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NONUNIQUENESS AND FRACTIONAL INDEX CONVOLUTION COMPLEMENTARITY PROBLEMS

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Abstract. Uniqueness of solutions of fractional index convolution complementarity problems (CCPs) has been shown for index $1+\alpha$ with $-1<\alpha\leq 0$ under mild assumptions, but not for $0<\alpha<1$. Here a family of counterexamples is given showing that uniqueness generally fails for $0<\alpha<1$. These results show that uniqueness is expected to fail for convolution complementarity problems of the type that arise in connection with solutions of impact problems for Kelvin-Voigt viscoelastic rods.

1. Convolution complementarity problems

A *convolution complementarity problem* (CCP) is the task, given functions $m : [0,\infty)\to\mathbb{R}^{n\times n}$ and $q:[0,\infty)\to\mathbb{R}^n$, of finding a function $z:[0,\infty)\to\mathbb{R}^n$ where

$$K \ni z(t) \perp \int_0^t m(t-\tau)\, z(\tau)\, d\tau + q(t) \in K^* \quad \text{for almost all } t\geq 0, \tag{1.1}$$

where K is a closed and convex cone ($x\in K$ and $\alpha\geq 0$ implies $\alpha x\in K$) and K^* is its dual cone:

$$K^* = \{y\in\mathbb{R}^n \mid x^Ty\geq 0 \text{ for all } x\in K\}. \tag{1.2}$$

Most commonly $K=\mathbb{R}^n_+$, for which $K^*=\mathbb{R}^n_+=K$. Also note that "$a\perp b$" means that a and b are orthogonal: $a^Tb=0$. Convolution complementarity problems were introduced by this name in [5], although this concept was used by Petrov and Schatzman [4].

One reason for studying CCPs is their use in studying mechanical impact problems. In particular, Petrov and Schatzman [4] studied the problem of a visco-elastic rod impacting a rigid obstacle:

$$\rho u_{tt} = Eu_{xx} + \beta u_{txx} + f(t,x), \quad x\in(0,L), \tag{1.3}$$

$$N(t) = -\left[Eu_x(t,0) + \beta u_{tx}(t,0)\right], \tag{1.4}$$

$$0 = -\left[Eu_x(t,L) + \beta u_{tx}(t,L)\right], \tag{1.5}$$

$$0\leq N(t)\perp u(t,0)\geq 0. \tag{1.6}$$

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Here $u(t,x)$ is the displacement at time t and position $x \in (0,L)$; (1.3) is the equation for one-dimensional Kelvin–Voigt visco-elasticity; (1.4) is the boundary condition for a contact force $N(t)$ applied at $x = 0$; (1.5) is the boundary condition for a free end at $x = L$; and finally, (1.6) is the Signorini-type contact condition at $x = 0$, indicating that separation ($u(t,0) > 0$) implies no contact force ($N(t) = 0$) while a positive contact force ($N(t) > 0$) implies contact ($u(t,0) = 0$). Because the system is time-invariant, $u(t,0)$ can be represented as $\widehat{u}(t,0) + \int_0^t m(t-\tau)N(\tau)\,d\tau$ where $\widehat{u}(t,x)$ is the solution of the linear system with $N(t) \equiv 0$ and no contact conditions, and the kernel function $m(t) \sim m_0 t^{1/2}$ as $t \downarrow 0$ with $m_0 > 0$. While existence of solutions has been demonstrated for these problems [4, 6], uniqueness has not. This paper shows why.

The *index* of a CCP is the number β where $(d/dt)^\beta m(t) = m_0\,\delta(t) + m_1(t)$ with δ the Dirac-δ function, and $\int_{[0,\epsilon)} \|(d/dt)^\beta m_1(t)\|\,dt \to 0$ as $\epsilon \downarrow 0$, and m_0 is an invertible matrix. If we allow fractional derivatives in the sense of [2], then β need not be an integer. Typically, for index β we have $m(t) \sim m_0\,t^{\beta-1}$ as $t \downarrow 0$. Basic results for fractional index CCPs with index $0 < \beta < 1$ were published in [9]. In particular, combining the results of [5], [9], and [6] we can say that under fairly mild regularity and positivity conditions (related to the index), solutions exist for $0 \le \beta < 2$ and are unique for $0 \le \beta \le 1$. These results can be extended to prove existence of solutions for index $\beta = 2$. However, it is known that solutions are not unique in general for $\beta = 2$. Neither existence nor uniqueness hold in general for $\beta > 2$ (see [8, §3.2.5]). For clarity as to what exactly has been proven for $1 < \beta < 2$, we quote the main results of [6, §8]:

Theorem 1.1. *If $m(t) = m_0 t^{\beta-1} + m_1(t)$ for $t \ge 0$ with $m_0 > 0$, m_1 Lipschitz, $1 < \beta < 2$, $\alpha = \beta - 1$, $q' \in H^{\alpha/2}(0,T^*)$ with $T^* > 0$, and $q(0) \ge 0$, then there is a solution $z(\cdot) \in H^{-\alpha/2}(0,T^*)$ of*

$$0 \le z(t) \perp (m * z)(t) + q(t) \ge 0 \quad \text{for all } t \ge 0.$$

As yet, an open question has been whether uniqueness holds for $1 < \beta < 2$. This paper answers this question in the negative: there are functions $q(\cdot)$ for which there are at least two solutions for $z(\cdot)$ with $m(t) = t^\alpha$ for $0 < \alpha < 1$ where $\alpha = \beta - 1$. The construction of a counter-example to uniqueness is somewhat involved. It proceeds in a similar manner to Mandelbaum's counter-example to uniqueness for certain differential complementarity problems [3]: we first prove equivalence of uniqueness of solutions for (1.1) for $n = 1$ to non-existence of a non-zero function $\zeta : [0,\infty) \to \mathbb{R}$ satisfying

$$\zeta(t)(m * \zeta)(t) \le 0 \quad \text{for all } t \ge 0. \tag{1.7}$$

Given such a ζ we are able to construct both a function $q(\cdot)$ a pair of solutions $z_1(\cdot)$ and $z_2(\cdot)$ of (1.1). The next task is then to construct a suitable $\zeta(\cdot) \not\equiv 0$ satisfying (1.7) for $m(t) = t^\alpha$.

We define the *floor* of a real number z to be $\lfloor z \rfloor = \max\{k \in \mathbb{Z} \mid k \le z\}$, and the *ceiling* of z to be $\lceil z \rceil = \min\{k \in \mathbb{Z} \mid k \ge z\}$.

2. Mandelbaum's condition for CCPs

In [3], Mandebaum considered differential complementarity problems of the form

$$\frac{dw}{dt}(t) = Mz(t) + q'(t), \quad w(0) = q(0), \tag{2.1}$$

$$0 \le w(t) \perp z(t) \ge 0 \tag{2.2}$$

for all t. He was able to show that multiple solutions may exist even for $M = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$ which is positive definite, but not symmetric. The tool that Mandelbaum used was the following theorem.

Theorem 2.1. *The system* (2.1), (2.2) *has a unique solution if and only if*$\omega(t) \circ \zeta(t) \le 0$ *and* $d\omega/dt(t) = M\zeta(t)$ *for* $t \ge 0$ *and* $\omega(0) = 0$ *implies that* $\zeta(t) = 0$ *for all* $t \ge 0$.

Note that "$a \circ b$" is the Hadamard product given by $(a \circ b)_i = a_i b_i$ for all i. In the scalar case ($n = 1$), the Hadamard product reduces to the ordinary product of real numbers.

Theorem 2.2. *The system* (1.1) *with* $n = 1$ *has unique solutions for all* $q(\cdot)$ *if and only if* $\zeta(t)(m * \zeta)(t) \le 0$ *for all* $t \ge 0$ *implies* $\zeta(t) = 0$ *for all* $t \ge 0$.

Proof. The proof is based on Mandelbaum's proof. The sufficiency of the condition for uniqueness can be shown via the contrapositive: if the system (1.1) has two distinct solutions $z_1(\cdot)$ and $z_2(\cdot)$ then we can set $\zeta(t) = z_1(t) - z_2(t)$ not identically zero where

$$\begin{aligned}
\zeta(t)(m * \zeta)(t) &= (z_1(t) - z_2(t))(m * z_1 + q - m * z_2 - q)(t) \\
&= z_1(t)(m * z_1 + q)(t) - z_1(t)(m * z_2 + q)(t) \\
&\quad - z_2(t)(m * z_1 + q)(t) + z_2(t)(m * z_2 + q)(t) \\
&= -z_1(t)(m * z_2 + q)(t) - z_2(t)(m * z_1 + q)(t) \le 0
\end{aligned}$$

for all $t \ge 0$, since $z_1(t)$, $z_2(t) \ge 0$, $(m * z_1 + q)(t), (m * z_2 + q)(t) \ge 0$ and $z_1(t)(m * z_1 + q)(t) = z_2(t)(m * z_2 + q)(t) = 0$.

To show necessity, we again use the contrapositive, and suppose that there is a function $\zeta(\cdot)$ which is not everywhere zero and $\zeta(t)(m * \zeta)(t) \le 0$ for all $t \ge 0$. Let $\omega = m * \zeta$. Note that $\omega(t)\zeta(t) \le 0$. We wish to find functions $q(\cdot)$, $z_1(\cdot)$, and $z_2(\cdot)$ such that $z_1(\cdot)$ and $z_2(\cdot)$ are both solutions to (1.1). Let $E^+ = \{t \ge 0 \mid \omega(t) > 0\}$, $E^- = \{t \ge 0 \mid \omega(t) < 0\}$, and $E^0 = \{t \ge 0 \mid \omega(t) = 0\}$. Let $w_1(t) = \max(\omega(t), 0)$ and $w_2(t) = \max(-\omega(t), 0)$. For $t \in E^+$ we set $z_1(t) = 0$ and $z_2(t) = -\zeta(t) > 0$; for $t \in E^-$ we set $z_1(t) = \zeta(t) \ge 0$ and $z_2(t) = 0$; for $t \in E^0$ we set $z_1(t) = \max(\zeta(t), 0)$ and $z_2(t) = \max(-\zeta(t), 0)$. Then $\zeta(t) = z_1(t) - z_2(t)$ and $z_1(t)$, $z_2(t)$, $w_1(t)$, $w_2(t) \ge 0$ for all $t \ge 0$. For $t \in E^+$, $w_1(t)z_1(t) = 0$ since $z_1(t) = 0$, and $w_2(t)z_2(t) = 0$ since $w_2(t) = 0$; for $t \in E^-$, $w_1(t)z_1(t) = 0$ since $w_1(t) = 0$, and $w_2(t)z_2(t) = 0$ since $z_2(t) = 0$; for $t \in E^0$, $w_1(t)z_1(t) = w_2(t)z_2(t) = 0$ since $w_1(t) = w_2(t) = 0$. Thus both $(z_1(\cdot), w_1(\cdot))$ and $(z_2(\cdot), w_2(\cdot))$ satisfy the complementarity conditions. We now check the dynamic conditions.

Let $q(t) = w_1(t) - (m * z_1)(t)$ for all $t \ge 0$. Then, clearly, $w_1(t) = (m * z_1)(t) + q(t)$. On the other hand, $w_1(t) - w_2(t) = \omega(t)$ and $z_1(t) - z_2(t) = \zeta(t)$ for all $t \ge 0$, so

$$\begin{aligned}
w_2(t) &= w_1(t) - \omega(t) \\
&= (m * z_1)(t) + q(t) - (m * \zeta)(t) \\
&= (m * (z_1 - \zeta))(t) + q(t) \\
&= (m * z_2)(t) + q(t).
\end{aligned}$$

Thus the dynamic conditions also hold, and we have two distinct solutions of (1.1), as we wanted. □

This theorem can be extended to the $n > 1$ case by working componentwise.

3. Constructing the counter-example

Much like the examples given for related non-smooth dynamical systems [1, 3, 7], there is a self-similar structure to the counter-example created here. The counter-example involves non-analytic $q(\cdot)$. The construction begins with a "bump" function $\theta : \mathbb{R} \to \mathbb{R}$ where $\theta(s) \geq 0$ for all $s \in \mathbb{R}$, $\operatorname{supp}\theta \subseteq [-1, +1]$, $\int_{-\infty}^{+\infty} \theta(s)\, ds = 1$, and θ is C^∞.

Let $\psi_\alpha(t) = t^\alpha$ for $t > 0$ and $\psi_\alpha(t) = 0$ for $t \leq 0$. We will consider $0 < \alpha < 1$; the CCP

$$0 \leq z(t) \perp (\psi_\alpha * z)(t) + q(t) \geq 0 \tag{3.1}$$

then has index $1 + \alpha$. The case $\alpha = \frac{1}{2}$ corresponds to the viscoelastic impact problem in Petrov and Schatzman [4] where, asymptotically, $m(t) \sim m_0\sqrt{t}$ as $t \downarrow 0$. The case $m(t) = t^\alpha$ has additional structure that we will exploit in the construction here. We will construct a function $\zeta(t)$ satisfying $\zeta(t)(\psi_\alpha * \zeta)(t) \leq 0$ for all $t \geq 0$ and $\zeta(t) = 0$ for $t < 0$.

Let $\zeta_1(s; \eta) = \eta^{-1}\, \theta(\eta^{-1}(s - \widehat{s}))$ where $\eta > 0$ and $\widehat{s}$ are parameters to be determined. We set

$$\zeta(t; \eta) = \sum_{k \in \mathbb{Z}} (-1)^k \mu^{-k} \zeta_1(\gamma^k t; \eta) \tag{3.2}$$

where $0 < \mu$, $1 < \gamma$ are to be determined. Let $\widehat{s} = \frac{1}{2}(1 + \gamma)$. Note that $\zeta_1(s; \eta) \to \delta(s - \widehat{s})$ as $\eta \downarrow 0$ in the sense of distributions where δ is the "Dirac-δ function". If we write

$$\widehat{\zeta}(t) = \sum_{k \in \mathbb{Z}} (-1)^k \mu^{-k} \gamma^{-k} \delta(t - \gamma^{-k}\widehat{s}),$$

then $\zeta(\cdot; \eta) \to \widehat{\zeta}$ as $\eta \downarrow 0$ in the sense of distributions, and in terms of weak* convergence of measures.

Note that

$$\begin{aligned}
\zeta(\gamma t; \eta) &= \sum_{k \in \mathbb{Z}} (-1)^k \mu^{-k} \zeta_1(\gamma^{k+1} t; \eta) \\
&= \sum_{\ell \in \mathbb{Z}} (-1)^{\ell-1} \mu^{-\ell+1} \zeta_1(\gamma^\ell t; \eta) \quad (\ell = k + 1) \\
&= -\mu \sum_{\ell \in \mathbb{Z}} (-1)^\ell \mu^{-\ell} \zeta_1(\gamma^\ell t; \eta) = -\mu \zeta(t; \eta).
\end{aligned} \tag{3.3}$$

Also note that

$$\begin{aligned}
(\psi_\alpha * f(\gamma \cdot))(t) &= \int_0^t \psi_\alpha(t - \tau) f(\gamma\tau)\, d\tau \\
&= \int_0^{\gamma t} (t - \gamma^{-1}\sigma)^\alpha f(\sigma) \gamma^{-1}\, d\sigma \quad (\sigma = \gamma\tau) \\
&= \gamma^{-1-\alpha} \int_0^{\gamma t} (\gamma t - \sigma)^\alpha f(\sigma)\, d\sigma \\
&= \gamma^{-1-\alpha} (\psi_\alpha * f)(\gamma t).
\end{aligned}$$

Thus $-\mu\gamma^{1+\alpha}(\psi_\alpha * \zeta(\cdot;\eta))(t) = (\psi_\alpha * \zeta(\cdot;\eta))(\gamma t)$. From these relationships, if $\zeta(t;\eta)(\psi_\alpha * \zeta(\cdot;\eta))(t) \le 0$ for $1 \le t \le \gamma$, then $\zeta(t;\eta)(\psi_\alpha * \zeta(\cdot;\eta))(t) \le 0$ for all $t > 0$. The reason is that $\zeta(\gamma t;\eta) = (-\mu\zeta(t;\eta))$ and so $\zeta(\gamma t;\eta)(\psi_\alpha * \zeta(\cdot;\eta))(\gamma t) = (-\mu)(-\mu\gamma^{1+\alpha})\zeta(t;\eta)(\psi_\alpha * \zeta(\cdot;\eta))(t)$ and therefore

$$\operatorname{sign}\zeta(t;\eta)(\psi_\alpha * \zeta(\cdot;\eta))(t) = \operatorname{sign}\zeta(\gamma t;\eta)(\psi_\alpha * \zeta(\cdot;\eta))(\gamma t).$$

Once we know that $\zeta(t;\eta)(\psi_\alpha * \zeta(\cdot;\eta))(t) \le 0$ for all $t \in [1,\gamma]$, it follows that $\zeta(t;\eta)(\psi_\alpha * \zeta(\cdot;\eta))(t) \le 0$ for all $t > 0$.

Since $\operatorname{supp}\zeta \cap [1,\gamma] = \widehat{s} + [-\eta,+\eta]$, it is sufficient to check that $\zeta(t;\eta)(\psi_\alpha * \zeta(\cdot;\eta))(t) \le 0$ for $t \in \widehat{s} + [-\eta,+\eta]$; since $\zeta(t;\eta) \ge 0$ for $1 \le t \le \gamma$, it suffices to check that $(\psi_\alpha * \zeta(\cdot;\eta))(t) \le 0$ for $t \in \widehat{s} + [-\eta,+\eta]$. We will consider the limit as $\eta \downarrow 0$, so it becomes a matter of ensuring simply that $(\psi_\alpha * \zeta(\cdot;\eta))(\widehat{s}) < 0$. There are some additional technical issues that must be addressed, but this will be done later.

Now we compute $\psi_\alpha * \zeta(\cdot;\eta)$:

$$\begin{aligned}
(\psi_\alpha * \zeta(\cdot;\eta))(t) &= \sum_{k\in\mathbb{Z}} (-1)^k \mu^{-k} (\psi_\alpha * \zeta_1(\gamma^k\cdot;\eta))(t) \\
&= \sum_{k\in\mathbb{Z}} (-1)^k \mu^{-k} (\gamma^k)^{-1-\alpha} (\psi_\alpha * \zeta_1(\cdot;\eta))(\gamma^k t) \\
&= \sum_{k=\lfloor \ln t/\ln\gamma \rfloor}^{\infty} (-1)^k (\mu\gamma^{1+\alpha})^{-k} (\psi_\alpha * \zeta_1(\cdot;\eta))(\gamma^k t)
\end{aligned}$$

since $\zeta_1(s;\eta) = 0$ for $s \le 1$ and therefore $(\psi_\alpha * \zeta_1(\cdot;\eta))(s) = 0$ for $s \le 1$. In particular, for $1 \le t \le \gamma$,

$$(\psi_\alpha * \zeta(\cdot;\eta))(t) = \sum_{k=0}^{\infty} (-1)^k (\mu\gamma^{1+\alpha})^{-k} (\psi_\alpha * \zeta_1(\cdot;\eta))(\gamma^k t).$$

For this sum to converge, we need $\mu\gamma > 1$: asymptotically $(\psi_\alpha * \zeta_1(\cdot;\eta))(s) \sim s^\alpha$ as $s \to \infty$, so $(\psi_\alpha * \zeta_1(\cdot;\eta))(\gamma^k t) \sim (\gamma^\alpha)^k t^\alpha$ as $k \to \infty$. Furthermore, $(\psi_\alpha * \zeta_1(\cdot;\eta))(s) \to \psi_\alpha(s - \widehat{s}) = (s - \widehat{s})^\alpha$ as $\eta \downarrow 0$. So for $1 \le t \le \gamma$,

$$\begin{aligned}
(\psi_\alpha * \zeta(\cdot;\eta))(t) &\to \sum_{k=0}^{\infty} (-1)^k (\mu\gamma^{1+\alpha})^{-k} (\gamma^k t - \widehat{s})^\alpha \quad \text{as } \eta \downarrow 0 \\
&= \sum_{k=0}^{\infty} (-1)^k (\mu\gamma)^{-k} (t - \gamma^{-k}\widehat{s})^\alpha.
\end{aligned}$$

In particular, for $t = \widehat{s}$,

$$(\psi_\alpha * \zeta(\cdot;\eta))(\widehat{s}) \to \sum_{k=0}^{\infty} (-1)^k (\mu\gamma)^{-k} (1 - \gamma^{-k})^\alpha (\widehat{s})^\alpha \quad \text{as } \eta \downarrow 0.$$

Note that the term in the sum with $k = 0$ is zero, and so can be ignored in the limit as $\eta \downarrow 0$. So we now want to evaluate the sum

$$\widehat{v}(\mu,\gamma) := \sum_{k=1}^{\infty} (-1)^k (\mu\gamma)^{-k} (1 - \gamma^{-k})^\alpha, \tag{3.4}$$

and check that the value is negative. Note that if $\mu\gamma = \rho > 1$ is held fixed, then $\widehat{v}(\mu,\gamma) = \sum_{k=1}^{\infty} (-1)^k \rho^{-k} (1 - \gamma^{-k})^\alpha \to \sum_{k=1}^{\infty} (-1)^k \rho^{-k} = -\rho^{-1}/(1 + \rho^{-1}) < 0$ as

$\gamma \to \infty$. Thus for sufficiently large $\gamma > 1$ with $\mu\gamma = \rho > 1$ fixed, we have $\widehat{v}(\mu,\gamma) < 0$ as we want. Also, $\rho\widehat{v}(\mu,\gamma) \to -(1-\gamma^{-1})^\alpha$ as $\rho \to \infty$ with fixed $\gamma > 1$.

3.1. **Regularity of ζ and $\psi_\alpha * \zeta$, and choice of parameters.** First we consider the question of how to ensure that $\zeta \in L^1(0,\gamma)$: Since $\|\zeta_1(\cdot;\eta)\| = 1$ independently of $\eta > 0$, we have

$$\|\zeta(\cdot;\eta)\|_{L^1(0,\gamma)} \le \sum_{k=0}^{\infty} (\mu\gamma)^{-k} = \frac{1}{1-\rho^{-1}}$$

which is finite as long as $\rho = \mu\gamma > 1$. Note that this bound is independent of $\eta > 0$. Also, ψ_α is uniformly Hölder continuous: $|\psi_\alpha(t) - \psi_\alpha(s)| = |t^\alpha - s^\alpha| \le |t-s|^\alpha$ for any $s,t \in \mathbb{R}$ as $0 < \alpha < 1$. Combining these results shows that for $s,t \in [0,\gamma]$, $|(\psi_\alpha*\zeta(\cdot;\eta))(t) - (\psi_\alpha*\zeta(\cdot;\eta))(s)| \le |t-s|^\alpha \|\zeta(\cdot;\eta)\|_{L^1(0,\gamma)}$. That is, $(\psi_\alpha * \zeta(\cdot;\eta))\,|_{[0,\gamma]}$ is uniformly Hölder continuous, independently of $\eta > 0$.

Thus, provided (3.4) is negative, for sufficiently small $\eta > 0$, we have $\zeta(t;\eta)(\psi_\alpha * \zeta(\cdot;\eta))(t) \le 0$ for all $1 \le t \le \gamma$. To see this rigorously, recall that $\zeta(t) \ne 0$ for $1 \le t \le \gamma$ only if $|t - \widehat{s}| < \eta$. Choose $\eta > 0$ sufficiently small so that $|(\psi_\alpha * \zeta(\cdot;\eta))(\widehat{s}) - \widehat{v}(\mu,\gamma)| \le \frac{1}{4}|\widehat{v}(\mu,\gamma)|$. Now for$|t-\widehat{s}| \le \eta$,

$$\begin{aligned}
|(\psi_\alpha * \zeta(\cdot;\eta))(t) - \widehat{v}(\mu,\gamma)| &\le |(\psi_\alpha * \zeta(\cdot;\eta))(t) - (\psi_\alpha * \zeta(\cdot;\eta))(\widehat{s})| + \frac{1}{4}|\widehat{v}(\mu,\gamma)| \\
&\le |t-\widehat{s}|^\alpha \|\zeta(\cdot;\eta)\|_{L^1(0,\gamma)} + \frac{1}{4}|\widehat{v}(\mu,\gamma)| \\
&\le \eta^\alpha \|\zeta(\cdot;\eta)\|_{L^1(0,\gamma)} + \frac{1}{4}|\widehat{v}(\mu,\gamma)|.
\end{aligned}$$

Choose $\eta > 0$ sufficiently small so that it also satisfies $\eta^\alpha \|\zeta(\cdot;\eta)\|_{L^1(0,\gamma)} \le \frac{1}{4}|\widehat{v}(\mu,\gamma)|$. Then $\zeta(t;\eta) \ne 0$ and $1 \le t \le \gamma$ imply that $(\psi_\alpha * \zeta(\cdot;\eta))(t) \le \frac{1}{2}\widehat{v}(\mu,\gamma) < 0$. Since $\zeta(t;\eta) \ge 0$ for $1 \le t \le \gamma$, we have $\zeta(t;\eta)(\psi_\alpha * \zeta(\cdot;\eta))(t) \le 0$ for all $1 \le t \le \gamma$.

Consequently, from the self-similarity property (3.3), $\zeta(t;\eta)(\psi_\alpha * \zeta(\cdot;\eta))(t) \le 0$ for all $t \ge 0$.

If we allow $\mu > 1$ we can get much stronger regularity on ζ. If $\mu > 1$ then by the Weierstass M-test (see, e.g., [10, Thm. 3.106, p. 141]), $\zeta(\cdot;\eta)$ is continuous. Furthermore, if $\mu\gamma^{-p} > 1$, ζ is p-times continuously differentiable for $p = 1, 2, \dots$, again by the Weierstass M-test but applied to $\zeta^{(p)}(\cdot;\eta)$. This is equivalent to the condition that $\rho\gamma^{-p-1} > 1$.

If we set $\rho = 2\gamma^{mp+1}$, then

$$\begin{aligned}
\gamma^{p+1}\,\widehat{v}(\mu,\gamma) &= \gamma^{p+1}\sum_{k=1}^{\infty}(-1)^k \rho^{-k}(1-\gamma^{-k})^\alpha \\
&= \gamma^{p+1}\sum_{k=1}^{\infty}(-1)^k (2\gamma^{m+1})^{-k}(1-\gamma^{-k})^\alpha \\
&= \sum_{k=1}^{\infty}(-1)^k \frac{1}{2}(2\gamma^{p+1})^{-k+1}(1-\gamma^{-k})^\alpha \\
&\to -\frac{1}{2} \quad \text{as } \gamma \to \infty.
\end{aligned}$$

So for sufficiently large $\gamma > 1$, $\widehat{v}(\mu,\gamma) < 0$. Then $\mu\gamma = \rho = 2\gamma^{p+1}$, so we set $\mu = 2\gamma^p$. We then choose $\eta > 0$ sufficiently small so that $\zeta(t;\eta)(\psi_\alpha * \zeta(\cdot;\eta))(t) \le 0$ for $1 \le t \le$

γ. Since $\zeta(\gamma^{-k}t;\eta) = (-\mu)^{-k}\zeta(t;\eta)$ and $(\psi_\alpha * \zeta(\cdot;\eta))(\gamma^{-k}t) = (-\mu\gamma^{1+\alpha})^{-k}(\psi_\alpha * \zeta(\cdot;\eta))(t)$, we have $\zeta(t;\eta)(\psi_\alpha * \zeta(\cdot;\eta))(t) \le 0$ for $\gamma^{-k} \le t \le \gamma^{-k+1}$ for any $k \in \mathbb{Z}$; thus $\zeta(t;\eta)(\psi_\alpha * \zeta(\cdot;\eta))(t) = 0$ for any $t > 0$. In addition, $(\psi_\alpha * \zeta(\cdot;\eta))(0) = 0$, so $\zeta(t;\eta)(\psi_\alpha * \zeta(\cdot;\eta))(t) \le 0$ for all $t \ge 0$, and there is a counter-example to uniqueness as we wanted. Furthermore, the counter-example is in C^p.

4. Extension to general $m(t) \sim m_0 t^\alpha$

Here we assume not only that $0 < \alpha < 1$ but also that $m_0 > 0$. If $m_0 < 0$ so that $m(t) < 0$ for $0 \le t \le T_1$ with $T_1 > 0$ and $z_1(t)$ is a positive smooth function of t, then for $q_1(t) = -(m * z_1)(t)$ not only is $z(t) = z_1(t)$ for $t \ge 0$ a solution to

$$0 \le z(t) \perp (m * z)(t) + q_1(t) \ge 0 \quad \text{for all } t \ge 0,$$

but $z(t) = 0$ for $0 \le t \le T_1$ is also a solution as $q_1(t) > 0$ for $0 \le t \le T_1$.

The assumptions made on m are that $m(t) \sim m_0 t^\alpha$, $m'(t) \sim m_0 \alpha t^{\alpha-1}$ as $t \downarrow 0$, and $m'(t)$ is continuous in t away from $t = 0$. This implies that on bounded sets, $m(\cdot)$ is uniformly Hölder continuous: given a bounded interval $[a, b]$, there is an M where $|m(t) - m(s)| \le M\,|t - s|^\alpha$ for all $s, t \in [a, b]$.

Note that dividing $m(t)$ by $m_0 > 0$ does not affect the existence of multiple solutions as (1.1) is equivalent to

$$0 \le z(t) \perp ((m/m_0) * z)(t) + q(t)/m_0 \ge 0 \quad \text{for all } t \ge 0.$$

So we consider without loss of generality the case where $m(t) \sim t^\alpha$. As in Section 2 we look for a non-zero function $\zeta : [0, \infty) \to \mathbb{R}$ where $\zeta(t)(m * \zeta)(t) \le 0$ for all $t \ge 0$. The constructed ζ from the previous Section will also work here with some small modifications.

Let $r(t) = (m(t)/\psi_\alpha(t)) - 1$. Note that $r(t) \to 0$ as $t \downarrow 0$. Using (3.2) to define $\zeta(\cdot)$,

$$\zeta(t) = \sum_{k\in\mathbb{Z}} (-1)^k \mu^{-k} \zeta_1(\gamma^k t;\eta),$$

we can show that for $\gamma^{-j} \le t < \frac{1}{2}\gamma^{-j}(1+\gamma)$,

$$\begin{aligned}(m * \zeta)(t) &= \sum_{k=j}^{\infty} (-1)^k \mu^{-k} (m * \zeta_1(\gamma^k \cdot;\eta))(t) \\ &\to \sum_{k=j+1}^{\infty} (-1)^k \mu^{-k} \gamma^{-k} m(t - \gamma^{-k+j}\widehat{s}) \quad \text{as } \eta \downarrow 0,\end{aligned}$$

using $(m * \zeta_1(\cdot;\eta))(s) \to m(s - \widehat{s})$ as $\eta \downarrow 0$, and $m(0) = 0$. We need to distinguish between the value and the limit. First, note that if $\operatorname{supp} g \subseteq [\widehat{s} - \rho, \widehat{s} + \rho]$ and g is non-negative, then for continuous f,

$$\Big| \int_{-\infty}^{+\infty} f(s)\, g(s)\, ds - f(\widehat{s}) \int_{\widehat{s}-\rho}^{\widehat{s}+\rho} g(s)\, ds \Big| \le \max_{s:|s-\widehat{s}|\le\rho} |f(s) - f(\widehat{s})| \int_{\widehat{s}-\rho}^{\widehat{s}+\rho} g(s)\, ds.$$

Then

$$|(m * \zeta_1(\gamma^k \cdot;\eta))(t) - \gamma^{-k} m(t - \gamma^{-k}\widehat{s})| \le M(\gamma^{-k}\eta)^\alpha \gamma^{-k} = M\eta^\alpha (\gamma^{1+\alpha})^{-k}.$$

So, for $t = \gamma^{-j}\widehat{s}$,

$$\Big|(m * \zeta)(\gamma^{-j}\widehat{s}) - \sum_{k=j}^{\infty} (-1)^k \mu^{-k} \gamma^{-k} m((1 - \gamma^{-k+j})\gamma^{-j}\widehat{s})\Big|$$

$$\le \sum_{k=j}^{\infty} \mu^{-k} (\gamma^{1+\alpha})^{-k}\, M\eta^{\alpha} = \frac{(\mu\gamma^{1+\alpha})^{-j} M\eta^{\alpha}}{1-(\mu\gamma^{1+\alpha})^{-1}}.$$

Note that

$$\begin{aligned}
&\sum_{k=j+1}^{\infty} (-1)^k \mu^{-k}\gamma^{-k} m((1-\gamma^{-k+j})\gamma^{-j}\widehat{s}) \\
&= (-1)^j(\mu\gamma)^{-j} \sum_{\ell=1}^{\infty} (-1)^{\ell}(\mu\gamma)^{-\ell} m((1-\gamma^{-\ell})\gamma^{-j}\widehat{s}) \\
&= (-1)^j(\mu\gamma)^{-j} \sum_{\ell=1}^{\infty} (-1)^{\ell}(\mu\gamma)^{-\ell} ((1-\gamma^{-\ell})\gamma^{-j}\widehat{s})^{\alpha}[1+r((1-\gamma^{-\ell})\gamma^{-j}\widehat{s})] \\
&= (-1)^j(\mu\gamma^{1+\alpha})^{-j}\widehat{s}^{\alpha} \sum_{\ell=1}^{\infty} (-1)^{\ell}(\mu\gamma)^{-\ell} (1-\gamma^{-\ell})^{\alpha}[1+r((1-\gamma^{-\ell})\gamma^{-j}\widehat{s})].
\end{aligned}$$

Since $r(t) \to 0$ as $t \downarrow 0$, for every $\epsilon > 0$ there is a $\delta > 0$ where $0 < t < \delta$ implies $|r(t)| < \epsilon$. Thus for $j \ge -\ln(\delta/\widehat{s})/\ln\gamma$, $|r((1-\gamma^{-\ell})\gamma^{-j}\widehat{s})| < \epsilon$, and so

$$\Big|\sum_{\ell=1}^{\infty} (-1)^{\ell}(\mu\gamma)^{-\ell}(1-\gamma^{-\ell})^{\alpha} r((1-\gamma^{-\ell})\gamma^{-j}\widehat{s})\Big| \le \frac{\epsilon}{1-(\mu\gamma)^{-1}}.$$

Since $\gamma^{-j} \le t \le \gamma^{-j+1}$ and $\zeta(t) \ne 0$ implies $|t-\gamma^{-j}\widehat{s}| \le \gamma^{-j}\eta$, we can use the bound $|(m*\zeta)(t) - (m*\zeta)(\gamma^{-j}\widehat{s})| \le M(\eta\gamma^{-j})^{\alpha}\|\zeta\|_{L^1(0,\gamma^{-j+1})} \le M\eta^{\alpha}\gamma^{-\alpha j}(\mu\gamma)^{-j}/(1-(\mu\gamma)^{-1})$ for $|t-\gamma^{-j}\widehat{s}| \le \gamma^{-j}\eta$. Thus for $\gamma^{-j} \le t \le \gamma^{-j+1}$ and $\zeta(t) \ne 0$,

$$\begin{aligned}
&|(m*\zeta)(t) - (-1)^j\widehat{s}^{\alpha}(\mu\gamma^{1+\alpha})^{-j}\widehat{v}(\mu,\gamma)| \\
&\le \frac{M\eta^{\alpha}(\mu\gamma^{1+\alpha})^{-j}}{1-(\mu\gamma)^{-1}} + \frac{(\mu\gamma^{1+\alpha})^{-j}M\eta^{\alpha}}{1-(\mu\gamma^{1+\alpha})^{-1}} + \frac{\widehat{s}^{\alpha}(\mu\gamma^{1+\alpha})^{-j}\epsilon}{1-(\mu\gamma)^{-1}} \\
&\le (\mu\gamma^{1+\alpha})^{-j}\Big[\frac{M\eta^{\alpha}}{1-(\mu\gamma)^{-1}} + \frac{M\eta^{\alpha}}{1-(\mu\gamma^{1+\alpha})^{-1}} + \frac{\widehat{s}^{\alpha}\epsilon}{1-(\mu\gamma)^{-1}}\Big].
\end{aligned}$$

Note that $\gamma > 1$ so that $\mu\gamma^{1+\alpha} > \mu\gamma > 1$. By choosing $\eta > 0$ and $\epsilon > 0$ sufficiently small, we can guarantee that the sign of $(m*\zeta)(t)$ for $\gamma^{-j} \le t \le \gamma^{-j+1}$ and $\zeta(t) \ne 0$ is the sign of $(-1)^j\widehat{v}(\mu,\gamma)$. After choosing $\eta > 0$ and $\epsilon > 0$ so that this holds, we can ensure that $\zeta(t)(m*\zeta)(t) \le 0$ for $\gamma^{-j} \le t \le \gamma^{-j+1}$ where $j \ge J := \lceil -\ln(\delta/\widehat{s})/\ln\gamma\rceil$. Thus $\zeta(t)(m*\zeta)(t) \le 0$ for all $0 < t \le \gamma^{-J}$. By setting $\widehat{\zeta}(t) = \zeta(t)$ for $0 \le t \le \gamma^{-J}$ and $\widehat{\zeta}(t) = 0$ for $t \ge \gamma^{-J}$ (noting that $\zeta(t) = 0$ in a neighborhood of γ^{-k} for any $k \in \mathbb{Z}$), we see that $\widehat{\zeta}(t)(m*\widehat{\zeta})(t) \le 0$ for all $t \ge 0$, and thus we have non-uniqueness of solutions for (1.1) where $m(t) \sim m_0 t^{\alpha}$ and $m'(t) \sim m_0\alpha t^{\alpha-1}$ as $t \downarrow 0$ provided $m_0 > 0$ and $0 < \alpha < 1$.

5. Conclusions

Non-uniqueness of convolution complementarity problems of the form (1.1) with convolution kernel $m(t) \sim m_0 t^{\alpha}$ and $m'(t) \sim m_0\alpha t^{\alpha-1}$ with $m_0 > 0$ and $0 < \alpha < 1$ has been demonstrated via a generalization of a result of Mandelbaum. Note that the counter-examples can belong to any space C^p, $p = 1, 2, 3, \ldots$. Counter-examples must have infinitely many oscillations in a finite time interval, and so cannot be analytic. The main non-uniqueness result is of particular interest for questions of contact mechanics, as the perpendicular impact of a Kelvin–Voigt viscoelastic rod on a rigid obstacle can be model by such a CCP (see [4]). Note that this non-uniqueness holds in spite of the existence of an energy balance for this situation [4]. By contrast, the perpendicular impact of a purely elastic rod on a rigid obstacle does have uniqueness of solutions, by using CCP formulations but with $\alpha = 0$ [5]. Multidimensional contact problems then either have a problem of existence (for purely elastic bodies) or with uniqueness (for Kelvin–Voigt viscoelastic bodies). How this can be resolved is a subject for future investigation.

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