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FRICTIONAL CONTACT PROBLEMS FOR ELECTRO-VISCOELASTIC MATERIALS WITH LONG-TERM MEMORY, DAMAGE, AND ADHESION

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Abstract. We consider a quasistatic contact problem between two electro-viscoelastic bodies with long-term memory and damage. The contact is frictional and is modelled with a version of normal compliance condition and the associated Coulomb's law of friction in which the adhesion of contact surfaces is taken into account. We derive a variational formulation for the model and prove an existence and uniqueness result of the weak solution. The proof is based on arguments of evolutionary variational inequalities, a classical existence and uniqueness result on parabolic inequalities, and Banach fixed point theorem.

1. Introduction

The aim of this article is to study a quasistatic frictional contact problem with adhesion between two electro-viscoelastic bodies. We use the electro-viscoelastic constitutive law with long-term memory and damage given by

$$\boldsymbol{\sigma}^\ell = \mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\ell) + \mathcal{G}^\ell\boldsymbol{\varepsilon}(\mathbf{u}^\ell) + (\mathcal{E}^\ell)^*\nabla\varphi^\ell + \int_0^t \mathcal{F}^\ell\big(t-s, \varepsilon(\mathbf{u}^\ell(s)), \zeta^\ell(s)\big)\,ds, \tag{1.1}$$

where $\mathbf{u}^\ell$ the displacement field, $\boldsymbol{\sigma}^\ell$ and $\boldsymbol{\varepsilon}(\mathbf{u}^\ell)$ represent the stress and the linearized strain tensor, respectively. Here $\mathcal{A}^\ell$ is a given nonlinear operator, $\mathcal{F}^\ell$ is the relaxation operator, and $\mathcal{G}^\ell$ represents the elasticity operator.$E(\varphi^\ell) = -\nabla\varphi^\ell$ is the electric field, $\mathcal{E}^\ell$ represents the third order piezoelectric tensor, $(\mathcal{E}^\ell)^*$ is its transposition. In (1.1) and everywhere in this paper the dot above a variable represents derivative with respect to the time variable t. It follows from (1.1) that at each time moment, the stress tensor $\boldsymbol{\sigma}^\ell(t)$ is split into three parts: $\boldsymbol{\sigma}^\ell(t) = \boldsymbol{\sigma}^\ell_V(t) + \boldsymbol{\sigma}^\ell_E(t) + \boldsymbol{\sigma}^\ell_R(t)$, where $\boldsymbol{\sigma}^\ell_V(t) = \mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\ell(t))$ represents the purely viscous part of the stress, $\boldsymbol{\sigma}^\ell_E(t) = (\mathcal{E}^\ell)^*\nabla\varphi^\ell(t)$ represents the electric part of the stress and $\boldsymbol{\sigma}^\ell_R(t)$ satisfies the *rate-type* elastic relation

$$\boldsymbol{\sigma}^\ell_R(t) = \mathcal{G}^\ell\boldsymbol{\varepsilon}(\mathbf{u}^\ell(t)) + \int_0^t \mathcal{F}^\ell\big(t-s, \varepsilon(\mathbf{u}^\ell(s)), \zeta^\ell(s)\big)\,ds. \tag{1.2}$$

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Various results, example and mechanical interpretations in the study of elastic materials of the form (1.2) can be found in [2, 24] and references therein. Note also that when $\mathcal{F}^\ell = 0$ the constitutive law (1.1) becomes the Kelvin-Voigt electro-viscoelastic constitutive relation

$$\boldsymbol{\sigma}^\ell(t) = \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\ell(t)) + \mathcal{G}^\ell \boldsymbol{\varepsilon}(\mathbf{u}^\ell(t)) + (\mathcal{E}^\ell)^* \nabla \varphi^\ell(t). \tag{1.3}$$

Quasistatic contact problems with Kelvin-Voigt materials of the form (1.3) can be found in [19, 20, 25]. The normal compliance contact condition was first considered in [14] in the study of dynamic problems with linearly elastic and viscoelastic materials and then it was used in various references, see e.g. [11, 19]. This condition allows the interpenetration of the body's surface into the obstacle and it was justified by considering the interpenetration and deformation of surface asperities.

Processes of adhesion are important in many industrial settings where parts, usually nonmetallic, are glued together. For this reason, adhesive contact between deformable bodies, when a glue is added to prevent relative motion of the surfaces, has received recently increased attention in the mathematical literature. Analysis of models for adhesive contact can be found in [4, 15, 16] and recently in the monographs [17, 18]. The novelty in all these papers is the introduction of a surface internal variable, the bonding field, denoted in this paper by β. It describes the point wise fractional density of adhesion of active bonds on the contact surface, and some times it is called *the intensity of adhesion*. Following [10], the bonding field satisfies the restriction $0 \le \beta \le 1$, when $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active, when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion, when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active. The damage is an extremely important topic in engineering, since it affects directly the useful life of the designed structure or component. There is a very large engineering literature on this topic. Models taking into account the influence of internal damage of the material on the contact process have been investigated mathematically. General models for damage were derived in [5, 6] from the virtual power principle. Mathematical analysis of one-dimensional problems can be found in [7]. The three-dimensional case has been investigated in [12]. In all these papers the damage of the material is described with a damage function ζ^ℓ, restricted to have values between zero and one. When $\zeta^\ell = 1$, there is no damage in the material, when $\zeta^\ell = 0$, the material is completely damaged, when $0 < \zeta^\ell < 1$ there is partial damage and the system has a reduced load carrying capacity. Contact problems with damage have been investigated in [7, 20, 21, 23]. In this paper the inclusion used for the evolution of the damage field is

$$\dot{\zeta}^\ell - \kappa^\ell \Delta \zeta^\ell + \partial \psi_{K^\ell}(\zeta^\ell) \ni \phi^\ell\big(\boldsymbol{\sigma}^\ell - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\ell), \boldsymbol{\varepsilon}(\mathbf{u}^\ell), \zeta^\ell\big), \tag{1.4}$$

where K^ℓ denotes the set of admissible damage functions defined by

$$K^\ell = \{\xi \in H^1(\Omega^\ell); 0 \le \xi \le 1, \text{ a.e. in } \Omega^\ell\}, \tag{1.5}$$

κ^ℓ is a positive coefficient, $\partial \psi_{K^\ell}$ represents the subdifferential of the indicator function of the set K^ℓ and ϕ^ℓ is a given constitutive function which describes the sources of the damage in the system. In this article we consider a mathematical frictional contact problem between two electro-viscoelastic bodies with constitutive law with long-term memory and damage. The contact is modelled with normal compliance where the adhesion of the contact surfaces is taken into account and is modelled with a surface variable, the bonding field. We derive a variational formulation of the problem and prove the existence of a unique weak solution.

This article is organized as follows. In Section 2 we describe the mathematical models for the frictional contact problem between two electro-viscoelastics bodies with long-term memory and damage. The contact is modelled with normal compliance and adhesion. In Section 3 we introduce some notation, list the assumptions on the problem's data, and derive the variational formulation of the model. We state our main result, the existence of a unique weak solution to the model in Theorem 4.1. The proof of the theorem is provided in Section 4, where it is carried out in several steps and is based on arguments of evolutionary variational inequalities, a classical existence and uniqueness result on parabolic inequalities, differential equations and the Banach fixed point theorem.

2. Problem Statement

Let us consider two electro-viscoelastic bodies with long-term memory occupying two bounded domains Ω^1, Ω^2 of the space $\mathbb{R}^d(d=2,3)$. For each domain Ω^ℓ, the boundary Γ^ℓ is assumed to be Lipschitz continuous, and is partitioned into three disjoint measurable parts Γ_1^ℓ, Γ_2^ℓ and Γ_3^ℓ, on one hand, and on two measurable parts Γ_a^ℓ and Γ_b^ℓ, on the other hand, such that $\operatorname{meas}\Gamma_1^\ell>0$, $\operatorname{meas}\Gamma_a^\ell>0$. Let $T>0$ and let $[0,T]$ be the time interval of interest. The body Ω^ℓ is subjected to $\boldsymbol{f}_0^\ell$ forces and volume electric charges of density q_0^ℓ. The bodies are assumed to be clamped on $\Gamma_1^\ell\times(0,T)$. The surface tractions $\boldsymbol{f}_2^\ell$ act on $\Gamma_2^\ell\times(0,T)$. We also assume that the electrical potential vanishes on $\Gamma_a^\ell\times(0,T)$ and a surface electric charge of density q_2^ℓ is prescribed on $\Gamma_b^\ell\times(0,T)$. The two bodies can enter in contact along the common part $\Gamma_3^1=\Gamma_3^2=\Gamma_3$. The bodies are in adhesive contact with an obstacle, over the contact surface Γ_3. With the assumption above, the classical formulation of the friction contact problem with adhesion and damage between two electro-viscoelastics bodies with long-term memory is following.

Problem P. For $\ell=1,2$, find a displacement field $\mathbf{u}^\ell:\Omega^\ell\times(0,T)\to\mathbb{R}^d$, a stress field $\boldsymbol{\sigma}^\ell:\Omega^\ell\times(0,T)\to\mathbb{S}^d$, an electric potential $\varphi^\ell:\Omega^\ell\times(0,T)\to\mathbb{R}$, a damage $\zeta^\ell:\Omega^\ell\times(0,T)\to\mathbb{R}$, a bonding $\beta:\Gamma_3\times(0,T)\to\mathbb{R}$ and an electric displacement field $\boldsymbol{D}^\ell:\Omega^\ell\times(0,T)\to\mathbb{R}^d$ such that

$$\begin{gathered}\boldsymbol{\sigma}^\ell=\mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\ell)+\mathcal{G}^\ell\boldsymbol{\varepsilon}(\mathbf{u}^\ell)+(\mathcal{E}^\ell)^*\nabla\varphi^\ell+\int_0^t\mathcal{F}^\ell\big(t-s,\varepsilon(\mathbf{u}^\ell(s)),\zeta^\ell(s)\big)\,ds,\\ \text{in }\Omega^\ell\times(0,T),\end{gathered}\tag{2.1}$$

$$\boldsymbol{D}^\ell=\mathcal{E}^\ell\varepsilon(\mathbf{u}^\ell)-\mathcal{B}^\ell\nabla\varphi^\ell\quad\text{in }\Omega^\ell\times(0,T),\tag{2.2}$$

$$\dot{\zeta}^\ell-\kappa^\ell\Delta\zeta^\ell+\partial\psi_{K^\ell}(\zeta^\ell)\ni\phi^\ell\big(\boldsymbol{\sigma}^\ell-\mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\ell),\boldsymbol{\varepsilon}(\mathbf{u}^\ell),\zeta^\ell\big)\quad\text{in }\Omega^\ell\times(0,T),\tag{2.3}$$

$$\operatorname{Div}\boldsymbol{\sigma}^\ell+\boldsymbol{f}_0^\ell=0\quad\text{in }\Omega^\ell\times(0,T),\tag{2.4}$$

$$\operatorname{div}\boldsymbol{D}^\ell-q_0^\ell=0\quad\text{in }\Omega^\ell\times(0,T),\tag{2.5}$$

$$\mathbf{u}^\ell=0\quad\text{on }\Gamma_1^\ell\times(0,T),\tag{2.6}$$

$$\boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell=\boldsymbol{f}_2^\ell\quad\text{on }\Gamma_2^\ell\times(0,T),\tag{2.7}$$

$$\left.\begin{aligned}\sigma_\nu^1=\sigma_\nu^2\equiv\sigma_\nu,\\ \sigma_\nu=-p_\nu([u_\nu])+\gamma_\nu\beta^2R_\nu([u_\nu])\end{aligned}\right\}\quad\text{on }\Gamma_3\times(0,T),\tag{2.8}$$

$$\left.\begin{array}{c}\boldsymbol{\sigma}_\tau^1 = -\boldsymbol{\sigma}_\tau^2 \equiv \boldsymbol{\sigma}_\tau,\\ \|\boldsymbol{\sigma}_\tau + \gamma_\tau\beta^2\boldsymbol{R}_\tau([\mathbf{u}_\tau])\| \le \mu p_\nu([u_\nu]),\\ \|\boldsymbol{\sigma}_\tau + \gamma_\tau\beta^2\boldsymbol{R}_\tau([\mathbf{u}_\tau])\| < \mu p_\nu([u_\nu]) \Rightarrow [\mathbf{u}_\tau] = 0,\\ \|\boldsymbol{\sigma}_\tau + \gamma_\tau\beta^2\boldsymbol{R}_\tau([\mathbf{u}_\tau])\| = \mu p_\nu([u_\nu])\\ \Rightarrow \exists\lambda \ge 0 \text{ such that } \boldsymbol{\sigma}_\tau + \gamma_\tau\beta^2\boldsymbol{R}_\tau([\mathbf{u}_\tau]) = -\lambda[\mathbf{u}_\tau]\end{array}\right\} \text{ on } \Gamma_3 \times (0,T), \tag{2.9}$$

$$\dot\beta = -\Big(\beta\big(\gamma_\nu(R_\nu([u_\nu]))^2 + \gamma_\tau|\boldsymbol{R}_\tau([\mathbf{u}_\tau])|^2\big) - \varepsilon_a\Big)_+ \quad \text{on } \Gamma_3 \times (0,T), \tag{2.10}$$

$$\varphi^\ell = 0 \quad \text{on } \Gamma_a^\ell \times (0,T), \tag{2.11}$$

$$\boldsymbol{D}^\ell \cdot \boldsymbol{\nu}^\ell = q_2^\ell \quad \text{on } \Gamma_b^\ell \times (0,T), \tag{2.12}$$

$$\frac{\partial \zeta^\ell}{\partial \nu^\ell} = 0 \quad \text{on } \Gamma^\ell \times (0,T), \tag{2.13}$$

$$\mathbf{u}^\ell(0) = \mathbf{u}_0^\ell, \quad \zeta^\ell(0) = \zeta_0^\ell \quad \text{in } \Omega^\ell, \tag{2.14}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3. \tag{2.15}$$

First, equations (2.1) and (2.2) represent the electro-viscoelastic constitutive law with long term-memory and damage, the evolution of the damage is governed by the inclusion of parabolic type given by the relation (2.3). Equations (2.4) and (2.5) are the equilibrium equations for the stress and electric-displacement fields, respectively, in which "Div" and "div" denote the divergence operator for tensor and vector valued functions, respectively. Next, the equations (2.6) and (2.7) represent the displacement and traction boundary condition, respectively. Condition (2.8) represents the normal compliance conditions with adhesion where γ_ν is a given adhesion coefficient, p_ν is a given positive function which will be described below and $[u_\nu] = u_\nu^1 + u_\nu^2$ stands for the displacements in normal direction, in this condition the interpenetrability between two bodies, that is $[u_\nu]$ can be positive on Γ_3. The contribution of the adhesive to the normal traction is represented by the term $\gamma_\nu\beta^2 R_\nu([u_\nu])$, the adhesive traction is tensile and is proportional, with proportionality coefficient γ_ν, to the square of the intensity of adhesion and to the normal displacement, but as long as it does not exceed the bond length L. The maximal tensile traction is $\gamma_\nu\beta^2 L$. R_ν is the truncation operator defined by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L,\\ -s & \text{if } -L \le s \le 0,\\ 0 & \text{if } s > 0.\end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The introduction of the operator R_ν. together with the operator $\boldsymbol{R}_\tau$ defined below, is motivated by mathematical arguments but it is not restrictive for physical point of view, since no restriction on the size of the parameter L is made in what follows. Condition (2.9) are a non local Coulomb's friction law conditions coupled with adhesive, where $[\mathbf{u}_\tau] = \mathbf{u}_\tau^1 - \mathbf{u}_\tau^2$ stands for the jump of the displacements in tangential direction. $\boldsymbol{R}_\tau$ is the truncation operator given by

$$\boldsymbol{R}_\tau(\mathbf{v}) = \begin{cases} \mathbf{v} & \text{if } |\mathbf{v}| \le L,\\ L\frac{\mathbf{v}}{|\mathbf{v}|} & \text{if } |\mathbf{v}| > L.\end{cases}$$

This condition shows that the shear on the contact surface depends on the bonding field and on the tangential displacement, but as long as it does not exceed the bond length L.

Next, the equation (2.10) represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [3], see also [22, 23] for more details. Here, besides γ_ν, two new adhesion coefficients are involved, γ_τ and ε_a. Notice that in this model once debonding occurs bonding cannot be reestablished since, as it follows from (2.10), $\dot{\beta} \le 0$. (2.11) and (2.12) represent the electric boundary conditions. The relation (2.13) represents a homogeneous Neumann boundary condition where $\frac{\partial \zeta^\ell}{\partial \nu^\ell}$ is the normal derivative of ζ^ℓ. (2.14) represents the initial displacement field and the initial damage field. Finally, (2.15) represents the initial condition in which β_0 is the given initial bonding field.

3. Variational formulation and the main result

In this section, we list the assumptions on the data and derive a variational formulation for the contact problem. To this end, we need to introduce some notation and preliminary material. Here and below, $\mathbb{S}^d$ represent the space of second-order symmetric tensors on $\mathbb{R}^d$. We recall that the inner products and the corresponding norms on $\mathbb{S}^d$ and $\mathbb{R}^d$ are given by

$$\mathbf{u}^\ell.\mathbf{v}^\ell = u_i^\ell.v_i^\ell, \quad |\mathbf{v}^\ell| = (\mathbf{v}^\ell.\mathbf{v}^\ell)^{\frac{1}{2}}, \quad \forall \mathbf{u}^\ell, \mathbf{v}^\ell \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma}^\ell.\boldsymbol{\tau}^\ell = \sigma_{ij}^\ell.\tau_{ij}^\ell, \quad |\boldsymbol{\tau}^\ell| = (\boldsymbol{\tau}^\ell \cdot \boldsymbol{\tau}^\ell)^{\frac{1}{2}}, \quad \forall \boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell \in \mathbb{S}^d.$$

Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted. Now, to proceed with the variational formulation, we need the following function spaces:

$$H^\ell = \{\mathbf{v}^\ell = (v_i^\ell); v_i^\ell \in L^2(\Omega^\ell)\}, \quad H_1^\ell = \{\mathbf{v}^\ell = (v_i^\ell); v_i^\ell \in H^1(\Omega^\ell)\},$$
$$\mathcal{H}^\ell = \{\boldsymbol{\tau}^\ell = (\tau_{ij}^\ell); \tau_{ij}^\ell = \tau_{ji}^\ell \in L^2(\Omega^\ell)\}, \quad \mathcal{H}_1^\ell = \{\boldsymbol{\tau}^\ell = (\tau_{ij}^\ell) \in \mathcal{H}^\ell; \operatorname{div} \boldsymbol{\tau}^\ell \in H^\ell\}.$$

The spaces H^ℓ, H_1^ℓ, $\mathcal{H}^\ell$ and $\mathcal{H}_1^\ell$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\mathbf{u}^\ell, \mathbf{v}^\ell)_{H^\ell} = \int_{\Omega^\ell} \mathbf{u}^\ell.\mathbf{v}^\ell dx, \quad (\mathbf{u}^\ell, \mathbf{v}^\ell)_{H_1^\ell} = \int_{\Omega^\ell} \mathbf{u}^\ell.\mathbf{v}^\ell dx + \int_{\Omega^\ell} \nabla \mathbf{u}^\ell.\nabla \mathbf{v}^\ell dx,$$
$$(\boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell)_{\mathcal{H}^\ell} = \int_{\Omega^\ell} \boldsymbol{\sigma}^\ell.\boldsymbol{\tau}^\ell dx, \quad (\boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell)_{\mathcal{H}_1^\ell} = \int_{\Omega^\ell} \boldsymbol{\sigma}^\ell.\boldsymbol{\tau}^\ell dx + \int_{\Omega^\ell} \operatorname{div} \boldsymbol{\sigma}^\ell. \operatorname{Div} \boldsymbol{\tau}^\ell dx$$

and the associated norms $\|\cdot\|_{H^\ell}$, $\|\cdot\|_{H_1^\ell}$, $\|\cdot\|_{\mathcal{H}^\ell}$, and $\|\cdot\|_{\mathcal{H}_1^\ell}$ respectively. Here and below we use the notation

$$\nabla \mathbf{u}^\ell = (u_{i,j}^\ell), \quad \varepsilon(\mathbf{u}^\ell) = (\varepsilon_{ij}(\mathbf{u}^\ell)), \quad \varepsilon_{ij}(\mathbf{u}^\ell) = \frac{1}{2}(u_{i,j}^\ell + u_{j,i}^\ell), \quad \forall u^\ell \in H_1^\ell,$$
$$\operatorname{Div} \boldsymbol{\sigma}^\ell = (\sigma_{ij,j}^\ell), \quad \forall \boldsymbol{\sigma}^\ell \in \mathcal{H}_1^\ell.$$

For every element $\mathbf{v}^\ell \in H_1^\ell$, we also use the notation $\mathbf{v}^\ell$ for the trace of $\mathbf{v}^\ell$ on Γ^ℓ and we denote by v_ν^ℓ and $\mathbf{v}_\tau^\ell$ the *normal* and the *tangential* components of $\mathbf{v}^\ell$ on the boundary Γ^ℓ given by

$$v_\nu^\ell = \mathbf{v}^\ell.\nu^\ell, \quad \mathbf{v}_\tau^\ell = \mathbf{v}^\ell - v_\nu^\ell \boldsymbol{\nu}^\ell.$$

Let H'_{Γ^ℓ} be the dual of $H_{\Gamma^\ell} = H^{\frac{1}{2}}(\Gamma^\ell)^d$ and let $(\cdot,\cdot)_{-\frac{1}{2},\frac{1}{2},\Gamma^\ell}$ denote the duality pairing between H'_{Γ^ℓ} and H_{Γ^ℓ}. For every element $\boldsymbol{\sigma}^\ell \in \mathcal{H}_1^\ell$ let $\boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell$ be the element of H'_{Γ^ℓ} given by

$$(\boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell, \mathbf{v}^\ell)_{-\frac{1}{2},\frac{1}{2},\Gamma^\ell} = (\boldsymbol{\sigma}^\ell, \varepsilon(\mathbf{v}^\ell))_{\mathcal{H}^\ell} + (\operatorname{Div}\boldsymbol{\sigma}^\ell, \mathbf{v}^\ell)_{H^\ell} \quad \forall \mathbf{v}^\ell \in H_1^\ell.$$

Denote by σ_ν^ℓ and $\boldsymbol{\sigma}_\tau^\ell$ the *normal* and the *tangential* traces of $\boldsymbol{\sigma}^\ell \in \mathcal{H}_1^\ell$, respectively. If $\boldsymbol{\sigma}^\ell$ is continuously differentiable on $\Omega^\ell \cup \Gamma^\ell$, then

$$\sigma_\nu^\ell = (\boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell)\cdot\boldsymbol{\nu}^\ell, \quad \boldsymbol{\sigma}_\tau^\ell = \boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell - \sigma_\nu^\ell\boldsymbol{\nu}^\ell,$$
$$(\boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell, \mathbf{v}^\ell)_{-\frac{1}{2},\frac{1}{2},\Gamma^\ell} = \int_{\Gamma^\ell} \boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell \cdot \mathbf{v}^\ell da$$

fore all $\mathbf{v}^\ell \in H_1^\ell$, where da is the surface measure element.

To obtain the variational formulation of the problem (2.1)–(2.15), we introduce for the bonding field the set

$$\mathcal{Z} = \big\{\theta \in L^\infty\big(0,T;L^2(\Gamma_3)\big); 0 \le \theta(t) \le 1\,\forall t \in [0,T], \text{ a.e. on } \Gamma_3\big\},$$

and for the displacement field we need the closed subspace of H_1^ℓ defined by

$$V^\ell = \big\{\mathbf{v}^\ell \in H_1^\ell; \mathbf{v}^\ell = 0 \text{ on } \Gamma_1^\ell\big\}.$$

Since $\operatorname{meas}\Gamma_1^\ell > 0$, the following Korn's inequality holds:

$$\|\varepsilon(\mathbf{v}^\ell)\|_{\mathcal{H}^\ell} \ge c_K\|\mathbf{v}^\ell\|_{H_1^\ell} \quad \forall \mathbf{v}^\ell \in V^\ell, \tag{3.1}$$

where the constant c_K denotes a positive constant which may depends only on Ω^ℓ, Γ_1^ℓ (see [17]). Over the space V^ℓ we consider the inner product given by

$$(\mathbf{u}^\ell, \mathbf{v}^\ell)_{V^\ell} = (\varepsilon(\mathbf{u}^\ell), \varepsilon(\mathbf{v}^\ell))_{\mathcal{H}^\ell}, \quad \forall \mathbf{u}^\ell, \mathbf{v}^\ell \in V^\ell, \tag{3.2}$$

and let $\|\cdot\|_{V^\ell}$ be the associated norm. It follows from Korn's inequality (3.1) that the norms $\|\cdot\|_{H_1^\ell}$ and $\|\cdot\|_{V^\ell}$ are equivalent on V^ℓ. Then $(V^\ell, \|\cdot\|_{V^\ell})$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (3.2), there exists a constant $c_0 > 0$, depending only on Ω^ℓ, Γ_1^ℓ and Γ_3 such that

$$\|\mathbf{v}^\ell\|_{L^2(\Gamma_3)^d} \le c_0\|\mathbf{v}^\ell\|_{V^\ell} \quad \forall \mathbf{v}^\ell \in V^\ell. \tag{3.3}$$

We also introduce the spaces

$$E_0^\ell = L^2(\Omega^\ell), \quad E_1^\ell = H^1(\Omega^\ell), \quad W^\ell = \{\psi^\ell \in E_1^\ell; \psi^\ell = 0 \text{ on } \Gamma_a^\ell\},$$
$$\mathcal{W}^\ell = \{\boldsymbol{D}^\ell = (D_i^\ell); D_i^\ell \in L^2(\Omega^\ell), \operatorname{div}\boldsymbol{D}^\ell \in L^2(\Omega^\ell)\}.$$

Since $\operatorname{meas}\Gamma_a^\ell > 0$, the following Friedrichs-Poincaré inequality holds:

$$\|\nabla\psi^\ell\|_{W^\ell} \ge c_F\|\psi^\ell\|_{H^1(\Omega^\ell)} \quad \forall \psi^\ell \in W^\ell, \tag{3.4}$$

where $c_F > 0$ is a constant which depends only on Ω^ℓ, Γ_a^ℓ. In the space W^ℓ, we consider the inner product

$$(\varphi^\ell, \psi^\ell)_{W^\ell} = \int_{\Omega^\ell} \nabla\varphi^\ell \cdot \nabla\psi^\ell dx \tag{3.5}$$

and let $\|\cdot\|_{W^\ell}$ be the associated norm. It follows from (3.4) that $\|\cdot\|_{H^1(\Omega^\ell)}$ and $\|\cdot\|_{W^\ell}$ are equivalent norms on W^ℓ and therefore $(W^\ell, \|\cdot\|_{W^\ell})$ is areal Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant $\mathbf{c}_0$, depending only on Ω^ℓ, Γ_a^ℓ and Γ_3, such that

$$\|\zeta^\ell\|_{L^2(\Gamma_3)} \le \mathbf{c}_0\|\zeta^\ell\|_{W^\ell} \quad \forall \zeta^\ell \in W^\ell. \tag{3.6}$$

The space $\mathcal{W}^\ell$ is real Hilbert space with the inner product

$$(\boldsymbol{D}^\ell, \boldsymbol{\Phi}^\ell)_{\mathcal{W}^\ell} = \int_{\Omega^\ell} \boldsymbol{D}^\ell \cdot \boldsymbol{\Phi}^\ell dx + \int_{\Omega^\ell} \operatorname{div} \boldsymbol{D}^\ell \cdot \operatorname{div} \boldsymbol{\Phi}^\ell dx,$$

where $\operatorname{div} \boldsymbol{D}^\ell = (\boldsymbol{D}^\ell_{i,i})$, and the associated norm $\|\cdot\|_{\mathcal{W}^\ell}$.

To simplify notation, we define the product spaces

$$\begin{gathered} \boldsymbol{V} = V^1 \times V^2, \quad H = H^1 \times H^2, \quad H_1 = H_1^1 \times H_1^2, \\ \mathcal{H} = \mathcal{H}^1 \times \mathcal{H}^2, \quad \mathcal{H}_1 = \mathcal{H}_1^1 \times \mathcal{H}_1^2, \quad E_0 = E_0^1 \times E_0^2, \\ E_1 = E_1^1 \times E_1^2, \quad W = W^1 \times W^2, \quad \mathcal{W} = \mathcal{W}^1 \times \mathcal{W}^2. \end{gathered}$$

The spaces $\boldsymbol{V}$, E_1, W and $\mathcal{W}$ are real Hilbert spaces endowed with the canonical inner products denoted by $(\cdot,\cdot)_{\boldsymbol{V}}$, $(\cdot,\cdot)_{E_1}$, $(\cdot,\cdot)_W$ and $(\cdot,\cdot)_{\mathcal{W}}$. The associate norms will be denoted by $\|\cdot\|_{\boldsymbol{V}}$, $\|\cdot\|_{E_1}$, $\|\cdot\|_W$ and $\|\cdot\|_{\mathcal{W}}$, respectively.

Finally, for any real Hilbert space X, we use the classical notation for the spaces $L^p(0,T;X)$, $W^{k,p}(0,T;X)$, where $1 \le p \le \infty$, $k \ge 1$. We denote by $C(0,T;X)$ and $C^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norms

$$\begin{gathered} \|f\|_{C(0,T;X)} = \max_{t\in[0,T]} \|f(t)\|_X, \\ \|f\|_{C^1(0,T;X)} = \max_{t\in[0,T]} \|f(t)\|_X + \max_{t\in[0,T]} \|\dot{f}(t)\|_X, \end{gathered}$$

respectively. Moreover, we use the dot above to indicate the derivative with respect to the time variable and, for areal number r, we use r_+ to represent its positive part, that is $r_+ = \max\{0, r\}$. For the convenience of the reader, we recall the following version of the classical theorem of Cauchy-Lipschitz (see, [23, p.48]).

Theorem 3.1. *Assume that $(X, \|\cdot\|_X)$ is a real Banach space and $T > 0$. Let $F(t,\cdot) : X \to X$ be an operator defined a.e. on $(0,T)$ satisfying the following conditions:*

(1) *There exists a constant $L_F > 0$ such that*
$$\|F(t,x) - F(t,y)\|_X \le L_F \|x-y\|_X \quad \forall x,y \in X, \text{ a.e. } t \in (0,T).$$
(2) *There exists $p \ge 1$ such that $t \mapsto F(t,x) \in L^p(0,T;X)$ for all $x \in X$.*

Then for any $x_0 \in X$, there exists a unique function $x \in W^{1,p}(0,T;X)$ such that

$$\begin{gathered} \dot{x}(t) = F(t, x(t)), \quad \text{a.e. } t \in (0,T), \\ x(0) = x_0. \end{gathered}$$

This theorem will be used in section4 to prove the unique solvability of the intermediate problem involving the bonding field.

In the study of the Problem P, we consider the following assumptions:

The *viscosity function* $\mathcal{A}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

(a) There exists $L_{\mathcal{A}^\ell} > 0$ such that $|\mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{\xi}_1) - \mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{\xi}_2)| \le L_{\mathcal{A}^\ell}|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|$ for all $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega^\ell$.
(b) There exists $m_{\mathcal{A}^\ell} > 0$ such that $(\mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{\xi}_1) - \mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{\xi}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \ge m_{\mathcal{A}^\ell}|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^2$ for all $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega^\ell$. (3.7)
(c) The mapping $\boldsymbol{x} \mapsto \mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{\xi})$ is Lebesgue measurable on Ω^ℓ, for any $\boldsymbol{\xi} \in \mathbb{S}^d$.
(d) The mapping $\boldsymbol{x} \mapsto \mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{0})$ is continuous on $\mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega^\ell$.

The *elasticity operator* $\mathcal{G}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

(a) There exists $L_{\mathcal{G}^\ell} > 0$ such that $|\mathcal{G}^\ell(\boldsymbol{x}, \boldsymbol{\xi}_1) - \mathcal{G}^\ell(\boldsymbol{x}, \boldsymbol{\xi}_2)| \le L_{\mathcal{G}^\ell}|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|$ for all $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega^\ell$.
(b) The mapping $\boldsymbol{x} \mapsto \mathcal{G}^\ell(\boldsymbol{x}, \boldsymbol{\xi})$ is Lebesgue measurable on Ω^ℓ, for any $\boldsymbol{\xi} \in \mathbb{S}^d$. (3.8)
(c) The mapping $\boldsymbol{x} \mapsto \mathcal{G}^\ell(\boldsymbol{x}, \mathbf{0})$ belongs to $\mathcal{H}^\ell$.

The *relaxation function* $\mathcal{F}^\ell : \Omega^\ell \times (0, T) \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies:

(a) There exists $L_{\mathcal{F}^\ell} > 0$ such that $|\mathcal{F}^\ell(\boldsymbol{x}, t, \boldsymbol{\xi}_1, d_1) - \mathcal{F}^\ell(\boldsymbol{x}, t, \boldsymbol{\xi}_2, d_2)| \le L_{\mathcal{F}^\ell}\big(|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| + |d_1 - d_2|\big)$, for all $t \in (0, T)$, $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d$, $d_1, d_2 \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Omega^\ell$.
(b) The mapping $\boldsymbol{x} \mapsto \mathcal{F}^\ell(\boldsymbol{x}, t, \boldsymbol{\xi}, d)$ is Lebesgue measurable in Ω^ℓ, for any $t \in (0, T)$, $\boldsymbol{\xi} \in \mathbb{S}^d$, $d \in \mathbb{R}$. (3.9)
(c) The mapping $t \mapsto \mathcal{F}^\ell(\boldsymbol{x}, t, \boldsymbol{\xi}, d)$ is continuous in $(0, T)$, for any $\boldsymbol{\xi} \in \mathbb{S}^d$, $d \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Omega^\ell$.
(d) The mapping $\boldsymbol{x} \mapsto \mathcal{F}^\ell(\boldsymbol{x}, t, \mathbf{0}, 0)$ belongs to $\mathcal{H}^\ell$, for all $t \in (0, T)$.

The *damage source function* $\phi^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ satisfies:

(a) There exists $L_{\phi^\ell} > 0$ such that $|\phi^\ell(\boldsymbol{x}, \boldsymbol{\eta}_1, \boldsymbol{\xi}_1, \alpha_1) - \phi^\ell(\boldsymbol{x}, \boldsymbol{\eta}_2, \boldsymbol{\xi}_2, \alpha_2)| \le L_{\phi^\ell}\big(|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2| + |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| + |\alpha_1 - \alpha_2|\big)$, for all $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ a.e. $\boldsymbol{x} \in \Omega^\ell$,
(b) The mapping $\boldsymbol{x} \mapsto \phi^\ell(\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{\xi}, \alpha)$ is Lebesgue measurable on Ω^ℓ, for any $\boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{S}^d$ and $\alpha \in \mathbb{R}$, (3.10)
(c) The mapping $\boldsymbol{x} \mapsto \phi^\ell(\boldsymbol{x}, \mathbf{0}, \mathbf{0}, 0)$ belongs to $L^2(\Omega^\ell)$,
(d) $\phi^\ell(\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{\xi}, \alpha)$ is bounded for all $\boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{S}^d$, $\alpha \in \mathbb{R}$ a.e. $\boldsymbol{x} \in \Omega^\ell$.

The *piezoelectric tensor* $\mathcal{E}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies:

(a) $\mathcal{E}^\ell(\boldsymbol{x}, \tau) = (e^\ell_{ijk}(\boldsymbol{x})\tau_{jk})$ for all $\tau = (\tau_{ij}) \in \mathbb{S}^d$ a.e. $\boldsymbol{x} \in \Omega^\ell$. (3.11)
(b) $e^\ell_{ijk} = e^\ell_{ikj} \in L^\infty(\Omega^\ell)$, $1 \le i, j, k \le d$.

Recall also that the transposed operator $(\mathcal{E}^\ell)^*$ is given by $(\mathcal{E}^\ell)^* = (e^{\ell,*}_{ijk})$ where $e^{\ell,*}_{ijk} = e^\ell_{kij}$ and the following equality hold

$$\mathcal{E}^\ell\sigma.\mathbf{v} = \sigma \cdot (\mathcal{E}^\ell)^*\mathbf{v} \quad \forall \sigma \in \mathbb{S}^d,\ \forall \mathbf{v} \in \mathbb{R}^d.$$

The *electric permittivity operator* $\mathcal{B}^\ell = (b^\ell_{ij}) : \Omega^\ell \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies:

(a) $\mathcal{B}^\ell(\boldsymbol{x}, \mathbf{E}) = (b^\ell_{ij}(\boldsymbol{x})E_j)$ for all $\mathbf{E} = (E_i) \in \mathbb{R}^d$, a.e. $\boldsymbol{x} \in \Omega^\ell$.
(b) $b^\ell_{ij} = b^\ell_{ji}$, $b^\ell_{ij} \in L^\infty(\Omega^\ell)$, $1 \le i, j \le d$. (3.12)
(c) There exists $m_{\mathcal{B}^\ell} > 0$, such that $\mathcal{B}^\ell\mathbf{E} \cdot \mathbf{E} \ge m_{\mathcal{B}^\ell}|\mathbf{E}|^2$ for all $\mathbf{E} = (E_i) \in \mathbb{R}^d$, a.e. $\boldsymbol{x} \in \Omega^\ell$.

The *normal compliance function* $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies:

(a) There exists $L_\nu > 0$ such that $|p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2)| \le L_\nu|r_1 - r_2|$ for all $r_1, r_2 \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Gamma_3$.
(b) $(p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2))(r_1 - r_2) \ge 0$ for all $r_1, r_2 \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Gamma_3$. (3.13)
(c) The mapping $\boldsymbol{x} \mapsto p_\nu(\boldsymbol{x}, r)$ is measurable on Γ_3 for all $r \in \mathbb{R}$.
(d) $p_\nu(\boldsymbol{x}, r) = 0$ for all $r \le 0$, a.e. $\boldsymbol{x} \in \Gamma_3$.

The following regularity is assumed on the density of volume forces, traction, volume electric charges and surface electric charges:

$$\begin{gathered}\mathbf{f}_0^\ell \in C(0,T;L^2(\Omega^\ell)^d), \quad \mathbf{f}_2^\ell \in C(0,T;L^2(\Gamma_2^\ell)^d),\\ q_0^\ell \in C(0,T;L^2(\Omega^\ell)), \quad q_2^\ell \in C(0,T;L^2(\Gamma_b^\ell)).\end{gathered} \tag{3.14}$$

The adhesion coefficients γ_ν, γ_τ and ε_a satisfy the conditions

$$\gamma_\nu, \gamma_\tau \in L^\infty(\Gamma_3), \quad \varepsilon_a \in L^2(\Gamma_3), \quad \gamma_\nu, \gamma_\tau, \varepsilon_a \geq 0, \quad \text{a.e. on } \Gamma_3. \tag{3.15}$$

The microcrack diffusion coefficient satisfies

$$\kappa^\ell > 0. \tag{3.16}$$

Finally, the friction coefficient and the initial data satisfy

$$\mu \in L^\infty(\Gamma_3), \quad \mu(x) \geq 0 \quad \text{a.e. on } \Gamma_3 \tag{3.17}$$

$$\mathbf{u}_0^\ell \in \boldsymbol{V}^\ell, \quad \zeta_0^\ell \in K^\ell, \quad \beta_0 \in L^2(\Gamma_3), \quad 0 \leq \beta_0 \leq 1, \text{ a.e. on } \Gamma_3. \tag{3.18}$$

where K^ℓ is the set of admissible damage functions defined in (1.5).

Using the Riesz representation theorem, we define the linear mappings $\mathbf{f} = (\mathbf{f}^1, \mathbf{f}^2) : [0,T] \to \boldsymbol{V}$ and $q = (q^1, q^2) : [0,T] \to W$ as follows:

$$(\mathbf{f}(t), \mathbf{v})_{\boldsymbol{V}} = \sum_{\ell=1}^{2} \int_{\Omega^\ell} \mathbf{f}_0^\ell(t) \cdot \mathbf{v}^\ell \, dx + \sum_{\ell=1}^{2} \int_{\Gamma_2^\ell} \mathbf{f}_2^\ell(t) \cdot \mathbf{v}^\ell \, da \quad \forall \mathbf{v} \in \boldsymbol{V}, \tag{3.19}$$

$$(q(t), \zeta)_W = \sum_{\ell=1}^{2} \int_{\Omega^\ell} q_0^\ell(t) \zeta^\ell \, dx - \sum_{\ell=1}^{2} \int_{\Gamma_b^\ell} q_2^\ell(t) \zeta^\ell \, da \quad \forall \zeta \in W. \tag{3.20}$$

Next, we define the mappings $a : E_1 \times E_1 \to \mathbb{R}$, $j_{ad} : L^2(\Gamma_3) \times \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$, $j_{\nu c} : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ and $j_{fr} : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$, respectively, by

$$a(\zeta, \xi) = \sum_{\ell=1}^{2} \kappa^\ell \int_{\Omega^\ell} \nabla \zeta^\ell \cdot \nabla \xi^\ell dx, \tag{3.21}$$

$$j_{ad}(\beta, \mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} \Big(-\gamma_\nu \beta^2 R_\nu([u_\nu])[v_\nu] + \gamma_\tau \beta^2 \boldsymbol{R}_\tau([\mathbf{u}_\tau]) \cdot [\mathbf{v}_\tau] \Big) \, da, \tag{3.22}$$

$$j_{\nu c}(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} p_\nu([u_\nu])[v_\nu] \, da, \tag{3.23}$$

$$j_{fr}(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} \mu p_\nu([u_\nu]) \big\| [v_\tau] \big\| \, da \tag{3.24}$$

for all $\mathbf{u}, \mathbf{v} \in \boldsymbol{V}$ and $t \in [0,T]$. We note that conditions (3.14) imply

$$\mathbf{f} \in C(0,T;\boldsymbol{V}), \quad q \in C(0,T;W). \tag{3.25}$$

By a standard procedure based on Green's formula, we derive the following variational formulation of the mechanical (2.1)–(2.15).

Problem PV. Find a displacement field $\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^2) : [0,T] \to \boldsymbol{V}$, a stress field $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) : [0,T] \to \mathcal{H}$, an electric potential $\varphi = (\varphi^1, \varphi^2) : [0,T] \to W$, a damage $\zeta = (\zeta^1, \zeta^2) : [0, T] \to E_1$, a bonding $\beta : [0, T] \to L^\infty(\Gamma_3)$ and an electric displacement field $\boldsymbol{D} = (\boldsymbol{D}^1, \boldsymbol{D}^2) : [0, T] \to \mathcal{W}$ such that

$$\begin{gathered}\boldsymbol{\sigma}^\ell = \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\ell) + \mathcal{G}^\ell \boldsymbol{\varepsilon}(\mathbf{u}^\ell) + (\mathcal{E}^\ell)^* \nabla \varphi^\ell + \int_0^t \mathcal{F}^\ell\big(t - s, \varepsilon(\mathbf{u}^\ell(s)), \zeta^\ell(s)\big)\, ds, \\ \text{in } \Omega^\ell \times (0, T),\end{gathered} \tag{3.26}$$

$$\boldsymbol{D}^\ell = \mathcal{E}^\ell \varepsilon(\mathbf{u}^\ell) - \mathcal{B}^\ell \nabla \varphi^\ell \quad \text{in } \Omega^\ell \times (0, T), \tag{3.27}$$

$$\begin{aligned}&\sum_{\ell=1}^{2} (\boldsymbol{\sigma}^\ell, \varepsilon(\mathbf{v}^\ell) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\ell(t)))_{\mathcal{H}^\ell} + j_{ad}(\beta(t), \mathbf{u}(t), \mathbf{v} - \dot{\mathbf{u}}(t)) \\ &+ j_{\nu c}(\mathbf{u}(t), \mathbf{v} - \dot{\mathbf{u}}(t)) + j_{fr}(\mathbf{u}(t), \mathbf{v}) - j_{fr}(\mathbf{u}(t), \dot{\mathbf{u}}(t)) \\ &\geq (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_{\boldsymbol{V}} \quad \forall \mathbf{v} \in \boldsymbol{V}, \quad \text{a.e. } t \in (0, T),\end{aligned} \tag{3.28}$$

$$\begin{aligned}&\zeta(t) \in K, \\ &\sum_{\ell=1}^{2} (\dot{\zeta}(t), \xi - \zeta(t))_{L^2(\Omega^\ell)} + a(\zeta(t), \xi - \zeta(t)) \\ &\geq \sum_{\ell=1}^{2} \Big(\phi^\ell\big(\boldsymbol{\sigma}^\ell(t) - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\ell(t)), \boldsymbol{\varepsilon}(\mathbf{u}^\ell(t)), \zeta^\ell(t)\big), \xi^\ell - \zeta^\ell(t)\Big)_{L^2(\Omega^\ell)}, \\ &\forall \xi \in K, \text{a.e. } t \in (0, T),\end{aligned} \tag{3.29}$$

$$\begin{gathered}\sum_{\ell=1}^{2} (\mathcal{B}^\ell \nabla \varphi^\ell(t), \nabla \phi^\ell)_{H^\ell} - \sum_{\ell=1}^{2} (\mathcal{E}^\ell \varepsilon(\mathbf{u}^\ell(t)), \nabla \phi^\ell)_{H^\ell} = (q(t), \phi)_W, \\ \forall \phi \in W, \text{ a.e. } t \in (0, T),\end{gathered} \tag{3.30}$$

$$\dot{\beta}(t) = -\Big(\beta(t)\big(\gamma_\nu (R_\nu([u_\nu(t)]))^2 + \gamma_\tau |\boldsymbol{R}_\tau([\mathbf{u}_\tau(t)])|^2\big) - \varepsilon_a\Big)_+ \quad \text{a.e. } (0, T), \tag{3.31}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \zeta(0) = \zeta_0, \quad \beta(0) = \beta_0. \tag{3.32}$$

We notice that the variational Problem PV is formulated in terms of a displacement field, a stress field, an electrical potential, a damage, a bonding and an electric displacement field. The existence of the unique solution of Problem PV is stated and proved in the next section.

Remark 3.2. We note that, in Problem P and in Problem PV, we do not need to impose explicitly the restriction $0 \leq \beta \leq 1$. Indeed, equation (3.31) guarantees that $\beta(x, t) \leq \beta_0(x)$ and, therefore, assumption (3.18) shows that $\beta(x, t) \leq 1$ for $t \geq 0$, a.e. $x \in \Gamma_3$. On the other hand, if $\beta(x, t_0) = 0$ at time t_0, then it follows from (3.31) that $\dot{\beta}(x, t) = 0$ for all $t \geq t_0$ and therefore, $\beta(x, t) = 0$ for all $t \geq t_0$, a.e. $x \in \Gamma_3$. We conclude that $0 \leq \beta(x, t) \leq 1$ for all $t \in [0, T]$, a.e. $x \in \Gamma_3$.

Below in this section β, β_1, β_2 denote elements of $L^2(\Gamma_3)$ such that $0 \leq \beta, \beta_1, \beta_2 \leq 1$ a.e. $x \in \Gamma_3$, $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{v}$ represent elements of $\boldsymbol{V}$ and $C > 0$ represents generic constants which may depend on Ω^ℓ, Γ_3, p_ν, γ_ν, γ_τ and L. First, we note that the functional j_{ad} and $j_{\nu c}$ are linear with respect to the last argument and, therefore,

$$\begin{gathered}j_{ad}(\beta, \mathbf{u}, -\mathbf{v}) = -j_{ad}(\beta, \mathbf{u}, \mathbf{v}), \\ j_{\nu c}(\mathbf{u}, -\mathbf{v}) = -j_{\nu c}(\mathbf{u}, \mathbf{v}).\end{gathered} \tag{3.33}$$

Next, using (3.23) and (3.13)(b) implies

$$j_{\nu c}(\mathbf{u}_1, \mathbf{v}_2) - j_{\nu c}(\mathbf{u}_1, \mathbf{v}_1) + j_{\nu c}(\mathbf{u}_2, \mathbf{v}_1) - j_{\nu c}(\mathbf{u}_2, \mathbf{v}_2) \le 0, \quad \forall \mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in \boldsymbol{V}, \tag{3.34}$$

and use (3.24), (3.13)(a), keeping in mind (3.3), we obtain

$$\begin{aligned} &j_{fr}(\mathbf{u}_1, \mathbf{v}_2) - j_{fr}(\mathbf{u}_1, \mathbf{v}_1) + j_{fr}(\mathbf{u}_2, \mathbf{v}_1) - j_{fr}(\mathbf{u}_2, \mathbf{v}_2) \\ &\le c_0^2 L_\nu \|\mu\|_{L^\infty(\Gamma_3)} \|\mathbf{u}_1 - \mathbf{u}_2\|_{\boldsymbol{V}} \|\mathbf{v}_1 - \mathbf{v}_2\|_{\boldsymbol{V}} \quad \forall \mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in \boldsymbol{V}. \end{aligned} \tag{3.35}$$

Inequalities (3.33)–(3.35) will be used in various places in the rest of this article. Our main existence and uniqueness result that we state now and prove in the next section is the following.

Theorem 3.3. *Assume that* (3.7)–(3.18) *hold. Then there exists a unique solution of Problem PV. Moreover, the solution satisfies*

$$\mathbf{u} \in C^1(0, T; \boldsymbol{V}), \tag{3.36}$$

$$\boldsymbol{\sigma} \in C(0, T; \mathcal{H}_1), \tag{3.37}$$

$$\varphi \in C(0, T; W), \tag{3.38}$$

$$\zeta \in H^1(0, T; E_0) \cap L^2(0, T; E_1), \tag{3.39}$$

$$\beta \in W^{1,\infty}(0, T; L^2(\Gamma_3)) \cap \mathcal{Z}, \tag{3.40}$$

$$\boldsymbol{D} \in C(0, T; \mathcal{W}). \tag{3.41}$$

The functions $\mathbf{u}$, $\boldsymbol{\sigma}$, φ, ζ, β and $\boldsymbol{D}$ which satisfy (3.26)-(3.32) are called a weak solution of the contact Problem P. We conclude that, under the assumptions (3.7)–(3.18), the mechanical problem (2.1)–(2.15) has a unique weak solution satisfying (3.36)–(3.41).

4. Proof of Theorem 3.3

The proof of Theorem 3.3 is carried out in several steps and is based on the following abstract result for evolutionary variational inequalities.

Let X be a real Hilbert space with the inner product $(\cdot,\cdot)_X$ and the associated norm $\|\cdot\|_X$, and consider the problem of finding $\mathbf{u} : [0, T] \to X$ such that

$$\begin{gathered} (A\dot{\mathbf{u}}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_X + (B\mathbf{u}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_X + j(\mathbf{u}(t), \mathbf{v}) - j(\mathbf{u}(t), \dot{\mathbf{u}}(t)) \\ \ge (f(t), \mathbf{v} - \dot{\mathbf{u}}(t))_X \quad \forall \mathbf{v} \in X,\ t \in [0, T], \\ \mathbf{u}(0) = \mathbf{u}_0. \end{gathered} \tag{4.1}$$

To study problem (4.1) we need the following assumptions: The operator $A : X \to X$ is Lipschitz continuous and strongly monotone, i.e.,

(a) There exists a positive constant L_A such that

$$\|A\mathbf{u}_1 - A\mathbf{u}_2\|_X \le L_A \|\mathbf{u}_1 - \mathbf{u}_2\|_X \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in X,$$

(b) There exists a positive constant m_A such that

$$(A\mathbf{u}_1 - A\mathbf{u}_2,\ \mathbf{u}_1 - \mathbf{u}_2)_X \ge m_A \|\mathbf{u}_1 - \mathbf{u}_2\|_X \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in X. \tag{4.2}$$

The nonlinear operator $B : X \to X$ is Lipschitz continuous, i.e., there exists a positive constant L_B such that

$$\|B\mathbf{u}_1 - B\mathbf{u}_2\|_X \le L_B \|\mathbf{u}_1 - \mathbf{u}_2\|_X \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in X. \tag{4.3}$$

The functional $j : X \times X \to \mathbb{R}$ satisfies:

(a) $j(\mathbf{u}, \cdot)$ is convex and l.S.C. on X for all $u \in X$.
(b) There exists $m_j > 0$ such that

$$\begin{aligned}&j(\mathbf{u}_1, \mathbf{v}_2) - j(\mathbf{u}_1, \mathbf{v}_1) + j(\mathbf{u}_2, \mathbf{v}_1) - j(\mathbf{u}_2, \mathbf{v}_2)\\ &\le m_j \|\mathbf{u}_1 - \mathbf{u}_2\|_X \|\mathbf{v}_1 - \mathbf{v}_2\|_X \quad \forall \mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in X.\end{aligned}$$

Finally, we assume that

$$f \in C(0, T; X), \tag{4.5}$$

$$\mathbf{u}_0 \in X. \tag{4.6}$$

The following existence, uniqueness result and regularity was proved in [8] and may be found in [9, p.230–234].

Theorem 4.1. *Let (4.7)–(4.6) hold. Then:*

(1) *There exists a unique solution $\mathbf{u} \in C^1(0, T; X)$ of Problem (4.1).*
(2) *If, moreover, $\mathbf{u}_1$ and $\mathbf{u}_2$ are two solutions of (4.1) corresponding to the data $f_1, f_2 \in C(0, T; X)$, then there exists $c > 0$ such that*

$$\|\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\|_X \le c\big(\|f_1(t) - f_2(t)\|_X + \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_X\big), \tag{4.7}$$

for all $t \in [0, T]$.

We turn now to the proof of Theorem 3.3 which will be carried out in several steps and is based on arguments of nonlinear equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities and fixed-point arguments. To this end, we assume in what follows that (3.7)–(3.18) hold, and we consider that C is a generic positive constant which depends on Ω^ℓ, Γ_1^ℓ, Γ_1^ℓ, Γ_3, p_ν, p_τ, $\mathcal{A}^\ell$, $\mathcal{B}^\ell$, $\mathcal{G}^\ell$, $\mathcal{F}^\ell$, $\mathcal{E}^\ell$, γ_ν, γ_τ, ϕ^ℓ, κ^ℓ, and T. but does not depend on t nor of the rest of input data, and whose value may change from place to place. Let a $\eta = (\eta^1, \eta^2) \in C(0, T; \boldsymbol{V})$ be given. In the first step we consider the following variational problem.

Problem $\mathbf{PV}_\eta^u$. Find a displacement field $\mathbf{u}_\eta = (\mathbf{u}_\eta^1, \mathbf{u}_\eta^2) : [0, T] \to \boldsymbol{V}$ such that

$$\begin{aligned}&\sum_{\ell=1}^{2} (\mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta^\ell), \varepsilon(\mathbf{v}^\ell) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta^\ell(t)))_{\mathcal{H}^\ell} + \sum_{\ell=1}^{2} (\mathcal{G}^\ell \boldsymbol{\varepsilon}(\mathbf{u}_\eta^\ell), \varepsilon(\mathbf{v}^\ell) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta^\ell(t)))_{\mathcal{H}^\ell}\\ &+ j_{\nu c}(\mathbf{u}_\eta(t), \mathbf{v} - \dot{\mathbf{u}}_\eta(t)) + j_{fr}(\mathbf{u}_\eta(t), \mathbf{v}) - j_{fr}(\mathbf{u}_\eta(t), \dot{\mathbf{u}}_\eta(t)) + (\eta(t), \mathbf{v} - \dot{\mathbf{u}}_\eta(t))_{\boldsymbol{V}}\\ &\ge (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}_\eta(t))_{\boldsymbol{V}} \quad \forall \mathbf{v} \in \boldsymbol{V},\ t \in (0, T),\end{aligned} \tag{4.8}$$

$$\mathbf{u}_\eta(0) = \mathbf{u}_0. \tag{4.9}$$

We have the following result for the problem PV_η^u.

Lemma 4.2. (1) *There exists a unique solution $\mathbf{u}_\eta \in C^1(0, T; \boldsymbol{V})$ to the problem (4.8) and (4.9).*
(2) *If $\mathbf{u}_1$ and $\mathbf{u}_2$ are two solutions of (4.8) and (4.9) corresponding to the data η_1, $\eta_2 \in C(0, T; \boldsymbol{V})$, then there exists $c > 0$ such that*

$$\|\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\|_{\boldsymbol{V}} \le c\left(\|\eta_1(t) - \eta_2(t)\|_{\boldsymbol{V}} + \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\boldsymbol{V}}\right) \quad \forall t \in [0, T]. \tag{4.10}$$

Proof. We apply Theorem 4.1 where $X = \boldsymbol{V}$, with the inner product $(\cdot,\cdot)_{\boldsymbol{V}}$ and the associated norm $\|\cdot\|_{\boldsymbol{V}}$. We use the Riesz representation theorem to define the operators $A : \boldsymbol{V} \to \boldsymbol{V}$, and $B : \boldsymbol{V} \to \boldsymbol{V}$ by

$$(A\mathbf{u}, \mathbf{v})_{\boldsymbol{V}} = \sum_{\ell=1}^{2} (\mathcal{A}^\ell \boldsymbol{\varepsilon}(\mathbf{u}^\ell), \boldsymbol{\varepsilon}(\mathbf{v}^\ell))_{\mathcal{H}^\ell}, \tag{4.11}$$

$$(B\mathbf{u}, \mathbf{v})_{\boldsymbol{V}} = \sum_{\ell=1}^{2} (\mathcal{G}^\ell \boldsymbol{\varepsilon}(\mathbf{u}^\ell), \boldsymbol{\varepsilon}(\mathbf{v}^\ell))_{\mathcal{H}^\ell}, \tag{4.12}$$

for all $\mathbf{u}, \mathbf{v} \in \boldsymbol{V}$, and define the functions $\mathbf{f}_\eta : [0,T] \to \boldsymbol{V}$, $j : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ by

$$\mathbf{f}_\eta(t) = \mathbf{f}(t) - \boldsymbol{\eta}(t) \quad \forall t \in [0,T], \tag{4.13}$$

$$j(\mathbf{u}, \mathbf{v}) = j_{\nu c}(\mathbf{u}, \mathbf{v}) + j_{fr}(\mathbf{u}, \mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in \boldsymbol{V}. \tag{4.14}$$

Assumptions (3.7) and (3.8) imply that the operators A and B satisfy conditions (4.2) and (4.3), respectively.

It follows from (3.13), (3.17), (3.23) and (3.24) that the functional j, (4.14), satisfies condition (4.4)(a). We use again (3.34), (3.35) and (4.14) to find

$$\begin{aligned} &j(\mathbf{u}_1, \mathbf{v}_2) - j(\mathbf{u}_1, \mathbf{v}_1) + j(\mathbf{u}_2, \mathbf{v}_1) - j(\mathbf{u}_2, \mathbf{v}_2) \\ &\le c_0^2 L_\nu \|\mu\|_{L^\infty(\Gamma_3)} \|\mathbf{u}_1 - \mathbf{u}_2\|_{\boldsymbol{V}} \|\mathbf{v}_1 - \mathbf{v}_2\|_{\boldsymbol{V}} \quad \forall \mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in \boldsymbol{V}, \end{aligned} \tag{4.15}$$

which shows that the functional j satisfies condition (4.4)(b) on $X = \boldsymbol{V}$. Moreover, using (3.25) and, keeping in mind that $\boldsymbol{\eta} \in C(0,T;\boldsymbol{V})$, we deduce from (4.13) that $\mathbf{f}_\eta \in C(0,T;\boldsymbol{V})$, i.e., $\mathbf{f}_\eta$ satisfies (4.5). Finally, we note that (3.18) shows that condition (4.6) is satisfied. Using now (4.11)–(4.14) we find that Lemma 4.2 is a direct consequence of Theorem 4.1. □

In the second step, we use the displacement field $\mathbf{u}_\eta$ obtained in Lemma 4.2 and we consider the following variational problem.

Problem $\mathbf{PV}_\eta^\varphi$. Find the electric potential $\varphi_\eta : [0,T] \to W$ such that

$$\sum_{\ell=1}^{2} (\mathcal{B}^\ell \nabla \varphi_\eta^\ell(t), \nabla \phi^\ell)_{H^\ell} - \sum_{\ell=1}^{2} (\mathcal{E}^\ell \varepsilon(\mathbf{u}_\eta^\ell(t)), \nabla \phi^\ell)_{H^\ell} = (q(t), \phi)_W \tag{4.16}$$

for all $\phi \in W$, a.e. $t \in (0,T)$. We have the following result.

Lemma 4.3. *Problem* PV_η^φ *has a unique solution* φ_η *which satisfies the regularity* (3.38).

Proof. We define a bilinear form: $b(\cdot,\cdot) : W \times W \to \mathbb{R}$ such that

$$b(\varphi, \phi) = \sum_{\ell=1}^{2} (\mathcal{B}^\ell \nabla \varphi^\ell, \nabla \phi^\ell)_{H^\ell} \quad \forall \varphi, \phi \in W. \tag{4.17}$$

We use (3.4), (3.5), (3.12) and (4.17) to show that the bilinear form $b(\cdot,\cdot)$ is continuous, symmetric and coercive on W, moreover using (3.20) and the Riesz representation Theorem we may define an element $q_\eta : [0,T] \to W$ such that

$$(q_\eta(t), \phi)_W = (q(t), \phi)_W + \sum_{\ell=1}^{2} (\mathcal{E}^\ell \varepsilon(\mathbf{u}_\eta^\ell(t)), \nabla \phi^\ell)_{H^\ell} \quad \forall \phi \in W, t \in (0,T).$$

We apply the Lax-Milgram Theorem to deduce that there exists a unique element $\varphi_\eta(t) \in W$ such that

$$b(\varphi_\eta(t), \phi) = (q_\eta(t), \phi)_W \quad \forall \phi \in W. \tag{4.18}$$

We conclude that φ_η is a solution of Problem PV_η^φ. Let $t_1, t_2 \in [0,T]$, it follows from (4.16) that

$$\|\varphi_\eta(t_1) - \varphi_\eta(t_2)\|_W \le C\big(\|\mathbf{u}_\eta(t_1) - \mathbf{u}_\eta(t_2)\|_{\boldsymbol{V}} + \|q(t_1) - q(t_2)\|_W\big). \tag{4.19}$$

We also note that assumptions (3.25) and $\mathbf{u}_\eta \in C^1(0,T;\boldsymbol{V})$, inequality (4.19) implies that $\varphi_\eta \in C(0,T;W)$. $\square$

In the third step, we use the displacement field $\mathbf{u}_\eta$ obtained in Lemma4.2 and we consider the following initial-value problem.

Problem $\mathbf{PV}_\eta^\beta$. Find the adhesion $\beta_\eta : [0,T] \to L^2(\Gamma_3)$ such that

$$\begin{gathered}\dot\beta_\eta(t) = -\Big(\beta_\eta(t)\big(\gamma_\nu(R_\nu([u_{\eta\nu}(t)]))^2 + \gamma_\tau|\boldsymbol{R}_\tau([\mathbf{u}_{\eta\tau}(t)])|^2\big) - \varepsilon_a\Big)_+, \\ \text{a.e. } t \in (0,T),\end{gathered} \tag{4.20}$$

$$\beta_\eta(0) = \beta_0. \tag{4.21}$$

We have the following result.

Lemma 4.4. *There exists a unique solution $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathcal{Z}$ to Problem PV_η^β.*

Proof. For simplicity we suppress the dependence of various functions on Γ_3, and note that the equalities and inequalities below are valid a.e. on Γ_3. Consider the mapping $F_\eta : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$F_\eta(t,\beta) = -\Big(\beta\big[\gamma_\nu(R_\nu([u_{\eta\nu}(t)]))^2 + \gamma_\tau|\boldsymbol{R}_\tau([\mathbf{u}_{\eta\tau}(t)])|^2\big] - \varepsilon_a\Big)_+,$$

for all $t \in [0,T]$ and $\beta \in L^2(\Gamma_3)$. It follows from the properties of the truncation operator R_ν and $\boldsymbol{R}_\tau$ that F_η is Lipschitz continuous with respect to the second variable, uniformly in time. Moreover, for all $\beta \in L^2(\Gamma_3)$, the mapping $t \to F_\eta(t,\beta)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Thus using the Cauchy-Lipschitz theorem given in Theorem 3.1 we deduce that there exists a unique function $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ solution to Problem PV_η^β. Also, the arguments used in Remark 3.2 show that $0 \le \beta_\eta(t) \le 1$ for all $t \in [0,T]$, a.e. on Γ_3. Therefore, from the definition of the set $\mathcal{Z}$, we find that $\beta_\eta \in \mathcal{Z}$, which concludes the proof of the lemma. $\square$

In the forth step we let $\theta \in C(0,T;E_0)$ be given and consider the following variational problem for the damage.

Problem $\mathbf{PV}_\theta^\zeta$. Find a damage $\zeta_\theta = (\zeta_\theta^1, \zeta_\theta^2) : [0,T] \to E$ such that $\zeta_\theta(t) \in K$ and

$$\begin{aligned}&\sum_{\ell=1}^{2}(\dot\zeta_\theta^\ell(t), \xi^\ell - \zeta_\theta^\ell(t))_{L^2(\Omega^\ell)} + a(\zeta_\theta(t), \xi - \zeta_\theta(t)) \\ &\ge \sum_{\ell=1}^{2}\big(\theta^\ell(t), \xi^\ell - \zeta_\theta^\ell(t)\big)_{L^2(\Omega^\ell)}, \quad \forall \xi \in K, \quad \text{a.e. } t \in (0,T),\end{aligned} \tag{4.22}$$

where $K = K^1 \times K^2$. The following abstract result for parabolic variational inequalities (see, e.g., [23, p.47]).

Theorem 4.5. *Let $X \subset Y = Y' \subset X'$ be a Gelfand triple. Let F be a nonempty, closed, and convex set of X. Assume that $a(\cdot,\cdot) : X \times X \to \mathbb{R}$ is a continuous and symmetric bilinear form such that for some constants $\alpha > 0$ and c_0,*

$$a(v,v) + c_0\|v\|_Y^2 \geq \alpha\|v\|_X^2 \quad \forall v \in X.$$

Then, for every $u_0 \in F$ and $f \in L^2(0,T;Y)$, there exists a unique function $u \in H^1(0,T;Y) \cap L^2(0,T;X)$ such that $u(0) = u_0$, $u(t) \in F$ for all $t \in [0,T]$, and

$$(\dot{u}(t), v - u(t))_{X' \times X} + a(u(t), v - u(t)) \geq (f(t), v - u(t))_Y \quad \forall v \in F \text{ a.e. } t \in (0,T).$$

We prove next the unique solvability of Problem PV_θ^ζ.

Lemma 4.6. *There exists a unique solution ζ_θ of Problem PV_θ^ζ and it satisfies*

$$\zeta_\theta \in H^1(0,T;E_0) \cap L^2(0,T;E_1).$$

Proof. The inclusion mapping of $(E_1, \|\cdot\|_{E_1})$ into $(E_0, \|\cdot\|_{E_0})$ is continuous and its range is dense. We denote by E_1' the dual space of E_1 and, identifying the dual of E_0 with itself, we can write the Gelfand triple

$$E_1 \subset E_0 = E_0' \subset E_1'.$$

We use the notation $(\cdot,\cdot)_{E_1' \times E_1}$ to represent the duality pairing between E' and E_1. We have

$$(\zeta,\xi)_{E_1' \times E_1} = (\zeta,\xi)_{E_0} \quad \forall \zeta \in E_0, \xi \in E_1,$$

and we note that K is a closed convex set in E_1. Then, using (3.16), (3.21) and the fact that $\zeta_0 \in K$ in (3.18), it is easy to see that Lemma 4.6 is a straight consequence of Theorem 4.5. □

Finally as a consequence of these results and using the properties of the operator $\mathcal{E}^\ell$, the operator $\mathcal{F}^\ell$, the functional j_{ad} and the functional ϕ^ℓ, for $t \in [0,T]$, we consider the element

$$\Lambda(\eta,\theta)(t) = \big(\Lambda^1(\eta,\theta)(t), \Lambda^2(\eta,\theta)(t)\big) \in \boldsymbol{V} \times E_0, \tag{4.23}$$

defined by the equations

$$\begin{aligned}
&(\Lambda^1(\eta,\theta)(t), \mathbf{v})_{\boldsymbol{V}} \\
&= \sum_{\ell=1}^{2} \Big(\int_0^t \mathcal{F}^\ell\big(t-s, \varepsilon(\mathbf{u}_\eta^\ell(s)), \zeta_\theta^\ell(s)\big)\, ds, \varepsilon(\mathbf{v}^\ell)\Big)_{\mathcal{H}^\ell} \\
&\quad + \sum_{\ell=1}^{2} \big((\mathcal{E}^\ell)^* \nabla \varphi_\eta^\ell, \varepsilon(\mathbf{v}^\ell)\big)_{\mathcal{H}^\ell} + j_{ad}(\beta_\eta(t), \mathbf{u}_\eta(t), \mathbf{v}), \quad \forall \mathbf{v} \in \boldsymbol{V},
\end{aligned} \tag{4.24}$$

$$\Lambda^2(\eta,\theta)(t) = \Big(\phi^1\big(\boldsymbol{\sigma}_{\eta\theta}^1(t), \boldsymbol{\varepsilon}(\mathbf{u}_\eta^1(t)), \zeta_\theta^1(t)\big),\ \phi^2\big(\boldsymbol{\sigma}_{\eta\theta}^2(t), \boldsymbol{\varepsilon}(\mathbf{u}_\eta^2(t)), \zeta_\theta^2(t)\big)\Big). \tag{4.25}$$

Here, for every $(\eta,\theta) \in C(0,T;\boldsymbol{V} \times E_0)$, $\mathbf{u}_\eta$, φ_η, β_η and ζ_θ represent the displacement field, the potential electric field and bonding field obtained in Lemmas 4.2, 4.3, 4.4 and 4.6 respectively, and $\boldsymbol{\sigma}_{\eta\theta}^\ell$ denote by

$$\boldsymbol{\sigma}_{\eta\theta}^\ell(t) = \mathcal{G}^\ell \boldsymbol{\varepsilon}(\mathbf{u}_\eta^\ell(t)) + (\mathcal{E}^\ell)^* \nabla \varphi_\eta^\ell + \int_0^t \mathcal{F}^\ell\big(t-s, \varepsilon(\mathbf{u}_\eta^\ell(s)), \zeta_\theta^\ell(s)\big)\, ds, \tag{4.26}$$

in $\Omega^\ell \times (0,T)$. We have the following result.

Lemma 4.7. *There exists a unique* $(\eta^*,\theta^*)\in C(0,T;\boldsymbol{V}\times E_0)$ *such that* $\Lambda(\eta^*,\theta^*)=(\eta^*,\theta^*)$.

Proof. Let (η_1,θ_1), $(\eta_2,\theta_2)\in C(0,T;\boldsymbol{V}\times E_0)$ and denote by $\mathbf{u}_i$, φ_i, β_i, ζ_i and $\boldsymbol{\sigma}_i$ the functions obtained in Lemmas 4.2, 4.3, 4.4, 4.6 and the relation (4.26) respectively, for $(\eta,\theta)=(\eta_i,\theta_i)$, $i=1,2$. Let $t\in[0,T]$. We use (3.9), (3.10), (3.11), (3.22) and the definition of R_ν, $\boldsymbol{R}_\tau$, we have

$$
\begin{aligned}
&\|\Lambda^1(\eta_1,\theta_1)(t)-\Lambda^1(\eta_2,\theta_1)(t)\|_{\boldsymbol{V}}^2\\
&\le\sum_{\ell=1}^{2}\|(\mathcal{E}^\ell)^*\nabla\varphi_1^\ell(t)-(\mathcal{E}^\ell)^*\nabla\varphi_2^\ell(t)\|_{\mathcal{H}^\ell}^2\\
&\quad+\sum_{\ell=1}^{2}\int_0^t\big\|\mathcal{F}^\ell\big(t-s,\varepsilon(\mathbf{u}_1^\ell(s)),\zeta_1^\ell(s)\big)-\mathcal{F}^\ell\big(t-s,\varepsilon(\mathbf{u}_2^\ell(s)),\zeta_2^\ell(s)\big)\big\|_{\mathcal{H}^\ell}^2\,ds\\
&\quad+C\|\beta_1^2(t)R_\nu([u_{1\nu}(t)])-\beta_2^2(t)R_\nu([u_{2\nu}(t)])\|_{L^2(\Gamma_3)}^2\\
&\quad+C\|\beta_1^2(t)\boldsymbol{R}_\tau([\mathbf{u}_{1\tau}(t)])-\beta_2^2(t)\boldsymbol{R}_\tau([\mathbf{u}_{2\tau}(t)])\|_{L^2(\Gamma_3)}^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|\Lambda^1(\eta_1,\theta_1)(t)-\Lambda^1(\eta_2,\theta_1)(t)\|_{\boldsymbol{V}}^2\\
&\le C\Big(\int_0^t\|\mathbf{u}_1(s)-\mathbf{u}_2(s))\|_{\boldsymbol{V}}^2\,ds+\int_0^t\|\zeta_1(s)-\zeta_2(s))\|_{E_0}^2\,ds\\
&\quad+\|\varphi_1(t)-\varphi_2(t)\|_W^2+\|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)}^2\Big).
\end{aligned}\tag{4.27}
$$

Recall that $u_{\eta\nu}^\ell$ and $\mathbf{u}_{\eta\tau}^\ell$ denote the normal and the tangential component of the function $\mathbf{u}_\eta^\ell$ respectively. By similar arguments, from (4.25), (4.26) and (3.10) it follows that

$$
\begin{aligned}
&\|\Lambda^2(\eta_1,\theta_1)(t)-\Lambda^2(\eta_2,\theta_1)(t)\|_{E_0}^2\\
&\le C\Big(\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_{\boldsymbol{V}}^2+\int_0^t\|\mathbf{u}_1(s)-\mathbf{u}_2(s))\|_{\boldsymbol{V}}^2\,ds\\
&\quad+\|\zeta_1(t)-\zeta_2(t))\|_{E_0}^2+\int_0^t\|\zeta_1(s)-\zeta_2(s))\|_{E_0}^2\,ds+\|\varphi_1(t)-\varphi_2(t)\|_W^2\Big).
\end{aligned}\tag{4.28}
$$

It follows now from (4.27) and (4.28) that

$$
\begin{aligned}
&\|\Lambda(\eta_1,\theta_1)(t)-\Lambda(\eta_2,\theta_1)(t)\|_{\boldsymbol{V}\times E_0}^2\\
&\le C\Big(\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_{\boldsymbol{V}}^2+\int_0^t\|\mathbf{u}_1(s)-\mathbf{u}_2(s))\|_{\boldsymbol{V}}^2\,ds+\|\zeta_1(t)-\zeta_2(t))\|_{E_0}^2\\
&\quad+\int_0^t\|\zeta_1(s)-\zeta_2(s))\|_{E_0}^2\,ds+\|\varphi_1(t)-\varphi_2(t)\|_W^2+\|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)}^2\Big).
\end{aligned}\tag{4.29}
$$

Also, since

$$\mathbf{u}_i^\ell(t)=\int_0^t\dot{\mathbf{u}}_i^\ell(s)ds+\mathbf{u}_0^\ell(t),\quad t\in[0,T],\ \ell=1,2,$$

we have

$$\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_{\boldsymbol{V}}\le\int_0^t\|\dot{\mathbf{u}}_1(s)-\dot{\mathbf{u}}_2(s))\|_{\boldsymbol{V}}\,ds$$

and using this inequality in (4.10) yields

$$\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_{\boldsymbol{V}} \le C\Big(\int_0^t \|\eta_1(s)-\eta_2(s))\|_{\boldsymbol{V}}\,ds+\int_0^t \|\mathbf{u}_1(s)-\mathbf{u}_2(s))\|_{\boldsymbol{V}}\,ds\Big). \tag{4.30}$$

Next, we apply Gronwall's inequality to deduce

$$\|\mathbf{u}_1(t)-\mathbf{u}_2(t))\|_{\boldsymbol{V}} \le C\int_0^t \|\eta_1(s)-\eta_2(s)\|_{\boldsymbol{V}}\,ds \quad \forall t\in[0,T]. \tag{4.31}$$

On the other hand, from the Cauchy problem (4.20)–(4.21) we can write

$$\beta_i(t)=\beta_0-\int_0^t \Big(\beta_i(s)\big(\gamma_\nu(R_\nu([u_{i\nu}(s)]))^2+\gamma_\tau|\boldsymbol{R}_\tau([\mathbf{u}_{i\tau}(s)])|^2\big)-\varepsilon_a\Big)_+ ds$$

and then

$$\begin{aligned}
&\big\|\beta_1(t)-\beta_2(t)\big\|_{L^2(\Gamma_3)}\\
&\le C\int_0^t \big\|\beta_1(s)R_\nu([u_{1\nu}(s)])^2-\beta_2(s)R_\nu([u_{2\nu}(s)])^2\big\|_{L^2(\Gamma_3)}ds\\
&\quad +C\int_0^t \big\|\beta_1(s)|\boldsymbol{R}_\tau([\mathbf{u}_{1\tau}(s)])|^2-\beta_2(s)|\boldsymbol{R}_\tau([\mathbf{u}_{2\tau}(s)])|^2\big\|_{L^2(\Gamma_3)}ds.
\end{aligned}$$

Using the definition of R_ν and $\boldsymbol{R}_\tau$ and writing $\beta_1=\beta_1-\beta_2+\beta_2$, we obtain

$$\begin{aligned}
&\big\|\beta_1(t)-\beta_2(t)\big\|_{L^2(\Gamma_3)}\\
&\le C\Big(\int_0^t \|\beta_1(s)-\beta_2(s)\|_{L^2(\Gamma_3)}ds+\int_0^t \big\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\big\|_{L^2(\Gamma_3)^d}ds\Big).
\end{aligned} \tag{4.32}$$

Next, we apply Gronwall's inequality to deduce

$$\|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)} \le C\int_0^t \|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_{L^2(\Gamma_3)^d}ds.$$

and from the relation (3.3) we obtain

$$\|\beta_1(t)-\beta_2(t)\|^2_{L^2(\Gamma_3)} \le C\int_0^t \|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|^2_V ds. \tag{4.33}$$

We use now (4.16), (3.4), (3.11) and (3.12) to find

$$\|\varphi_1(t)-\varphi_2(t)\|^2_W \le C\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|^2_V. \tag{4.34}$$

From (4.22) we deduce that

$$(\dot\zeta_1-\dot\zeta_2,\zeta_1-\zeta_2)_{E_0}+a(\zeta_1-\zeta_2,\zeta_1-\zeta_2)\le\big(\theta_1-\theta_2,\zeta_1-\zeta_2\big)_{E_0}, \quad \text{a.e. } t\in(0,T).$$

Integrating the previous inequality with respect to time, using the initial conditions $\zeta_1(0)=\zeta_2(0)=\zeta_0$ and inequality $a(\zeta_1-\zeta_2,\zeta_1-\zeta_2)\ge 0$, we find

$$\frac12\|\zeta_1(t)-\zeta_2(t)\|^2_{E_0}\le\int_0^t \big(\theta_1(s)-\theta_2(s),\zeta_1(s)-\zeta_2(s)\big)_{E_0}\,ds,$$

which implies that

$$\|\zeta_1(t)-\zeta_2(t)\|^2_{E_0}\le\int_0^t \|\theta_1(s)-\theta_2(s)\|^2_{E_0}\,ds+\int_0^t \|\zeta_1(s)-\zeta_2(s)\|^2_{E_0}\,ds.$$

This inequality combined with Gronwall's inequality leads to

$$\|\zeta_1(t)-\zeta_2(t)\|^2_{E_0}\le C\int_0^t \|\theta_1(s)-\theta_2(s)\|^2_{E_0}\,ds \quad \forall t\in[0,T]. \tag{4.35}$$

We substitute (4.31), (4.33), (4.34) and (4.35) in (4.29) to obtain

$$\|\Lambda(\eta_1,\theta_1)(t)-\Lambda(\eta_2,\theta_1)(t)\|^2_{\boldsymbol{V}\times E_0}\le C\int_0^t \|(\eta_1,\theta_1)(s)-(\eta_2,\theta_1)(s)\|^2_{\boldsymbol{V}\times E_0}\,ds.$$

Reiterating this inequality m times we obtain

$$\|\Lambda^m(\eta_1,\theta_1)-\Lambda^m(\eta_2,\theta_1)\|^2_{C(0,T;\boldsymbol{V}\times E_0)}\le \frac{C^mT^m}{m!}\|(\eta_1,\theta_1)-(\eta_2,\theta_1)\|^2_{C(0,T;\boldsymbol{V}\times E_0)}.$$

Thus, for m sufficiently large, the operator $\Lambda^m(\cdot,\cdot)$ is a contraction on the Banach space $C(0,T;\boldsymbol{V}\times E_0)$, and so $\Lambda(\cdot,\cdot)$ has a unique fixed point. □

Now, we have all the ingredients to prove Theorem 3.3.

Proof of Existence. Let $(\eta^*,\theta^*)\in C(0,T;\boldsymbol{V}\times E_0)$ be the fixed point of $\Lambda(\cdot,\cdot)$ and denote

$$\mathbf{u}_*=\mathbf{u}_{\eta^*},\quad \varphi_*=\varphi_{\eta^*},\quad \zeta_*=\zeta_{\theta^*},\quad \beta_*=\beta_{\eta^*}, \tag{4.36}$$

$$\boldsymbol{\sigma}^\ell_*=\mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\ell_*)+\mathcal{G}^\ell\boldsymbol{\varepsilon}(\mathbf{u}^\ell_*)+(\mathcal{E}^\ell)^*\nabla\varphi^\ell_*+\int_0^t\mathcal{F}^\ell\big(t-s,\varepsilon(\mathbf{u}^\ell_*(s)),\zeta^\ell_*(s)\big)\,ds, \tag{4.37}$$

$$\boldsymbol{D}^\ell_*=\mathcal{E}^\ell\varepsilon(\mathbf{u}^\ell_*)-\mathcal{B}^\ell\nabla\varphi^\ell_*. \tag{4.38}$$

We prove that the $\{\mathbf{u}_*,\boldsymbol{\sigma}_*,\varphi_*,\zeta_*,\beta_*,\boldsymbol{D}_*\}$ satisfies (3.26)–(3.32) and the regularities (3.36)–(3.41). Indeed, we write (4.8) for $\eta=\eta^*$ and use (4.36) to find

$$\begin{aligned}
&\sum_{\ell=1}^2(\mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\ell_*),\varepsilon(\mathbf{v}^\ell)-\boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\ell_*(t)))_{\mathcal{H}^\ell}+\sum_{\ell=1}^2(\mathcal{G}^\ell\boldsymbol{\varepsilon}(\mathbf{u}^\ell_*),\varepsilon(\mathbf{v}^\ell)-\boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\ell_*(t)))_{\mathcal{H}^\ell}\\
&+j_{\nu c}(\mathbf{u}_*(t),\mathbf{v}-\dot{\mathbf{u}}_*(t))+j_{fr}(\mathbf{u}_*(t),\mathbf{v})-j_{fr}(\mathbf{u}_*(t),\dot{\mathbf{u}}_*(t))+(\eta^*(t),\mathbf{v}-\dot{\mathbf{u}}_*(t)))_{\boldsymbol{V}}\\
&\ge(\mathbf{f}(t),\mathbf{v}-\dot{\mathbf{u}}_*(t))_{\boldsymbol{V}}\quad\forall\mathbf{v}\in\boldsymbol{V},\ \text{a.e. } t\in[0,T].
\end{aligned}\tag{4.39}$$

We use equalities $\Lambda^1(\eta^*,\theta^*)=\eta^*$ and $\Lambda^2(\eta^*,\theta^*)=\theta^*$ it follows from (4.24) and (4.25) that

$$\begin{aligned}
(\eta^*(t),\mathbf{v})_{\boldsymbol{V}}=&\sum_{\ell=1}^2\big((\mathcal{E}^\ell)^*\nabla\varphi^\ell_*(t),\varepsilon(\mathbf{v}^\ell)\big)_{\mathcal{H}^\ell}+j_{ad}(\beta_*(t),\mathbf{u}_*(t),\mathbf{v})\\
&+\sum_{\ell=1}^2\Big(\int_0^t\mathcal{F}^\ell\big(t-s,\varepsilon(\mathbf{u}^\ell_*(s)),\zeta^\ell_*(s)\big)\,ds,\varepsilon(\mathbf{v}^\ell)\Big)_{\mathcal{H}^\ell},\\
&\forall\mathbf{v}\in\boldsymbol{V},\ \text{a.e. } t\in(0,T),
\end{aligned}\tag{4.40}$$

$$\theta^\ell_*(t)=\phi^\ell\big(\boldsymbol{\sigma}^\ell_*(t)-\mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\mathbf{u}}^\ell_*(t)),\boldsymbol{\varepsilon}(\mathbf{u}^\ell_*(t)),\zeta^\ell_*(t)\big),\quad \text{a.e. } t\in(0,T),\ \ell=1,2. \tag{4.41}$$

We now substitute (4.40) in (4.39) to obtain

$$
\begin{aligned}
&\sum_{\ell=1}^{2}(\mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_*^\ell)(t),\varepsilon(\mathbf{v}^\ell)-\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_*^\ell(t)))_{\mathcal{H}^\ell}+\sum_{\ell=1}^{2}(\mathcal{G}^\ell\boldsymbol{\varepsilon}(\mathbf{u}_*^\ell)(t),\ \varepsilon(\mathbf{v}^\ell)-\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_*^\ell(t)))_{\mathcal{H}^\ell}\\
&+\sum_{\ell=1}^{2}\Big(\int_0^t\mathcal{F}^\ell\big(t-s,\varepsilon(\mathbf{u}_*^\ell(s)),\zeta_*^\ell(s)\big)\,ds,\varepsilon(\mathbf{v}^\ell)-\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_*^\ell(t))\Big)_{\mathcal{H}^\ell}\\
&+j_{ad}(\beta_*(t),\mathbf{u}_*(t),\mathbf{v}-\dot{\mathbf{u}}_*(t))+j_{\nu c}(\mathbf{u}_*(t),\mathbf{v}-\dot{\mathbf{u}}_*(t))+j_{fr}(\mathbf{u}_*(t),\mathbf{v})\\
&-j_{fr}(\mathbf{u}_*(t),\dot{\mathbf{u}}_*(t))+\sum_{\ell=1}^{2}\big((\mathcal{E}^\ell)^*\nabla\varphi_*^\ell(t),\varepsilon(\mathbf{v}^\ell)-\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_*^\ell(t))\big)_{\mathcal{H}^\ell}\\
&\geq(\mathbf{f}(t),\mathbf{v}-\dot{\mathbf{u}}_*(t))_{\boldsymbol{V}}\quad\forall\mathbf{v}\in\boldsymbol{V}\text{ a.e. }t\in[0,T],
\end{aligned}
\tag{4.42}
$$

and we substitute (4.41) in (4.22) to have $\zeta_*(t)\in K$ and

$$
\begin{aligned}
&\sum_{\ell=1}^{2}(\dot{\zeta}_*^\ell(t),\xi^\ell-\zeta_*^\ell(t))_{L^2(\Omega^\ell)}+a(\zeta_*(t),\xi-\zeta_*(t))\\
&\geq\sum_{\ell=1}^{2}\Big(\phi^\ell\big(\boldsymbol{\sigma}_*^\ell(t)-\mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_*^\ell(t)),\boldsymbol{\varepsilon}(\mathbf{u}_*^\ell(t)),\zeta_*^\ell(t)\big),\xi^\ell-\zeta_*^\ell(t)\Big)_{L^2(\Omega^\ell)},
\end{aligned}
\tag{4.43}
$$

for all $\xi\in K$, a.e. $t\in(0,T)$. We write now (4.16) for $\eta=\eta^*$ and use (4.36) to see that

$$
\sum_{\ell=1}^{2}(\mathcal{B}^\ell\nabla\varphi_*^\ell(t),\nabla\phi^\ell)_{H^\ell}-\sum_{\ell=1}^{2}(\mathcal{E}^\ell\varepsilon(\mathbf{u}_*^\ell(t)),\nabla\phi^\ell)_{H^\ell}=(q(t),\phi)_W \tag{4.44}
$$

for all $\phi\in W$, $t\in[0,T]$. Additionally, we use $\mathbf{u}_{\eta^*}$ in (4.20) and (4.36) to find

$$
\dot{\beta}_*(t)=-\Big(\beta_*(t)\big(\gamma_\nu(R_\nu([u_{*\nu}(t)]))^2+\gamma_\tau|\boldsymbol{R}_\tau([\mathbf{u}_{*\tau}(t)])|^2\big)-\varepsilon_a\Big)_+, \tag{4.45}
$$

a.e. $t\in[0,T]$. Relations (4.36), (4.37), (4.38), (4.42), (4.43), (4.44) and (4.45) allow us to conclude now that $\{\mathbf{u}_*,\boldsymbol{\sigma}_*,\varphi_*,\zeta_*,\beta_*,\boldsymbol{D}_*\}$ satisfies (3.26)–(3.31). Next, (3.32) and the regularity (3.36), (3.38)–(3.40) follow from Lemmas 4.2, 4.3, 4.4 and 4.6. Since $\mathbf{u}_*$, φ_* and ζ_* satisfies (3.36), (3.38) and (3.39), respectively, it follows from (4.37) that

$$
\boldsymbol{\sigma}_*\in C(0,T;\mathcal{H}). \tag{4.46}
$$

For $\ell=1,2$, we choose $\mathbf{v}=\dot{\mathbf{u}}\pm\phi$ in (4.42), with $\phi=(\phi^1,\phi^2)$, $\phi^\ell\in D(\Omega^\ell)^d$ and $\phi^{3-\ell}=0$, to obtain

$$
\operatorname{Div}\boldsymbol{\sigma}_*^\ell(t)=-\boldsymbol{f}_0^\ell(t)\quad\forall t\in[0,T],\quad\ell=1,2, \tag{4.47}
$$

where $D(\Omega^\ell)$ is the space of infinitely differentiable real functions with a compact support in Ω^ℓ. The regularity (3.37) follows from (3.14), (4.46) and (4.47). Let now $t_1,t_2\in[0,T]$, by (3.11), (3.12), (3.4) and (4.38), we deduce that

$$
\|\boldsymbol{D}_*(t_1)-\boldsymbol{D}_*(t_2)\|_H\leq C\left(\|\varphi_*(t_1)-\varphi_*(t_2)\|_W+\|\mathbf{u}_*(t_1)-\mathbf{u}_*(t_2)\|_{\boldsymbol{V}}\right).
$$

The regularity of $\mathbf{u}_*$ and φ_* given by (3.36) and (3.38) implies

$$
\boldsymbol{D}_*\in C(0,T;H). \tag{4.48}
$$

For $\ell = 1, 2$, we choose $\phi = (\phi^1, \phi^2)$ with $\phi^\ell \in D(\Omega^\ell)^d$ and $\phi^{3-\ell} = 0$ in (4.44) and using (3.20) we find

$$\operatorname{div} \boldsymbol{D}_*^\ell(t) = q_0^\ell(t) \quad \forall t \in [0, T], \quad \ell = 1, 2. \tag{4.49}$$

Property (3.41) follows from (3.14), (4.48) and (4.49). □

Finally we conclude that the weak solution $\{\mathbf{u}_*, \boldsymbol{\sigma}_*, \varphi_*, \zeta_*, \beta_*, \boldsymbol{D}_*\}$ of the piezoelectric contact Problem PV has the regularity (3.36)–(3.41), which concludes the existence part of Theorem 3.3.

Proof of Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator $\Lambda(\cdot, \cdot)$ defined by (4.24)-(4.25) and the unique solvability of the Problems PV^u_η, PV^φ_η, PV^β_η, and PV^ζ_η. □

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