(-1)^{l-1}(u_l)_{\alpha_l}\Big).\tag{3.1}$$

Set
$$c_k = \inf_{M_k} I(u), \ k = 1, 2, \dots .$$

Lemma 3.1. *c_k is attained, $k = 0, 1, 2, \dots$.*

Proof. By induction we prove that for each k there exists $u_k \in M_k$ such that
$$I(u_k) = c_k.$$
The case that $k = 0$ can be deduced by setting $\Omega = \mathbb{R}^N$ in Lemma 2.8. We suppose the claim is true for $k-1$ and discuss the case $k \geq 1$ in the following. For convenience, we divide the proof of the rest proof into four steps.

Step 1. I is bounded from below on M_k by a positive constant. Since
$$I(u) = I\Big(\sum_{l=1}^{k+1}(-1)^{l-1}u_l\Big) = \sum_{l=1}^{k+1} I_{\Omega^l}(u_l), \quad \forall u \in M_k.$$
We just need to prove that, for $l = 1, 2, \dots, k+1$, I_{Ω^l} is bounded from below on $M(\Omega^l)$ by a positive constant.

For any $u_l \in M(\Omega^l)$, we extend it to X by setting $u_l \equiv 0$ on $\mathbb{R}^N \setminus \Omega^l$. By (V1) and (A1) we have
$$I(u_l) \geq \frac{1}{2}\int_{\mathbb{R}^N} (C_1(1+u_l^2)|\nabla u_l|^2 + V_0 u_l^2)\,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^N} |u_l|^{p+1}\,\mathrm{d}x.$$
Let
$$\bar{I}(u_l) = \frac{1}{2}\int_{\mathbb{R}^N} (C_1(1+u_l^2)|\nabla u_l|^2 + V_0 u_l^2)\,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^N} |u_l|^{p+1}\,\mathrm{d}x.$$
Obviously,
$$\bar{c} := \inf_{u_l \in X(\Omega^l), u_l \neq 0} \max_{t>0} \bar{I}((u_l)_t) \leq \inf_{u_l \in X(\Omega^l), u_l \neq 0} \max_{t>0} I((u_l)_t) = c.$$
Let us define
$$\bar{M}(\Omega^l) = \{u_l \in X(\Omega^l) \setminus \{0\} : g'_{u_l}(1) = 0\} \quad \text{where } g_{u_l}(t) = \bar{I}((u_l)_t).$$
Similar to Lemma 2.6, we know that
$$\bar{c} = \inf_{u_l \in \bar{M}(\Omega^l)} \bar{I}_{\Omega^l}(u_l).$$
For any $u_l \in \bar{M}(\Omega^l)$,
$$\begin{aligned}
&\frac{N+2}{2}V_0 \int_{\Omega^l} u_l^2\,\mathrm{d}x + \frac{C_1(N+2)}{2}\int_{\Omega^l} |\nabla u_l|^2 u_l^2\,\mathrm{d}x \\
&\leq \frac{N+p+1}{p+1}\int_{\Omega^l} |u_l|^{p+1}\,\mathrm{d}x \\
&\leq \frac{N+2}{2}V_0 \int_{\Omega^l} u_l^2\,\mathrm{d}x + C\int_{\Omega^l} |u_l|^{\frac{4N}{N+2}}\,\mathrm{d}x,
\end{aligned}$$
for a suitable constant $C > 0$. So, by using the Sobolev's inequality,
$$\frac{C_1(N+2)}{2}\int_{\Omega^l} |\nabla u_l|^2 u_l^2\,\mathrm{d}x \leq C\int_{\Omega^l} |u_l|^{\frac{4N}{N+2}}\,\mathrm{d}x \leq C'\Big(\int_{\Omega^l} |\nabla u_l|^2 u_l^2\,\mathrm{d}x\Big)^{\frac{N}{N-2}},$$

this shows that $\int_{\Omega^l} |\nabla u_l|^2 u_l^2 \,\mathrm{d}x$ is bounded away from zero on $\bar{M}(\Omega^l)$. Since the functional $\bar{I}_{\Omega^l}$ restricted to $\bar{M}(\Omega^l)$ has the expression

$$\begin{aligned}\bar{I}_{\Omega^l}(u_l) &= \frac{C_1}{2}\frac{p+1}{N+p+1}\int_{\Omega^l} |\nabla u_l|^2 \,\mathrm{d}x + \frac{V_0}{2}\frac{p-1}{N+p+1}\int_{\Omega^l} u_l^2 \,\mathrm{d}x \\ &\quad + \frac{C_1(p-1)}{N+p+1}\int_{\Omega^l} |\nabla u_l|^2 |u_l|^2 \,\mathrm{d}x.\end{aligned}$$

We obtain that $\bar{c} > 0$, and hence $c > 0$. This implies that $d_X(M(\Omega^l), 0) > 0$. Then by Proposition 2.7, we get that I_{Ω^l} is bounded from below on $M(\Omega^l)$ by a positive constant.

Step 2. We suppose $\{u_m\}_{m\geq 1}$ be a minimizing sequence of c_k in M_k; that is

$$\lim_{m\to+\infty} I(u_m) = c_k, \quad u_m \in M_k, \quad m = 1, 2, \dots.$$

u_m corresponds to k nodes, $r_m^1, r_m^2, \dots, r_m^k$ with $0 < r_m^1 < r_m^2 < \dots < r_m^k < +\infty$. By Proposition 2.7, we know that $\{u_m\}$ is bounded in X. Set

$$\Omega_m^l = \{x \in \mathbb{R}^N : r_m^{l-1} < |x| < r_m^l\},$$
$$u_m^l = \begin{cases} u_m, & x \in \Omega_m^l, \\ 0, & x \notin \Omega_m^l. \end{cases}$$

By selecting a subsequence, we may assume that $\lim_{m\to+\infty} r_m^l = r^l$, and clearly $0 \leq r^1 \leq r^2 \leq \dots \leq r^k \leq +\infty$.

Next we prove that $r^l \neq r^{l-1}, l = 1, 2, \dots, k$. Here we denote $r^0 = 0$. If there exists some $l \in \{1, 2, \dots, k\}$ such that $r^l = r^{l-1}$, then $\lim_{m\to+\infty} r_m^l = \lim_{m\to+\infty} r_m^{l-1}$. We denote the measure of Ω_m^l by $|\Omega_m^l|$, so that $|\Omega_m^l| \to 0$ as $m \to +\infty$. From (A1) and the fact that $\{u_m\}$ is bounded in X, we have

$$\begin{aligned} I(u_m^l) &= \frac{1}{2}\int_{\Omega_m^l} \Big(\sum_{i,j=1}^N a_{ij}(u_m^l)\partial_i u_m^l \partial_j u_m^l + V(u_m^l)^2\Big)\,\mathrm{d}x - \frac{1}{p+1}\int_{\Omega_m^l} |u_m^l|^{p+1}\,\mathrm{d}x \\ &\leq \frac{1}{2}\int_{\Omega_m^l} \Big(C_2(1+(u_m^l)^2)|\nabla u_m^l|^2 + V_\infty (u_m^l)^2\Big)\,\mathrm{d}x - \frac{1}{p+1}\int_{\Omega_m^l} |u_m^l|^{p+1}\,\mathrm{d}x \\ &\leq C - \frac{1}{p+1}\int_{\Omega_m^l} |u_m^l|^{p+1}\,\mathrm{d}x. \end{aligned} \tag{3.2}$$

By using Hölder's inequality,

$$\int_{\Omega_m^l} |u_m^l|^2 \,\mathrm{d}x \leq \Big(\int_{\Omega_m^l} |u_m^l|^{p+1}\,\mathrm{d}x\Big)^{\frac{2}{p+1}} |\Omega_m^l|^{1-\frac{2}{p+1}},$$

i.e.,

$$\int_{\Omega_m^l} |u_m^l|^{p+1}\,\mathrm{d}x \geq \Big(\int_{\Omega_m^l} |u_m^l|^2 \,\mathrm{d}x\Big)^{\frac{p+1}{2}} |\Omega_m^l|^{\frac{1-p}{2}}. \tag{3.3}$$

Since $u_m^l \in M_k$,

$$
\begin{aligned}
&\frac{N+p+1}{p+1}\int_{\Omega_m^l} |u_m^l|^{p+1}\,\mathrm{d}x \\
&= \frac{N}{2}\int_{\Omega_m^l}\sum_{i,j=1}^N a_{ij}(u_m^l)\partial_i u_m^l\partial_j u_m^l\,\mathrm{d}x + \frac{1}{2}\int_{\Omega_m^l} u_m^l \sum_{i,j=1}^N a'_{ij}(u_m^l)\partial_i u_m^l\partial_j u_m^l\,\mathrm{d}x \\
&\quad + \frac{N+2}{2}\int_{\Omega_m^l} V(x)(u_m^l)^2\,\mathrm{d}x + \frac{1}{2}\int_{\Omega_m^l}\nabla V(x)\cdot x(u_m^l)^2\,\mathrm{d}x \\
&\geq \frac{C_1(N+b-2)}{2}\int_{\Omega_m^l}(1+(u_m^l)^2)|\nabla u_m^l|^2\,\mathrm{d}x + \frac{N+2}{2}V_0\int_{\Omega_m^l}(u_m^l)^2\,\mathrm{d}x \\
&\quad - \frac{1}{2}C_0\int_{\Omega_m^l}(u_m^l)^2\,\mathrm{d}x \\
&\geq \frac{C_1(N+b-2)}{2}\int_{\Omega_m^l}(u_m^l)^2|\nabla u_m^l|^2\,\mathrm{d}x + \frac{(N+3-p)V_0}{2}\int_{\Omega_m^l}(u_m^l)^2\,\mathrm{d}x.
\end{aligned}
\tag{3.4}
$$

On the other hand,

$$
\frac{N+p+1}{p+1}\int_{\Omega_m^l}|u_m^l|^{p+1}\,\mathrm{d}x \leq \frac{(N+3-p)V_0}{2}\int_{\Omega_m^l}|u_m^l|^2\,\mathrm{d}x + C\int_{\Omega_m^l}|u_m^l|^{\frac{4N}{N+2}}\,\mathrm{d}x,
$$

for a suitable $C > 0$. So by using Sobolev's inequality

$$
\begin{aligned}
\frac{C_1(N+b-2)}{2}\int_{\Omega_m^l}|u_m^l|^2|\nabla u_m^l|^2\,\mathrm{d}x &\leq C\int_{\Omega_m^l}|u_m^l|^{\frac{4N}{N+2}}\,\mathrm{d}x \\
&\leq C'\int_{\Omega_m^l}|u_m^l|^2|\nabla u_m^l|^2\,\mathrm{d}x.
\end{aligned}
$$

This shows that $\int_{\Omega_m^l}|u_m^l|^2|\nabla u_m^l|^2\,\mathrm{d}x$ is bounded away from zero on M_k. This implies that

$$
\int_{\Omega_m^l}|u_m^l|^2\,\mathrm{d}x \geq \delta > 0.
$$

Then from (3.3) we obtain

$$
\int_{\Omega_m^l}|u_m^l|^{p+1}\,\mathrm{d}x \geq \Big(\int_{\Omega_m^l}|u_m^l|^2\,\mathrm{d}x\Big)^{\frac{p+1}{2}}|\Omega_m^l|^{\frac{1-p}{2}} \geq \delta^{\frac{p+1}{2}}|\Omega_m^l|^{\frac{1-p}{2}}.
$$

Note that $|\Omega_m^l| \to 0$ as $m \to +\infty$ and $p > 1$, we have

$$
\int_{\Omega_m^l}|u_m^l|^{p+1}\,\mathrm{d}x \to +\infty, \quad \text{as } m \to +\infty.
$$

This and (3.2) implies that $I(u_m^l) \to -\infty$ as $m \to +\infty$, which contradicts Step 1. Thus $r^l \neq r^{l-1}, l = 1, 2, \dots, k$.

Step 3. $r^k < +\infty$. If $r^k = +\infty$, then $\lim_{m\to+\infty} r^k_m = +\infty$. Since $u^k_m \in M(\Omega^k_m)$, from (V1), (V2), (A1) and (A2) we have

$$
\begin{aligned}
&I(u^k_m)\\
&= \frac{1}{2}\int_{\Omega^k_m} \Big(\sum_{i,j=1}^N a_{ij}(u^k_m)\partial_i u^k_m \partial_j u^k_m + V(x)(u^k_m)^2\Big)\,\mathrm{d}x - \frac{1}{p+1}\int_{\Omega^k_m} |u^k_m|^{p+1}\,\mathrm{d}x\\
&= \frac{1}{2}\int_{\Omega^k_m} \Big(\sum_{i,j=1}^N a_{ij}(u^k_m)\partial_i u^k_m \partial_j u^k_m + V(x)(u^k_m)^2\Big)\,\mathrm{d}x\\
&\quad - \frac{1}{N+p+1}\Big(\frac{N}{2}\int_{\Omega^k_m}\sum_{i,j=1}^N a_{ij}(u^k_m)\partial_i u^k_m \partial_j u^k_m\,\mathrm{d}x\\
&\quad + \frac{1}{2}\int_{\Omega^k_m} u^k_m \sum_{i,j=1}^N a'_{ij}(u^k_m)\partial_i u^k_m \partial_j u^k_m\,\mathrm{d}x + \frac{N+2}{2}\int_{\Omega^k_m} V(x)(u^k_m)^2\,\mathrm{d}x\\
&\quad + \frac{1}{2}\int_{\Omega^k_m} \nabla V(x)\cdot x (u^k_m)^2\,\mathrm{d}x\Big)\\
&\geq \Big(\frac{1}{2} - \frac{N}{2(N+p+1)} - \frac{p-1}{2(N+p+1)}\Big)\int_{\Omega^k_m}\sum_{i,j=1}^N a_{ij}(u^k_m)\partial_i u^k_m \partial_j u^k_m\,\mathrm{d}x\\
&\quad + \Big(\Big(\frac{1}{2} - \frac{N+2}{2(N+p+1)}\Big)V_0 - \frac{C_0}{2(N+p+1)}\Big)\int_{\Omega^k_m} (u^k_m)^2\,\mathrm{d}x\\
&\quad + \frac{b}{2(N+p+1)}\int_{\Omega^k_m} |\nabla u^k_m|^2\,\mathrm{d}x\\
&\geq \frac{1}{N+p+1}\int_{\Omega^k_m} C_1\big(1+(u^k_m)^2\big)|\nabla u^k_m|^2\,\mathrm{d}x + \frac{(p-1)V_0 - C_0}{2(N+p+1)}\int_{\Omega^k_m}(u^k_m)^2\,\mathrm{d}x\\
&\quad + \frac{b}{2(N+p+1)}\int_{\Omega^k_m} |\nabla u^k_m|^2\,\mathrm{d}x\\
&\geq C\eta^2(u^k_m),
\end{aligned}
\tag{3.5}
$$

where

$$
\eta^2(u^k_m) = \int_{\Omega^k_m} \big(1+(u^k_m)^2\big)|\nabla u^k_m|^2\,\mathrm{d}x + \int_{\Omega^k_m} (u^k_m)^2\,\mathrm{d}x.
$$

From Step 1 we know that $\int_{\Omega^k_m} |u^k_m|^2|\nabla u^k_m|^2\,\mathrm{d}x$ is bounded away from zero on $M(\Omega^k_m)$. Then there exists some $\delta_0 > 0$ such that

$$
\int_{\Omega^k_m} |u^k_m|^2\,\mathrm{d}x \geq \delta_0 > 0.
$$

This and (3.4) imply that there exists some $\delta_1 > 0$ such that

$$
\int_{\Omega^k_m} |u^k_m|^{p+1}\,\mathrm{d}x \geq \delta_1 > 0.
$$

Then from (3.2), we have

$$\begin{aligned} I(u_m^k) &\le C - \frac{1}{p+1}\int_{\Omega_m^k} |u_m^k|^{p+1}\,\mathrm{d}x \\ &\le C + C\int_{\Omega_m^k} |u_m^k|^{p+1}\,\mathrm{d}x \\ &\le C\int_{\Omega_m^k} |u_m^k|^{p+1}\,\mathrm{d}x\Big(\Big(\int_{\Omega_m^k} |u_m^k|^{p+1}\,\mathrm{d}x\Big)^{-1} + 1\Big) \\ &\le C\int_{\Omega_m^k} |u_m^k|^{p+1}\,\mathrm{d}x(\delta_1^{-1}+1) \\ &\le C\int_{\Omega_m^k} |u_m^k|^{p+1}\,\mathrm{d}x, \end{aligned} \tag{3.6}$$

for some suitable $C > 0$. It follows from (3.5),(3.6) and Lemma 2.1 that

$$\begin{aligned} \eta^2(u_m^k) &\le I(u_m^k) \\ &\le C\int_{\Omega_m^k} |u_m^k|^{p+1}\,\mathrm{d}x \\ &\le C\int_{\Omega_m^k} |u_m^k|^2|u_m^k|^{p-1}\,\mathrm{d}x \\ &\le C\|u_m^k\|^{p-1}\int_{\Omega_m^k} |u_m^k|^2|x|^{\frac{(1-N)(p-1)}{2}}\,\mathrm{d}x \\ &\le C\Big(\eta^2(u_m^k)\Big)^{\frac{p+1}{2}}|r_m^k|^{\frac{(1-N)(p-1)}{2}}. \end{aligned}$$

Thus

$$\eta^2(u_m^k) \ge C|r_m^k|^{N-1}. \tag{3.7}$$

From (3.7) we have

$$\eta^2(u_m^k) \to +\infty \quad \text{as } m \to +\infty.$$

So (3.5) implies

$$I(u_m^k) \to +\infty \quad \text{as } m \to +\infty. \tag{3.8}$$

By the inductive assumption and (3.8), for $\varepsilon > 0$ fixed we choose $M > 0$ such that

$$I(u_m^k) > c_k - c_{k-1} + \varepsilon, \quad |I(u_m) - c_k| < \varepsilon, \quad \text{as } m \ge M.$$

Then we may define $\hat{u}(x) \in M_{k-1}$ by

$$\hat{u}(x) = \begin{cases} u_m^s(x), & x \in \Omega_m^s \text{ as } s < k, \\ 0, & x \in \Omega_m^k. \end{cases}$$

Hence $I(\hat{u}) = I(u_m) - I(u_m^k) < c_k + \varepsilon - (c_k - c_{k-1} + \varepsilon) = c_{k-1}$ as $m \ge M$, which contradicts the fact that $c_{k-1} = \inf_{M_{k-1}} I(u)$. Then, we obtain $r^k < +\infty$.

Step 4. c_k is attained. By Proposition 2.7 we can find a subsequence (still denoted by $\{u_m\}$) such that

$$\begin{gathered} u_m \rightharpoonup u \quad \text{in } X, \\ u_m \to u \quad \text{in } L^{p+1}(\mathbb{R}^N). \end{gathered}$$

Set $\Omega^l=\{x\in\mathbb{R}^N | r^{l-1}<|x|<r^l\}$, for all $l=1,2,\dots,k+1$, $r^0=0$ and $r^{k+1}=+\infty$. Lemma 2.8 implies that $c=\inf_{u\in M(\Omega^l)} I(u)$ is attained by some positive function $\hat{u}^l$ which satisfies the boundary-value problem

$$
\begin{gathered}
-\sum_{i,j=1}^N \partial_j(a_{ij}(u)\partial_i u)+\frac{1}{2}\sum_{i,j=1}^N a'_{ij}(u)\partial_i u\partial_j u+V(x)u=|u|^{p-1}u, \quad x\in\Omega^l,\\
u|_{\partial\Omega^l}=0.
\end{gathered}
$$

Define $u_k=\sum_{l=1}^{k+1}(-1)^{l-1}\hat{u}^l(x)$, $(\hat{u}^l(x)=0, x\notin\Omega^l)$. Then, clearly, $u_k\in M_k$. Consider the coordinate transformations $\Phi_m:\mathbb{R}^N\to\mathbb{R}^N$, $m=1,2,\dots$, defined by

$$
\Phi_m(x)=\varphi_m(|x|)\frac{x}{|x|}, \quad x\in\mathbb{R}^N,
$$

where

$$
\varphi_m(r)=\frac{(r^l-r^{l-1})(r-r_m^{l-1})}{r_m^l-r_m^{l-1}}+r^{l-1}.
$$

For any $r\in\mathbb{R}$, clearly $\Phi_m(\Omega_m^l)=\Omega^l$. Let $y=\Phi_m(x)\in\Omega^l$, if $x\in\Omega_m^l$. It is easy to show that

$$
|\nabla u(y)|=(R_m^l)^{-1}|\nabla u(x)|, \tag{3.9}
$$

$$
\mathrm{d}y=|J_m^l|\,\mathrm{d}x, \tag{3.10}
$$

$$
a_m^l\le\Big(\frac{\Phi_m(r)}{r}\Big)^{N-1}\le A_m^l, \tag{3.11}
$$

where

$$
\begin{gathered}
R_m^l=\frac{r^l-r^{l-1}}{r_m^l-r_m^{l-1}}, \quad J_m^l=\big(\varphi_m(|x|)\big)^{N-1}\big(\varphi_m(|x|)\big)'|x|^{1-N},\\
a_m^l=\Big(\min\big\{\frac{r^l}{r_m^l},\frac{r^{l-1}}{r_m^{l-1}}\big\}\Big)^{N-1}, \quad A_m^l=\Big(\max\big\{\frac{r^l}{r_m^l},\frac{r^{l-1}}{r_m^{l-1}}\big\}\Big)^{N-1}.
\end{gathered}
$$

Clearly,

$$
a_m^lR_m^l\le|J_m^l|\le A_m^lR_m^l, \tag{3.12}
$$

and

$$
R_m^l\to1, \quad a_m^l\to1, \quad A_m^l\to1, \quad J_m^l\to1, \quad \text{as } m\to+\infty. \tag{3.13}
$$

Let

$$
\begin{aligned}
\gamma(t)&=\frac{N}{2}t^{N-1}\int_{\Omega^l}\sum_{i,j=1}^N a_{ij}(tu_m^l)\partial_i u_m^l\partial_j u_m^l\,\mathrm{d}y\\
&\quad+\frac{t^N}{2}\int_{\Omega^l}u_m^l\sum_{i,j=1}^N a'_{ij}(tu_m^l)\partial_i u_m^l\partial_j u_m^l\,\mathrm{d}y+\frac{N+2}{2}t^{N+1}\int_{\Omega^l}V(ty)(u_m^l)^2\,\mathrm{d}y\\
&\quad+\frac{t^{N+2}}{2}\int_{\Omega^l}\nabla V(ty)\cdot y(u_m^l)^2\,\mathrm{d}y-\frac{N+p+1}{p+1}t^{N+p}\int_{\Omega^l}|u_m^l|^{p+1}\,\mathrm{d}y.
\end{aligned}
$$

From Lemma 2.6, there exists some $t_m^l>0$, such that $\gamma(t_m^l)=0$, thus $(u_m^l)_{t_m^l}\in M(\Omega^l)$. Now we claim that

$$
t_m^l\to1 \quad \text{as } m\to+\infty,\ l=1,2,\dots,k. \tag{3.14}
$$

Indeed, since $\gamma(t_m^l)=0$, we have

$$
\begin{aligned}
&\frac{N}{2}(t_m^l)^{N-1}\int_{\Omega^l}\sum_{i,j=1}^N a_{ij}(t_m^l u_m^l)\partial_i u_m^l\partial_j u_m^l\,\mathrm{d}y\\
&+\frac{(t_m^l)^N}{2}\int_{\Omega^l}u_m^l\sum_{i,j=1}^N a'_{ij}(t_m^l u_m^l)\partial_i u_m^l\partial_j u_m^l\,\mathrm{d}y\\
&+\frac{N+2}{2}(t_m^l)^{N+1}\int_{\Omega^l}V(t_m^l y)(u_m^l)^2\,\mathrm{d}y+\frac{(t_m^l)^{N+2}}{2}\int_{\Omega^l}\nabla V(t_m^l y)\cdot y(u_m^l)^2\,\mathrm{d}y\\
&-\frac{N+p+1}{p+1}(t_m^l)^{N+p}\int_{\Omega^l}|u_m^l|^{p+1}\,\mathrm{d}y=0.
\end{aligned}
\tag{3.15}
$$

We can prove that there exists a constant $\tilde{t}>0$ such that

$$
0<t_m^l\le\tilde{t}<+\infty.
$$

By selecting a subsequence, we may assume that $\lim_{m\to+\infty}t_m^l=t_*^l$. Using (3.9)-(3.13), we have

$$
\begin{aligned}
&\lim_{m\to+\infty}\int_{\Omega^l}\sum_{i,j=1}^N a_{ij}(t_m^l u_m^l)\partial_i u_m^l\partial_j u_m^l\,\mathrm{d}y\\
&=\lim_{m\to+\infty}\int_{\Omega_m^l}\sum_{i,j=1}^N a_{ij}(t_m^l u_m^l)\partial_i u_m^l\partial_j u_m^l\,\mathrm{d}x,
\end{aligned}
\tag{3.16}
$$

$$
\begin{aligned}
&\lim_{m\to+\infty}\int_{\Omega^l}u_m^l\sum_{i,j=1}^N a'_{ij}(t_m^l u_m^l)\partial_i u_m^l\partial_j u_m^l\,\mathrm{d}y\\
&=\lim_{m\to+\infty}\int_{\Omega_m^l}u_m^l\sum_{i,j=1}^N a'_{ij}(t_m^l u_m^l)\partial_i u_m^l\partial_j u_m^l\,\mathrm{d}x,
\end{aligned}
\tag{3.17}
$$

$$
\lim_{m\to+\infty}\int_{\Omega^l}V(t_m^l y)(u_m^l)^2\,\mathrm{d}y=\lim_{m\to+\infty}\int_{\Omega_m^l}V(t_m^l x)(u_m^l)^2\,\mathrm{d}x,
\tag{3.18}
$$

$$
\lim_{m\to+\infty}\int_{\Omega^l}\nabla V(t_m^l y)\cdot y(u_m^l)^2\,\mathrm{d}y=\lim_{m\to+\infty}\int_{\Omega_m^l}\nabla V(t_m^l x)\cdot x(u_m^l)^2\,\mathrm{d}x,
\tag{3.19}
$$

$$
\lim_{m\to+\infty}\int_{\Omega^l}|u_m^l|^{p+1}\,\mathrm{d}y=\lim_{m\to+\infty}\int_{\Omega_m^l}|u_m^l|^{p+1}\,\mathrm{d}x.
\tag{3.20}
$$

Substituting (3.16)-(3.20) in (3.15) we find that

$$
\begin{aligned}
&\lim_{m\to+\infty}\Big(\frac{N}{2}(t_m^l)^{N-1}\int_{\Omega_m^l}\sum_{i,j=1}^N a_{ij}(t_m^l u_m^l)\partial_i u_m^l\partial_j u_m^l\,\mathrm{d}x\\
&+\frac{(t_m^l)^N}{2}\int_{\Omega_m^l}u_m^l\sum_{i,j=1}^N a'_{ij}(t_m^l u_m^l)\partial_i u_m^l\partial_j u_m^l\,\mathrm{d}x\\
&+\frac{N+2}{2}(t_m^l)^{N+1}\int_{\Omega_m^l}V(t_m^l x)(u_m^l)^2\,\mathrm{d}x+\frac{(t_m^l)^{N+2}}{2}\int_{\Omega_m^l}\nabla V(t_m^l x)\cdot x(u_m^l)^2\,\mathrm{d}x
\end{aligned}
$$

$$
- \frac{N+p+1}{p+1}(t_m^l)^{N+p}\int_{\Omega_m^l}|u_m^l|^{p+1}\,\mathrm{d}x\Big)=0. \tag{3.21}
$$

But for $u_m^l(x)\in M(\Omega_m^l)$, we know that

$$
\begin{aligned}
&\frac{N}{2}\int_{\Omega_m^l}\sum_{i,j=1}^N a_{ij}(u_m^l)\partial_i u_m^l\partial_j u_m^l\,\mathrm{d}x+\frac{1}{2}\int_{\Omega_m^l}u_m^l\sum_{i,j=1}^N a'_{ij}(u_m^l)\partial_i u_m^l\partial_j u_m^l\,\mathrm{d}x\\
&+\frac{N+2}{2}\int_{\Omega_m^l}V(t_m^l x)(u_m^l)^2\,\mathrm{d}x+\frac{1}{2}\int_{\Omega_m^l}\nabla V(x)\cdot x(u_m^l)^2\,\mathrm{d}x\\
&-\frac{N+p+1}{p+1}\int_{\Omega_m^l}|u_m^l|^{p+1}\,\mathrm{d}x=0.
\end{aligned} \tag{3.22}
$$

Set

$$
\begin{aligned}
h(s)=&\frac{N}{2}s^{N-1}\int_{\Omega_m^l}\sum_{i,j=1}^N a_{ij}(su_m^l)\partial_i u_m^l\partial_j u_m^l\,\mathrm{d}x\\
&+\frac{s^N}{2}\int_{\Omega_m^l}u_m^l\sum_{i,j=1}^N a'_{ij}(su_m^l)\partial_i u_m^l\partial_j u_m^l\,\mathrm{d}x\\
&+\frac{N+2}{2}s^{N+1}\int_{\Omega_m^l}V(sx)(u_m^l)^2\,\mathrm{d}x+\frac{s^{N+2}}{2}\int_{\Omega_m^l}\nabla V(sx)\cdot x(u_m^l)^2\,\mathrm{d}x\\
&-\frac{N+p+1}{p+1}s^{N+p}\int_{\Omega_m^l}|u_m^l|^{p+1}\,\mathrm{d}x.
\end{aligned} \tag{3.23}
$$

From the proof of Lemma 2.6, we know that $h(s)$ has only one zero on $(0,+\infty)$. So, from (3.21)-(3.23) we get that $t_*^l=1$. Moreover,

$$
\lim_{m\to+\infty}I((u_m^l)_{t_m^l})=\lim_{m\to+\infty}I(u_m^l).
$$

On the other hand, since $I(\hat{u}^l)=\inf_{M(\Omega_m^l)}I(u)$ and $(u_m^l)_{t_m^l}\in M(\Omega_m^l)$, we obtain

$$
I(\hat{u}^l)\le I((u_m^l)_{t_m^l}).
$$

Hence $\lim_{m\to+\infty}I((u_m^l)_{t_m^l})\ge I(\hat{u}^l)$, $l=1,2,\dots,k+1$. Thus

$$
c_k=\lim_{m\to+\infty}I(u_m)=\lim_{m\to+\infty}\sum_{l=1}^{k+1}I(u_m^l)\ge\sum_{l=1}^{k+1}I(\hat{u}^l)=I(u_k).
$$

Since $u_k\in M_k$, we have that $c_k=I(u_k)$, which means that c_k is attained. □

Proof of Theorem 1.1. By Lemma 3.1, there exists $u_k\in M_k$ which attains c_k. We will prove that u_k is indeed a solution to problem (1.1). For convenience, we denote $u:=u_k$. Thus we get k nodes: $r_1,r_2,\dots,r_k$, $0<r_1<r_2<\dots<r_k<+\infty$. Clearly, u satisfies (1.1) in $E=\{x\in\mathbb{R}^N:|x|\neq r_l,l=1,2,\dots,k+1\}$. We know already that u is of class C^2 on E and satisfies, for $x\in E$

$$
-\sum_{i,j=1}^N\partial_j(a_{ij}(u)\partial_i u)+\frac{1}{2}\sum_{i,j=1}^N a'_{ij}(u)\partial_i u\partial_j u+V(x)u=|u|^{p-1}u. \tag{3.24}
$$

We will prove that u satisfies (3.24) for all $x\in\mathbb{R}^N$.

We use an indirect argument. Assume that for some $l = 1, 2, \dots, k$, there exists $x_0 \in \mathbb{R}^N, |x_0| = r_l$ such that (3.24) does not hold. To complete the proof, it suffices to show that for $a_{ij}(u) = (1 + u^2)\delta_{ij}$, there exists a contradiction.

The existence of the contradiction can be proved similar to that as in [11], by a slight modification, their arguments worked also for $p \in (1, 3]$. We just sketch the proof. We set $r := |x|$ and treat the special case $a_{ij}(u) = (1+u^2)\delta_{ij}$ as an ordinary differential equation:

$$-(1 + u^2)(r^{N-1}u')' = r^{N-1}(|u|^{p-1} - V + |u'|^2)u,$$

where $'$ denotes $\frac{d}{dr}$. Then our assumption becomes to

$$u'_+ = \lim_{r\to r_l^+} u'(r) \neq \lim_{r\to r_l^-} u'(r) = u'_-.$$

Firstly, we construct some w such that $w \in M_k$. Let

$$\psi(h) = \int_{r_{l-1}}^{r_{l+1}} \Big(\frac{1}{2}(h'^2 + Vh^2 + h^2h'^2) - \frac{1}{p+1}|h|^{p+1}\Big)r^{N-1}\,\mathrm{d}r.$$

Then, according to the definition of u, there holds

$$\psi(u) \leq \psi(w).$$

However, under the assumption $u'_+ \neq u'_-$, we can prove that $\psi(w) < \psi(u)$ (cf. [11]). This is a contradiction. As a result, we complete the proof. □

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