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QUASILINEAR PROBLEMS WITH TWO PARAMETERS INCLUDING SUPERLINEAR AND GRADIENT TERMS

MANUELA C. REZENDE, CARLOS A. SANTOS

Abstract. In this article, we establish conditions for the existence of solutions for a quasilinear elliptic two-parameter problem with dependence on the gradient term in smooth bounded domains or in the whole space $\mathbb{R}^N$. We consider superlinear and asymptotically linear terms. Estimates on the values of two parameters for which the problem have solutions are provided.

1. Introduction

This article concerns the existence of solutions and estimates of the intervals of parameters for which the problem

$$\begin{gathered} -\Delta_p u = a(x)f(u) + \lambda b(x)g(u) + \mu V(x, \nabla u) \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

has a solution. Here, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < \infty$, denotes the usual p-Laplacian operator; $\lambda > 0$; $\mu \ge 0$ are real parameters; $f, g : (0,\infty) \to [0,\infty)$; $a, b : \Omega \to [0,\infty)$ with $a, b \ne 0$; $V : \Omega \times \mathbb{R}^N \to [0,\infty)$ are continuous functions satisfying appropriate hypotheses and either $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain or $\Omega = \mathbb{R}^N$. When $\Omega = \mathbb{R}^N$, the condition $u = 0$ on $\partial\Omega$ means that $u(x) \to 0$ when $|x| \to \infty$.

By a solution of (1.1) we mean a function $u = u_{\lambda,\mu} \in C^1(\Omega) \cap C(\overline{\Omega})$, with $u > 0$ in Ω, $u = 0$ on $\partial\Omega$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi \, dx = \int_\Omega [a(x)f(u) + \lambda b(x)g(u) + \mu V(x, \nabla u)]\phi \, dx,$$

for all $\phi \in C_0^\infty(\Omega)$.

In this article we say that a function $h : (0,\infty) \to [0,\infty)$ is $(p-1)$-sublinear at 0 or at $+\infty$, if $\lim_{s\to 0} h(s)/s^{p-1} = \infty$ or $\lim_{s\to\infty} h(s)/s^{p-1} = 0$ respectively; $(p-1)$-superlinear at 0 or at $+\infty$, if $\lim_{s\to 0} h(s)/s^{p-1} = 0$ or $\lim_{s\to\infty} h(s)/s^{p-1} = \infty$ respectively and $(p-1)$-asymptotically linear, if there are positive and finite numbers that correspond to the values of these limits. To abbreviate, we say sublinear, superlinear and asymptotically linear nonlinearities, respectively. In particular, a sublinear term h at 0 is called singular at 0, if $\lim_{s\to 0^+} h(s) = \infty$.

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For $\mu = 0$, problems like (1.1) have been intensively studied in recent years, including sublinear and superlinear nonlinearities terms at zero and/or infinity and singular terms at zero. In [5, 15, 13, 18] and references therein studies on the bounded domain case are found. For $\Omega = \mathbb{R}^N$, we refer the reader to [4, 14, 19, 21] and their references.

However, there are not many results in the case where the nonlinearities depend on the gradient of the solution, that is, $\mu \neq 0$, with $p \neq 2$. In general, variational techniques are not suitable to handle (1.1). In the case $p = 2$, an interesting exception can be seen in [6].

One of the novelties in this article is that we improve a regularization-monotonicity technique (see Section 2). This allows us to treat (1.1) with superlinear nonlinearities both in bounded domain and $\mathbb{R}^N$. This improvement also makes possible for us to study (1.1) in $\mathbb{R}^N$ by creating a sequence of solutions of (1.1) in bounded domains, locally bounded below by a positive function and bounded above by a carefully constructed function.

We emphasize that our results do not require any monotonicity condition and (or) singularity of the functions f and g. We are particularly interested in the cases where f and g may have singularity at 0. Problems including singular nonlinearities arise in electrical conductivity, the theory of pseudoplastic fluids, singular minimal surfaces, reaction-diffusion processes, the obtaining of various geophysical indexes and industrial processes, among others; see [3, 10] for a detailed discussion.

Considering the problem (1.1) in smooth bounded domains, we quote Zhang and Yu [27] who, in 2000, studied the problem

$$\begin{gathered} -\Delta u = u^{-\alpha} + \lambda + \mu|\nabla u|^q \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where $\mu, \lambda \geq 0$, $\alpha > 0$ and $q \in (0, 2]$. Using a change of variables, the authors proved that the problem (1.2) has classical solutions for $\mu\lambda < \lambda_1$, if $q = 2$ or $\mu \in [0, \mu^*)$, if $0 < q < 2$, with $\mu^* = \mu^*(q, \lambda)$, where $\lambda_1 > 0$ denotes the first eigenvalue of the Dirichlet problem in $W_0^{1,2}(\Omega)$.

Ghergu and Radulescu [11] considered

$$\begin{gathered} -\Delta u = h(u) + \lambda f(x, u) + \mu|\nabla u|^q \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

under the conditions $f > 0$ in $\overline{\Omega} \times (0, \infty)$, $\partial f/\partial s(x, s) \geq 0$, $s > 0$, $f(x, s)/s$ non-increasing in $s > 0$, $\lim_{s\to\infty} f(x, s)/s = 0$, $\lim_{s\to 0} h(s) = +\infty$, $h \in C^{0,\alpha}((0, \infty))$, $h > 0$ non-increasing and $\lambda = 1$. They proved that

(i) if $0 < q < 1$, then (1.3) has solution for each $\mu \geq 0$,
(ii) if $1 \leq q \leq 2$, there exists $\mu^* > 0$ such that (1.3) has a solution for $0 \leq \mu < \mu^*$. Moreover, if $1 < q \leq 2$, then $\mu^* < \infty$.

In 2010, Alves, Carrião and Faria [1] used the Galerkin method to study

$$\begin{gathered} -\Delta u = g(x, u) + \mu V(x, \nabla u) \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.4}$$

where g and V are locally Hölder continuous functions such that

$$b|s|^{r_1} \leq g(x, s) \leq a_1(x) + a_2(x)|s|^{r_2} + \frac{a_3(x)}{|s|^{r_3}}, \quad 0 \leq V(x, \xi) \leq a_5(x) + a_4(x)|\xi|^{r_4},$$

with $b > 0, r_i \in (0,1)$, $i = 1,\dots,4$ are constants and a_i, $i = 1,\dots,5$ are positive continuous functions. Under these conditions, it was shown that (1.4) has a solution for each $\mu \geq 0$.

Liu, Shi and Wei [16], still with $p = 2$, recently showed, by using Morse theory and an iterative method, existence of solution for a problem like (1.1) with terms that have asymptotically linear growth at zero and infinity. Considering singular terms at 0 and permitting $p \neq 2$, Loc and Schmitt [17] used the lower and upper solution method to show existence of solution for (1.1) with the nonlinearity of the gradient term bounded above by the natural growth.

For $\Omega = \mathbb{R}^N$ in the problem (1.1). In 2007, Ghergu and Radulescu [12] showed existence of solution for the problem

$$\begin{gathered} -\Delta u = a(x)[f(u) + g(u) + |\nabla u|^q] \quad \text{in } \mathbb{R}^N, \\ u > 0 \quad \text{in } \mathbb{R}^N, \quad \text{and} \quad u(x) \overset{|x|\to\infty}{\longrightarrow} 0, \end{gathered} \tag{1.5}$$

where $q \in (0,1)$, $f \in C^1((0,\infty))$ is positive and decreasing, $\lim_{s\to 0^+} f(s) = \infty$ and the function $g : [0,\infty) \to [0,\infty)$ satisfies

$$g' \geq 0, \quad \frac{g(s)}{s} \text{ is non-increasing in } s > 0, \quad \lim_{s\to 0^+} \frac{g(s)}{s} = +\infty \\ \text{and } \lim_{s\to\infty} \frac{g(s)}{s} = 0.$$

Concerning the function a, they assumed that $0 < a \in C^{0,\alpha}(\mathbb{R}^N)$ and

$$\int_0^\infty r\phi(r)\,dr < \infty, \quad \text{where } \phi(r) = \max_{|x|=r} a(x). \tag{1.6}$$

In the same year, Xue and Zhang in [24] assumed (1.6) and studied the problem (1.5) without requiring any monotonicity condition over f and g. They just assumed

$$\lim_{s\to 0^+} \frac{g(s)}{s} = +\infty, \quad \lim_{s\to\infty} \frac{g(s)}{s} = 0, \ \lim_{s\to 0^+} \frac{f(s)}{s} = +\infty, \quad \lim_{s\to\infty} \frac{f(s)}{s} = 0.$$

For the rest of this article, given $\sigma : (0,\infty) \to (0,\infty)$, we denote by $\sigma^i, \sigma_i \in [0,\infty]$ the following limits

$$\sigma^i := \lim_{s\to i} \sigma(s) \quad \text{and} \quad \sigma_i := \lim_{s\to i} \frac{\sigma(s)}{s^{p-1}}, \quad \text{for } i = 0 \text{ or } i = \infty$$

and we assume that there exists a $\rho \in C(\Omega) \cap L^\infty(\Omega)$, $\rho \geq 0$, $\rho \neq 0$ such that $\rho \leq a, b$. We denote by $\lambda_\Omega = \lambda_{1,\Omega}(\rho) > 0$ the first eigenvalue and by $\varphi_\Omega = \varphi_{1,\Omega} > 0$ the first eigenfunction of the problem

$$\begin{gathered} -\Delta_p \varphi = \lambda\rho(x)|\varphi|^{p-2}\varphi \quad \text{in } \Omega, \\ \varphi > 0 \quad \text{in } \Omega, \quad \varphi = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.7}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. Moreover, we denote by $\lambda_1(\rho) = \lim_{R\to\infty} \lambda_{1,B_R(0)}(\rho) \geq 0$, where $B_R(0)$ is the ball centered at the origin of $\mathbb{R}^N$ with radius $R > 0$.

Also, we let us assume:

(V1) $V(x,\xi) \leq \alpha(x)|\xi|^q + \beta(x)$ in $\Omega \times \mathbb{R}^N$ for some $0 \leq \alpha, \beta \in C(\Omega) \cap L^\infty(\Omega)$ and $q \geq 0$,

(M1) there exists $\omega_M \in C^1(\overline{\Omega})$ ($\omega_M \in C^1(\Omega) \cap W^{1,\infty}(\Omega)$ if $\Omega = \mathbb{R}^N$) satisfying
$$\begin{gathered} -\Delta_p \omega_M = M(x) \quad \text{in } \Omega, \\ \omega_M > 0 \quad \text{in } \Omega, \quad \omega_M = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.8}$$
where $M(x) := \max\{2a(x), 2b(x), \alpha(x), \beta(x)\}$, $x \in \Omega$,

(F1)
$$(F_0) \quad f_0 < 1/\|\omega_M\|^{p-1}_{L^\infty(\Omega)}, \quad \text{or} \quad (F_\infty) \quad f_\infty < 1/\|\omega_M\|^{p-1}_{L^\infty(\Omega)}.$$

Remark 1.1. With respect to the hypotheses (M1) and (F1), we note that:

(1) If $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, then (M1) occurs if, for example, $M \in L^q(\Omega)$ for some $q > N > 1$. See, for instance, Perera and Zhang [20]. This allows we take singular potentials of the type $a(x) = b(x) = 1/(1 - |x|)^\gamma$, with $\gamma < 1$ and $\Omega = B_1(0) \subset \mathbb{R}^N$ in (1.1).

(2) If $\Omega = \mathbb{R}^N$, it is known that (1.8) has a solution if M is a bounded continuous function and satisfies
$$M_\infty := \int_0^\infty \Big[s^{1-N} \int_0^s t^{N-1} \hat{M}(t) dt\Big]^{\frac{1}{p-1}} ds < \infty,$$
where $\hat{M}(t) = \max_{|x|=t} M(x)$, $t \geq 0$. The existence and L^∞-boundedness of a solution of (1.8) imply its regularity (see [8]). In addition, if we assume that $N \geq 3$ and
$$\int_1^\infty r^{\frac{1}{p-1}} \hat{M}^{\frac{1}{p-1}}(r) dr < \infty \quad \text{or} \quad \int_1^\infty r^{\frac{(p-2)N+1}{p-1}} \hat{M}(r) dr < \infty,$$
if $1 < p \leq 2$ or $p \geq 2$, respectively, then $M_\infty < \infty$. In [25], we have an example that shows that the converse of this fact is not true.

(3) Condition (F_0) holds if f is superlinear at 0 ($f_0 = 0$) and (F_∞) occurs if f is sublinear at ∞ ($f_\infty = 0$).

To state our results, we assume $0 < g_0 + f_0 \leq \infty$ and denote by
$$\lambda_* = \lambda_*(g_0) := \begin{cases} 0, & \text{if } g_0 = 0 \text{ and } f_0 > \lambda_\Omega(\rho), \\ \max\{0, \frac{\lambda_\Omega(\rho) - f_0}{g_0}\}, & \text{if } 0 < g_0 < \infty, \\ 0, & \text{if } g_0 = \infty, \end{cases}$$
where $\lambda_1(\rho) = \lambda_\Omega(\rho)$, if $\Omega = \mathbb{R}^N$. We have that $\lambda_* = 0$ in all the previous works, because $g_0 = \infty$ there.

Regarding problem (1.1) in bounded domains we have the following result.

Theorem 1.2. *Assume that* (F1), (M1), (V1) *with* $q \in [0, p]$ *hold. Then there exists* $\lambda^* \in (0, \infty]$ *such that for each* $\lambda_* < \lambda < \lambda^*$, *there exist* $\mu^* = \mu^*_\lambda > 0$ *and a* $u = u_{\lambda,\mu} \in C^1(\Omega) \cap C(\overline{\Omega})$ *solution of* (1.1) *for each* $0 \leq \mu < \mu^*$. *Additionally:*

(i) $u \geq c\varphi_\Omega$ *for some* $c > 0$,

(ii) *if* (F_i) *holds, for* $i \in \{0, \infty\}$, *then*
$$\lambda^* \geq \frac{1}{g_i}\Big(\frac{1}{\|\omega_M\|^{p-1}_{L^\infty(\Omega)}} - f_i\Big) := \lambda^i,$$

(iii) *there exists a constant* $d > 0$ *such that*
$$\mu^*_\lambda \geq d \min\big\{[f^i + \lambda g^i]^{\frac{p-1-q}{p-1}}, f^i + \lambda g^i\big\}, \quad \textit{if } q \in [0, p-1].$$

For $\Omega = \mathbb{R}^N$ and $1 < p < N$ our main result is the following.

Theorem 1.3. *Assume that* (F1), (M1), (V1) *with* $q \in [0, p-1]$ *hold. Then there exists* $\lambda^* \in (0, \infty]$ *such that for each* $\lambda_* < \lambda < \lambda^*$, *there exist* $\mu^* = \mu^*_\lambda > 0$ *and a* $u = u_{\lambda,\mu} \in C^1(\mathbb{R}^N)$ *solution of* (1.1) *for each* $0 \le \mu < \mu^*$. *Moreover, if* (F_i) *holds, for* $i \in \{0, \infty\}$, *then there is a constant* $d > 0$ *such that*

(i) $\lambda^* \ge \lambda^i$

(ii) $\mu^*_\lambda \ge d \min\big\{[f^i + \lambda g^i]^{\frac{p-1-q}{p-1}}, f^i + \lambda g^i\big\}$ *for* $0 < \lambda < \lambda^i$.

Remark 1.4. In the definition of λ_*, the possibility $f_0 > \lambda_1(\rho)$ does not permit (F_0) to occur, because $\lambda_\Omega(\rho) \ge \lambda_\Omega(M) \ge \|w_M\|^{1-p}_{L^\infty(\Omega)}$ and as a consequence of this, we have $\lambda_1(\rho) \ge \lambda_1(M) \ge \|w_M\|^{1-p}_{L^\infty(\mathbb{R}^N)}$ also (see Santos [21]). In this situation, (F_∞) should occur, as in [12] and [24].

Theorem 1.3 improves previous results principally because it addresses the p-Laplacian operator, obtains estimates for λ^* and μ^*, no monotonicity or growth restriction on the nonlinearities are required, the cases $q = 0$ and $q = p - 1$ are included and we assume the hypothesis (M1) that is weaker than (1.6). We point out that problem (1.1) has no solution for $p \ge N$ (see Serrin and Zou [22]).

This paper is organized as follows: In section 2 we construct several auxiliary functions for the terms f and g and we study their properties. Because of the singularities allowed on f and g, we regularize the problem (1.1) and we obtain an upper solution for it in bounded domain and in $\mathbb{R}^N$, in sections 3 and 5, respectively. After that, we use section 4 to prove Theorem 1.2. In section 6, we generalize this result for $\mathbb{R}^N$.

2. Auxiliary functions

To prove Theorems 1.2 and 1.3 we refine a regularization-motonicity technique used, among others, by Feng and Liu [9], Zhang [26] and Mohammed [19].

Observing that we do not assume monotonicity on the nonlinearities, we introduce a truncation of the terms f and g through a parameter $\gamma > 0$ and build auxiliary functions which allow us to obtain not only the monotonicity, but also the necessary regularity for the proof of our results. Parallel to this, the inclusion of a parameter $\theta < 1$, in this construction, makes it possible solving the problem (1.1) for the case $q > p - 1$.

Analyzing the behavior of these auxiliary functions, the parameters λ, γ, θ and the fact that the problem (1.8) has a solution, we determine a Λ^*-curve whose behavior allow us to find region of variation for the parameter λ, and consequently, obtain an estimate from below for that region.

With these purposes, let us define the continuous functions, depending on real parameter $\gamma > 0$, as

$$f_\gamma(s) := \begin{cases} f(s), & \text{if } 0 < s \le \gamma \\ \frac{f(\gamma)}{\gamma^{p-1}} s^{p-1}, & \text{if } s \ge \gamma \end{cases} \quad \text{and} \quad g_\gamma(s) := \begin{cases} g(s), & \text{if } 0 < s \le \gamma \\ \frac{g(\gamma)}{\gamma^{p-1}} s^{p-1}, & \text{if } s \ge \gamma. \end{cases}$$

Now, for each $s > 0$, defining the function

$$\zeta_{\lambda,\gamma}(s) = s^{p-1} \sup\big\{\frac{f_\gamma(t)}{t^{p-1}},\ t > s\big\} + \lambda s^{p-1} \sup\big\{\frac{g_\gamma(t)}{t^{p-1}},\ t > s\big\}, \quad \lambda \ge 0 \tag{2.1}$$

we obtain, from the above definitions, that

(i) $\frac{\zeta_{\lambda,\gamma}(s)}{s^{p-1}}$ is non-increasing in $s > 0$;

(ii) $\zeta_{\lambda,\gamma}(s) \geq f_\gamma(s) + \lambda g_\gamma(s)$, $s > 0$;
(iii) $\lim_{s\to\infty} \frac{\zeta_{\lambda,\gamma}(s)}{s^{p-1}} = \frac{f(\gamma)}{\gamma^{p-1}} + \lambda \frac{g(\gamma)}{\gamma^{p-1}}$.

Now, defining

$$H_{\lambda,\gamma}(s) = \frac{s^2}{\int_0^s \frac{t}{\zeta_{\lambda,\gamma}(t)^{\frac{1}{p-1}}} dt}, \; s > 0,$$

and using (i) and (iii) above, we have the following lemma.

Lemma 2.1. *The function H satisfies:*

(i) $H_{\lambda,\gamma} \in C^1((0,\infty),(0,\infty))$;
(ii) $\zeta_{\lambda,\gamma}(s) \leq [H_{\lambda,\gamma}(s)]^{p-1}$, $s > 0$;
(iii) $\frac{H_{\lambda,\gamma}(s)}{s}$ *is non-increasing in* $s > 0$;
(iv)
$$\lim_{s\to\infty} \frac{H_{\lambda,\gamma}(s)}{s} = \Big[\frac{f(\gamma)}{\gamma^{p-1}} + \lambda \frac{g(\gamma)}{\gamma^{p-1}}\Big]^{\frac{1}{p-1}}.$$

After these, introducing a parameter $\theta \in (0,1]$ and defining the function

$$\Gamma_\lambda(\gamma) = \Gamma_{\lambda,\theta}(\gamma) = \frac{\theta}{\gamma} \int_0^\gamma \frac{t^\theta}{H_{\lambda,\gamma}(t^\theta)} dt, \quad \gamma > 0 \tag{2.2}$$

we obtain, from the previously defined functions and their properties, the following result.

Lemma 2.2. *Suppose* (M1) *and* (F1) *hold. Then for each* $\theta \in (\|w_M\|_\infty f_i^{1/(p-1)}, 1]$, *for either* $i = 0$ *or* $i = \infty$, *we have:*

(i) $\lim_{\gamma\to\infty} \Gamma_{\lambda,\theta}(\gamma) = \frac{\theta}{(f_\infty + \lambda g_\infty)^{\frac{1}{p-1}}}$, *for each* $\lambda \geq 0$;
(ii) $\lim_{\gamma\to 0} \Gamma_{\lambda,\theta}(\gamma) = \frac{\theta}{(f_0 + \lambda g_0)^{\frac{1}{p-1}}}$, *for each* $\lambda \geq 0$;
(iii) $\Gamma_{\lambda,\theta}$ *is decreasing in* $\lambda > 0$, *for each* $\gamma > 0$;
(iv) *there exists a* $\tilde{\gamma} = \tilde{\gamma}(\Omega,\theta) > 0$ *such that* $\Gamma_{0,\theta}(\tilde{\gamma}) > \|\omega_M\|_{L^\infty(\Omega)}$.

By Lemma 2.2, we can define the nonempty set

$$\mathcal{A} = \mathcal{A}_{\Omega,\theta} := \{\gamma \in (0,\infty) : \Gamma_{0,\theta}(\gamma) > \|\omega_M\|_{L^\infty(\Omega)}\}.$$

Now, as a consequence of $\lim_{\lambda\to\infty} \Gamma_{\lambda,\theta}(\gamma) = 0$, $\lim_{\lambda\to 0} \Gamma_{\lambda,\theta}(\gamma) = \Gamma_{0,\theta}(\gamma)$ and of the above lemma, we have that the function $\Lambda^* = \Lambda^*_{\Omega,\theta} : \mathcal{A} \to (0,\infty)$ that associate for each $\gamma \in \mathcal{A}$ the unique number $\Lambda^*(\gamma)$ satisfying

$$\Gamma_{\Lambda^*(\gamma),\theta}(\gamma) = \|\omega_M\|_{L^\infty(\Omega)}, \tag{2.3}$$

is well defined.

Thus, we can define the positive number

$$\lambda^*_\theta(\Omega) := \sup\{\Lambda^*(\gamma) : \gamma \in \mathcal{A}\}. \tag{2.4}$$

After these, we infer the following lemma.

Lemma 2.3. *Suppose* (M1) *and* (F1) *hold. Then for each* $\theta \in (\|w_M\|_\infty f_i^{1/(p-1)}, 1]$, *we have*

$$\lambda^*_\theta(\Omega) \geq \frac{1}{g_i}\Big(\frac{\theta}{\|\omega_M\|^{p-1}_{L^\infty(\Omega)}} - f_i\Big) := \lambda^i_\theta.$$

Proof. If (F_0) occurs and $g_0 < \infty$, then for each $0 < \delta < \lambda_\theta^0$, from Lemma 2.2 (ii) it follows that

$$\begin{aligned}\liminf_{\gamma\to 0}(\Gamma_{\delta,\theta}(\gamma) - \|\omega_M\|_\infty) &= \frac{\theta}{(f_0+\delta g_0)^{\frac{1}{p-1}}} - \|\omega_M\|_\infty \\ &> \frac{\theta}{(f_0+\lambda_\theta^0 g_0)^{\frac{1}{p-1}}} - \|\omega_M\|_\infty = 0.\end{aligned}$$

Now, if (F_∞) occurs and $g_\infty < \infty$, using Lemma 2.2 (i), we have

$$\begin{aligned}\liminf_{\gamma\to \infty}(\Gamma_{\delta,\theta}(\gamma) - \|\omega_M\|_\infty) &= \frac{\theta}{(f_\infty+\delta g_\infty)^{\frac{1}{p-1}}} - \|\omega_M\|_\infty \\ &> \frac{\theta}{(f_\infty+\lambda_\theta^\infty g_\infty)^{\frac{1}{p-1}}} - \|\omega_M\|_\infty = 0,\end{aligned}$$

for each $0 < \delta < \lambda_\theta^\infty$.

So, in both cases, there exists a $\gamma_0 = \gamma_0(\delta) > 0$ such that $\Gamma_{\delta,\theta}(\gamma_0) > \|\omega_M\|_\infty$. As a consequence of this and Lemma 2.3(iii), we have that $\gamma_0 \in \mathcal{A}$, because $\Gamma_{0,\theta}(\gamma_0) > \Gamma_{\delta,\theta}(\gamma_0) > \|\omega_M\|_\infty$. So, from (2.3) there is a unique $\Lambda^*(\gamma_0)$ such that $\Gamma_{\Lambda^*(\gamma_0),\theta}(\gamma_0) = \|\omega_M\|_\infty$. Now, using $\Gamma_{\Lambda^*(\gamma_0),\theta}(\gamma_0) < \Gamma_{\delta,\theta}(\gamma_0)$ and Lemma 2.3(iii), we obtain $\Lambda^*(\gamma_0) > \delta$. So, by the arbitrariness of δ, it follows the proof of the Lemma. □

Now, defining

$$\eta_\lambda(s) = \eta_{\lambda,\theta}(s) = \frac{\theta}{\gamma}\int_0^s \frac{t^\theta}{H_{\lambda,\gamma}(t^\theta)}dt, \quad s > 0,\ \gamma \in \mathcal{A},\ \lambda > 0, \tag{2.5}$$

it follows that

$$\eta_{\lambda,\theta}(\gamma) = \Gamma_{\lambda,\theta}(\gamma) > \|\omega_M\|_\infty + \bar\sigma, \tag{2.6}$$

for each $0 < \lambda < \Lambda^*(\gamma)$, where $\bar\sigma = \bar\sigma(\lambda,\theta,\gamma) = \big(\Gamma_{\lambda,\theta}(\gamma) - \|\omega_M\|_\infty\big)/2 > 0$.

Besides this, the following lemma follows from the previous results.

Lemma 2.4. *Suppose* (M1) *and* (F1) *hold. Then, for each* $0 < \lambda < \lambda_\theta^*(\Omega)$ *given:*

(i) $[\bar\sigma, \|\omega_M\|_\infty + \bar\sigma] \subset \operatorname{Im}(\eta_\lambda)$;
(ii) $\eta_\lambda \in C^2((0,\infty), \operatorname{Im}(\eta_\lambda))$ *is increasing in* $s > 0$;
(iii) $\eta_\lambda^{-1} := \psi_\lambda \in C^2((\operatorname{Im}(\eta_\lambda)\backslash\{0\}, (0,\infty))$ *is increasing in* $s > 0$;
(iv) $\psi_\lambda'(s) = \frac{\gamma H_{\lambda,\gamma_0}(\psi_\lambda(s)^\theta)}{\theta\psi_\lambda(s)^\theta}$, $s > 0$;
(v) $\psi_\lambda''(s) \le 0$, $s > 0$;
(vi) η_λ *is decreasing in* λ.

3. An auxiliary problem

To solve the problem (1.1) with the gradient term in the presence of nonlinearities f and g already described, we will explore the behavior of the auxiliary λ,γ,θ-functions given in the previous section considering different intervals of variation for $q \in [0,p]$ and an appropriate division of the domain $\Omega \subset \mathbb{R}^N$. All this together with the behavior of the Λ^*-curve will allow us to determine a μ^*-curve whose behavior will define the region of variation of the parameter $\mu \ge 0$.

As a consequence of the hypotheses (M1), (F1) and of the behavior of Λ^*, μ^*-curves, we obtain a γ_0 which allow us to show the existence of solution (ϵ-uniformly limited in $L^\infty(\Omega)$) of the ϵ-family of problems (3.1) below, for appropriate $\lambda > 0$ and $\mu \geq 0$.

In this sense, we will construct a positive bounded upper solution for the ϵ-family of problems

$$\begin{gathered} -\Delta_p u = a(x)f(u+\epsilon) + \lambda b(x)g(u+\epsilon) + \mu V(x,\nabla u) \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.1}$$

for sufficiently small $\epsilon > 0$.

Proposition 3.1. *Assume* (F1), (M1), (V1) *with* $q \in [0,p]$ *hold. Then there exists a* $\lambda^* > 0$ *such that for each* $0 < \lambda < \lambda^*$*, there exist real numbers* $\overline{\sigma} = \overline{\sigma}(\lambda) > 0$ *and* $\mu^* = \mu^*_\lambda > 0$*, both independent of* ϵ*, such that if* $0 < \sigma \leq \overline{\sigma}$ *and* $0 \leq \mu < \mu^*$*, then there exists a* $v_\sigma = v_{\sigma,\lambda} \in C^1(\overline{\Omega})$ *upper solution of* (3.1). *Additionally:*

(i) $\psi_\lambda(\sigma)^{\theta_0} \leq v_\sigma \leq \gamma_0^{\theta_0}$ *for some* $\theta_0 = \theta_0(\lambda) \in (\|w_M\|_\infty f_i^{1/(p-1)}, 1]$ *and* $\gamma_0 = \gamma_0(\lambda) > 0$*;*

(ii) *if* (F_i) *holds, for* $i \in \{0,\infty\}$*, then*

$$\lambda^* \geq \frac{1}{g_i}\Big(\frac{1}{\|\omega_M\|^{p-1}_{L^\infty(\Omega)}} - f_i\Big) := \lambda^i;$$

(iii) *there exists a constant* $d > 0$ *such that for* $0 < \lambda < \lambda^i$*, we have*

$$\mu^*_\lambda \geq d \min\big\{[f^i + \lambda g^i]^{\frac{p-1-q}{p-1}}, f^i + \lambda g^i\big\} \quad \textit{if } q \in [0, p-1].$$

Proof. Because the possible singular behavior of the nonlinearities, we divide this proof into two parts, depending on the value of the exponent q of the gradient term in the hypothesis (V1).

Case one: $q \in [0, p-1]$. In this case, we pick $\theta_0 = 1$ and take $\theta = \theta_0$ in the functions $\Gamma_{\lambda,\theta}$ and $\eta_{\lambda,\theta}$. So, given $0 < \lambda < \lambda^* := \lambda^*_1(\Omega)$ we define, for each $\gamma > 0$, the positive number

$$\mu^*_\lambda(\gamma) = \mu^*_{\lambda,\Omega}(\gamma) := \min\Big\{\frac{[f(\gamma)+\lambda g(\gamma)]^{\frac{p-1-q}{p-1}}}{4\|\nabla\omega_M\|^q_{L^\infty(\Omega)}}, \frac{f(\gamma)+\lambda g(\gamma)}{4}\Big\}. \tag{3.2}$$

Now, we can define

$$\mu^*_\lambda = \mu^*_{\lambda,\Omega} := \sup\{\mu^*_\lambda(\gamma) : \gamma \in \mathcal{A} \text{ and } \lambda < \Lambda^*(\gamma)\} \in (0,\infty]. \tag{3.3}$$

So, from (2.4), there exists $\overline{\gamma} \in \mathcal{A}$ such that $\lambda < \Lambda^*(\overline{\gamma})$. That is, $\mu^*_\lambda \geq \mu^*_\lambda(\overline{\gamma}) > 0$.

Thus, given $0 \leq \mu < \mu^*_\lambda$ there is a $\gamma_0 = \gamma_0(\lambda) \in \mathcal{A}$ such that $\lambda < \Lambda^*(\gamma_0)$ and $\mu < \mu^*_\lambda(\gamma_0)$. Now, we fix this γ_0.

From the hypothesis (M1) and Lemma 2.4 (ii), we define $v_\sigma = v_{\sigma,\lambda} \in C^1(\overline{\Omega})$, increasing in σ, by

$$v_\sigma(x) := \psi_\lambda(\omega_M(x) + \sigma), \ x \in \overline{\Omega} \tag{3.4}$$

for each $0 < \sigma \leq \bar{\sigma}$, where $\bar{\sigma} = \bar{\sigma}(\lambda)$ is given in (2.6). So, $v_\sigma(x) > \psi_\lambda(\sigma)$ in Ω and $v_\sigma(x) = \psi_\lambda(\sigma)$ on $\partial\Omega$, because $\omega_M(x) > 0$ in Ω and $\omega_M(x) = 0$ on $\partial\Omega$.

Besides this, from (2.6), Lemma 2.4 (iii) and $0 < \lambda < \Lambda^*(\gamma_0)$ we have that $v_{\overline{\sigma}}(x) < \gamma_0, \ x \in \overline{\Omega}$. So, there exists an $\epsilon > 0$, which is sufficiently small, such that

$$\|v_\sigma\|_{L^\infty(\Omega)} < \gamma_0 - \epsilon, \ 0 < \sigma \leq \bar{\sigma}. \tag{3.5}$$

Now, it follows from (3.4), Lemmas 2.1, 2.4 and the assumption (M1), that

$$
\begin{aligned}
&\int_\Omega |\nabla v_\sigma|^{p-2}\nabla v_\sigma \nabla\phi\,dx\\
&= \int_\Omega [\psi_\lambda'(\omega_M)+\sigma]^{p-1}|\nabla\omega_M|^{p-2}\nabla\omega_M\nabla\phi\,dx\\
&= \int_\Omega |\nabla\omega_M|^{p-2}\nabla\omega_M\nabla([\psi_\lambda'(\omega_M+\sigma)]^{p-1}\phi)\,dx\\
&\quad -(p-1)\int_\Omega |\nabla\omega_M|^p[\psi_\lambda'(\omega_M+\sigma)]^{p-2}\psi_\lambda''(\omega_M+\sigma)\phi\,dx\\
&\ge \int_\Omega M(x)[\psi_\lambda'(\omega_M+\sigma)]^{p-1}\phi\,dx\\
&= \int_\Omega M(x)\gamma_0^{p-1}\Big[\frac{H_{\lambda,\gamma_0}(\psi_\lambda(\omega_M+\sigma))}{\psi_\lambda(\omega_M+\sigma)}\Big]^{p-1}\phi\,dx
\end{aligned}
\tag{3.6}
$$

for each $\phi\in C_0^\infty(\Omega)$, $\phi\ge 0$.

The study of this inequality will be divided in two parts. One of them will produce an estimate for $af+\lambda bg$ while the other will result in an estimate for μV.

We note that from the definitions and properties of the functions defined in the Section 2 and (3.5) that

$$
\begin{aligned}
&\frac12\int_\Omega M(x)\gamma_0^{p-1}\Big[\frac{H_{\lambda,\gamma_0}(\psi_\lambda(\omega_M+\sigma))}{\psi_\lambda(\omega_M+\sigma)}\Big]^{p-1}\phi\\
&\ge \frac12\int_\Omega M(x)\gamma_0^{p-1}\frac{\zeta_{\lambda,\gamma_0}(v_\sigma+\epsilon)}{(v_\sigma+\epsilon)^{p-1}}\phi\\
&\ge \frac12\int_\Omega M(x)\gamma_0^{p-1}\frac{\zeta_{\lambda,\gamma_0}(v_\sigma+\epsilon)}{(\gamma_0)^{p-1}}\phi\\
&\ge \int_\Omega [a(x)f(v_\sigma+\epsilon)+\lambda b(x)g(v_\sigma+\epsilon)]\phi
\end{aligned}
\tag{3.7}
$$

for each $\epsilon>0$ and $0<\sigma<\bar\sigma$.

On the other hand, from Lemma 2.1 (iii)-(iv) and $0\le q\le p-1$, it follows that

$$
\begin{aligned}
&\frac12\int_\Omega M(x)\gamma_0^{p-1}\Big[\frac{H_{\lambda,\gamma_0}(\psi_\lambda(\omega_M+\sigma))}{\psi_\lambda(\omega_M+\sigma)}\Big]^{p-1}\phi\,dx\\
&\ge \frac14\int_\Omega M(x)\gamma_0^{p-1}\Big[\frac{f(\gamma_0)}{\gamma_0^{p-1}}+\lambda\frac{g(\gamma_0)}{\gamma_0^{p-1}}\Big]\phi\,dx\\
&\quad +\frac14\int_\Omega M(x)\gamma_0^{p-1-q}\Big[\frac{H_{\lambda,\gamma_0}(\psi_\lambda(\omega_M+\sigma))}{\psi_\lambda(\omega_M+\sigma)}\Big]^{p-1-q}\Big[\frac{\gamma_0 H_{\lambda,\gamma_0}(\psi_\lambda(\omega_M+\sigma))}{\psi_\lambda(\omega_M+\sigma)}\Big]^{q}\phi\,dx\\
&\ge \frac{[f(\gamma_0)+\lambda g(\gamma_0)]}{4}\int_\Omega M(x)\phi\,dx\\
&\quad +\frac{\{[f(\gamma_0)+\lambda g(\gamma_0)]^{\frac{1}{p-1}}\}^{p-1-q}}{4}\int_\Omega M(x)[\psi_\lambda'(\omega_M+\sigma)]^q\phi\,dx\\
&\ge \frac{[f(\gamma_0)+\lambda g(\gamma_0)]}{4}\int_\Omega \beta(x)\phi\,dx\\
&\quad +\frac{[f(\gamma_0)+\lambda g(\gamma_0)]^{\frac{p-1-q}{p-1}}}{4\|\nabla\omega_M\|^q_{L^\infty(\Omega)}}\int_\Omega M(x)[\psi_\lambda'(\omega_M+\sigma)]^q|\nabla\omega_M|^q\phi\,dx.
\end{aligned}
$$

Now, using (V1) and (3.2) we can write

$$\begin{aligned} &\frac{1}{2}\int_\Omega M(x)\gamma_0^{p-1}\Big[\frac{H_{\lambda,\gamma_0}(\psi_\lambda(\omega_M+\sigma))}{\psi_\lambda(\omega_M+\sigma)}\Big]^{p-1}\phi\,dx \\ &\geq \mu_\lambda^*(\gamma_0)\int_\Omega \beta(x)\phi\,dx + \mu_\lambda^*(\gamma_0)\int_\Omega \alpha(x)|\nabla v_\sigma|^q\phi\,dx \geq \mu\int_\Omega V(x,\nabla v_\sigma)\phi\,dx. \end{aligned} \tag{3.8}$$

So, replacing (3.7) and (3.8) in (3.6), we conclude the proof of Proposition 3.1.

Case two: $q \in (p-1,p]$. If $g_i < \infty$, we define $\lambda^* := \liminf_{\theta\nearrow 1}\lambda_\theta^*(\Omega)$, where $\lambda_\theta^*(\Omega)$ was defined in (2.4). Note that, by Lemma 2.3, we have $\lambda^* \geq \lambda_\theta^i$ with $\theta = 1$. So, given $0<\lambda<\lambda^*$, there is a $\theta_0 = \theta_0(\lambda) \in (\|w_M\|_\infty f_i^{1/(p-1)}, 1)$ such that $0<\lambda<\lambda_{\theta_0}^*(\Omega)$. Now, we fix this θ_0 in the functions $\Gamma_{\lambda,\theta}$ and $\eta_{\lambda,\theta}$ defined in (2.2) and (2.5), respectively. So, if $g_i = \infty$, we choose a $\theta_0 \in (\|w_M\|_\infty f_i^{1/(p-1)}, 1)$ and we set $\lambda^* = \lambda_{\theta_0}^*(\Omega)$. In this case, we have $\lambda^* \geq \lambda_{\theta_0}^i$.

In both cases, given $0<\lambda<\lambda^*$, we set the positive number $\mu_\lambda^*(\gamma) := \mu_{\lambda,\Omega}^*(\gamma)$ by

$$\begin{aligned} \min\Big\{&\frac{\gamma^{p-1-q}}{4C_2\|\nabla\omega_M\|_{L^\infty(\Omega)}^q}\frac{\gamma^{(p-1)(\theta_0-1)}[f(\gamma)+\lambda g(\gamma)]}{4}, \\ &\frac{(1-\theta_0)(p-1)[\gamma H_{\lambda,\gamma}(1)]^{p-q}}{4\|\alpha\|_{L^\infty(\Omega)}}\Big\} \end{aligned} \tag{3.9}$$

for each $\gamma>0$ and for some constant $C_2 = C_2(\gamma)>0$ to be chosen posteriorly. Now, we define

$$\mu_\lambda^* = \mu_{\lambda,\Omega}^* := \sup\big\{\mu_\lambda^*(\gamma) : \gamma \in \mathcal{A} \text{ and } \lambda < \Lambda^*(\gamma)\big\}.$$

As in Case one, we claim that $\mu_\lambda^*>0$ and given $0\leq\mu<\mu_\lambda^*$, there is a $\gamma_0 = \gamma_0(\lambda) \in \mathcal{A}$ such that $\lambda<\Lambda^*(\gamma_0)$ and $\mu<\mu_\lambda^*(\gamma_0)$. From now on, we fix this γ_0.

Since $\omega_M \in C^1(\overline{\Omega})$ and $\partial\omega_M/\partial\nu < 0$ on $\partial\Omega$, there are $\delta_0>0$ sufficiently small and $k_0 = k_0(\delta_0)>0$ such that

$$|\nabla\omega_M|^p > k_0(\delta_0) \quad \text{for } x \in \Omega_{\delta_0}, \tag{3.10}$$

where $\Omega_{\delta_0} = \{x\in\Omega : \operatorname{dist}(x,\partial\Omega)<\delta_0\}$ and ν is the exterior normal to the $\partial\Omega$.

In a similar way to that done in (3.4), we obtain that

$$v_\sigma(x) := [\psi_\lambda(\omega_M(x)+\sigma)]^{\theta_0}, \quad x\in\overline{\Omega} \tag{3.11}$$

is well-defined, $\psi_\lambda(\sigma)^{\theta_0} \leq v_\sigma \in C^1(\overline{\Omega})$ and $\|v_\sigma\|_{L^\infty(\Omega)} < \gamma_0^{\theta_0}$, for each $0<\sigma\leq\bar\sigma$. In the last conclusion, we used Lemma 2.4 and the inequality (2.6).

That is, there is a sufficiently small $\epsilon>0$ such that

$$\|v_\sigma\|_{L^\infty(\Omega)} < \gamma_0^{\theta_0} - \epsilon. \tag{3.12}$$

Since $\lim_{s\to 0}\psi_\lambda(s) = 0$, we can take $0<\tilde\sigma<\bar\sigma$ sufficiently small such that

$$\psi_\lambda(\tilde\sigma)^{\theta_0} < \frac{1}{2} \quad \text{and} \quad \frac{k_0(\delta_0)}{2\psi_\lambda(\tilde\sigma)^{\theta_0}} > \|\nabla\omega_M\|_{L^\infty(\Omega)}^q. \tag{3.13}$$

So, from Lemma 2.4 (iii), it follows that $v_\sigma(x) < v_{\tilde\sigma}(x)$ in $\overline{\Omega}$, for each $0<\sigma<\tilde\sigma$. Moreover, since $v_{\tilde\sigma}(x) = \psi_\lambda(\tilde\sigma)^{\theta_0}$ on $\partial\Omega$, it follows from Lemma 2.4 (iii) again, that there exists a $\delta_1 = \delta_1(\tilde\sigma)>0$ sufficiently small such that

$$v_\sigma(x) < v_{\tilde\sigma}(x) < 2\psi_\lambda(\tilde\sigma)^{\theta_0}, \quad \text{for } x\in\Omega_{\delta_1},\ \sigma\in(0,\tilde\sigma). \tag{3.14}$$

Then, from (3.10), (3.13) and (3.14), we have

$$\frac{|\nabla\omega_M(x)|^p}{v_\sigma(x)} > \frac{k_0(\delta_0)}{2\psi_\lambda(\tilde\sigma)^{\theta_0}} > |\nabla\omega_M(x)|^q, \tag{3.15}$$

for each $x\in\Omega_\delta$, where $\delta=\min\{\delta_0,\delta_1\}>0$.

Now, given $\phi\in C_0^\infty(\Omega)$ with $\phi\geq 0$ and $0<\sigma<\tilde\sigma$, we take $\tau\in C_0^\infty(\Omega)$ defined by $\tau=1$ in $\Omega\backslash\Omega_\delta$ and $\tau=0$ in $\Omega_{\delta/2}$ with $0\leq\tau\leq 1$. So, writing $\phi=\tau\phi+(1-\tau)\phi$, we have that

$$\int_\Omega |\nabla v_\sigma|^{p-2}\nabla v_\sigma\nabla\phi = \int_{\Omega\backslash\Omega_{\delta/2}}|\nabla v_\sigma|^{p-2}\nabla v_\sigma\nabla(\tau\phi)+\int_{\Omega_\delta}|\nabla v_\sigma|^{p-2}\nabla v_\sigma\nabla(1-\tau)\phi. \tag{3.16}$$

In $\Omega\backslash\Omega_{\delta/2}$, it follows from the definition of v_σ, that

$$\begin{aligned}
&\int_{\Omega\backslash\Omega_{\delta/2}}|\nabla v_\sigma|^{p-2}\nabla v_\sigma\nabla(\tau\phi)\\
&=\int_{\Omega\backslash\Omega_{\delta/2}}|\nabla\omega_M|^{p-2}\nabla\omega_M\nabla\big\{{\theta_0}^{p-1}[\psi_\lambda(\omega_M+\sigma)]^{(\theta_0-1)(p-1)}[\psi_\lambda'(\omega_M+\sigma)]^{p-1}\tau\phi\big\}\\
&\quad-(\theta_0-1)(p-1)\int_{\Omega\backslash\Omega_{\delta/2}}|\nabla\omega_M|^p{\theta_0}^{p-1}\\
&\quad\times[\psi_\lambda(\omega_M+\sigma)]^{(\theta_0-1)(p-1)-1}[\psi_\lambda'(\omega_M+\sigma)]^p\tau\phi\\
&\quad-(p-1)\int_{\Omega\backslash\Omega_{\delta/2}}|\nabla\omega_M|^p{\theta_0}^{p-1}[\psi_\lambda(\omega_M+\sigma)]^{(\theta_0-1)(p-1)}\\
&\quad\times[\psi_\lambda'(\omega_M+\sigma)]^{p-2}\psi_\lambda''(\omega_M+\sigma)\tau\phi
\end{aligned}$$

Now, recalling that $\theta_0\in(\|w_M\|_\infty f_i^{1/(p-1)},1)$, $\psi_\lambda'\geq 0$, $\psi_\lambda''\leq 0$ (see Lemma 2.4) and noting that

$${\theta_0}^{p-1}[\psi_\lambda(\omega_M+\sigma)]^{(\theta_0-1)(p-1)}[\psi_\lambda'(\omega_M+\sigma)]^{p-1}\tau\phi\in W_0^{1,p}(\Omega),$$

it follows from (M1) and Lemma 2.4 (iv) that

$$\begin{aligned}
&\int_{\Omega\backslash\Omega_{\delta/2}}|\nabla v_\sigma|^{p-2}\nabla v_\sigma\nabla(\tau\phi)dx\\
&\geq\int_{\Omega\backslash\Omega_{\delta/2}}|\nabla\omega_M|^{p-2}\nabla\omega_M\nabla\big\{{\theta_0}^{p-1}[\psi_\lambda(\omega_M+\sigma)]^{(\theta_0-1)(p-1)}\\
&\quad\times[\psi_\lambda'(\omega_M+\sigma)]^{p-1}\tau\phi\big\}\\
&\geq\int_{\Omega\backslash\Omega_{\delta/2}}M(x){\theta_0}^{p-1}[\psi_\lambda(\omega_M+\sigma)]^{(\theta_0-1)(p-1)}[\psi_\lambda'(\omega_M+\sigma)]^{p-1}\tau\phi\\
&=\int_{\Omega\backslash\Omega_{\delta/2}}M(x){\theta_0}^{p-1}v_\sigma^{\frac{(\theta_0-1)(p-1)}{\theta_0}}\frac{\gamma_0^{p-1}}{\theta_0^{p-1}}\Big[\frac{H_{\lambda,\gamma_0}((\psi_\lambda(\omega_M+\sigma))^{\theta_0})}{(\psi_\lambda(\omega_M+\sigma))^{\theta_0}}\Big]^{p-1}\tau\phi.
\end{aligned}\tag{3.17}$$

As in Case one, the analysis of this inequality will be divided in two parts. So, from the properties of auxiliary functions, Lemma 2.1 (ii) and (3.12), we have

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega\setminus\Omega_{\delta/2}} |\nabla v_\sigma|^{p-2}\nabla v_\sigma \nabla\tau\phi\,dx \\
&\geq \frac{1}{2}\int_{\Omega\setminus\Omega_{\delta/2}} M(x)\gamma_0^{(\theta_0-1)(p-1)}\gamma_0^{p-1}\frac{\zeta_{\lambda,\gamma_0}(v_\sigma+\epsilon)}{(v_\sigma+\epsilon)^{p-1}}\tau\phi \\
&\geq \frac{1}{2}\int_{\Omega\setminus\Omega_{\delta/2}} M(x)\gamma_0^{(p-1)\theta_0}\frac{\zeta_{\lambda,\gamma_0}(v_\sigma+\epsilon)}{\gamma_0^{\theta_0(p-1)}}\tau\phi \\
&\geq \frac{1}{2}\int_{\Omega\setminus\Omega_{\delta/2}} M(x)[f_{\gamma_0}(v_\sigma+\epsilon)+\lambda g_{\gamma_0}(v_\sigma+\epsilon)]\tau\phi \\
&\geq \int_{\Omega\setminus\Omega_{\delta/2}} [a(x)f(v_\sigma+\epsilon)+\lambda b(x)g(v_\sigma+\epsilon)]\tau\phi,
\end{aligned}
\tag{3.18}
$$

for each $\lambda\in(0,\lambda^*)$, $\sigma\in(0,\tilde\sigma)$, $\epsilon>0$.

Now, denoting by

$$
v(x) := \lim_{\sigma\to 0} v_\sigma(x) = [\psi_\lambda(\omega_M(x))]^{\theta_0}, \quad x\in\overline{\Omega}, \tag{3.19}
$$

it follows from Lemma 2.1 (iii), $v_\sigma > v > 0$ in $\Omega\setminus\Omega_{\delta/2}$ and $q\in(p-1,p]$ that

$$
\begin{aligned}
\Big[\frac{H_{\lambda,\gamma_0}(v_\sigma)}{v_\sigma}\Big]^{q-(p-1)} &\leq \Big[\frac{H_{\lambda,\gamma_0}(v)}{v}\Big]^{q-(p-1)} \\
&\leq \Big\|\frac{H_{\lambda,\gamma_0}(v)}{v}\Big\|^{q-(p-1)}_{L^\infty(\Omega\setminus\Omega_{\delta/2})} \\
&= C_2\Big[\min_{\overline{\Omega}\setminus\Omega_{\delta/2}} v\Big]^{\frac{(\theta_0-1)(p-1-q)}{\theta_0}} \\
&\leq C_2 v^{\frac{(\theta_0-1)(p-1-q)}{\theta_0}} \\
&< C_2 v_\sigma^{\frac{(\theta_0-1)(p-1-q)}{\theta_0}}, \quad \text{for all } x\in\overline{\Omega\setminus\Omega}_{\delta/2},
\end{aligned}
\tag{3.20}
$$

where

$$
C_2 = \Big\|\frac{H_{\lambda,\gamma_0}(v)}{v}\Big\|^{q-(p-1)}_{L^\infty(\Omega\setminus\Omega_{\delta/2})}\Big/\Big[\min_{\overline{\Omega}\setminus\Omega_{\delta/2}} v\Big]^{\frac{(\theta_0-1)(p-1-q)}{\theta_0}} > 0
$$

is independent of σ.

Now we show that

$$
\begin{aligned}
\frac{1}{2}\int_{\Omega\setminus\Omega_{\delta/2}} |\nabla v_\sigma|^{p-2}\nabla v_\sigma\nabla\tau\phi\,dx &\geq \frac{\gamma_0^{(p-1)(\theta_0-1)}[f(\gamma_0)+\lambda g(\gamma_0)]}{4}\int_{\Omega\setminus\Omega_{\delta/2}}\beta(x)\tau\phi\,dx \\
&\quad + \frac{\gamma_0^{p-1-q}}{4C_2\|\nabla\omega_M\|^q_{L^\infty(\Omega)}}\int_{\Omega\setminus\Omega_{\delta/2}}\alpha(x)|\nabla v_\sigma|^q]\tau\phi\,dx
\end{aligned}
$$

and as a consequence of this, using (3.9), we obtain

$$
\frac{1}{2}\int_{\Omega\setminus\Omega_{\delta/2}} |\nabla v_\sigma|^{p-2}\nabla v_\sigma\nabla\tau\phi\,dx \geq \mu\int_{\Omega\setminus\Omega_{\delta/2}} V(x,\nabla v_\sigma)\tau\phi\,dx
$$

for each $0\leq\mu<\mu^*_\lambda$.

By (3.20) and Lemma 2.1 (iii)-(iv) in (3.17), we have

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega\setminus\Omega_{\delta/2}} |\nabla v_\sigma|^{p-2}\nabla v_\sigma \nabla\tau\phi\,dx\\
&\geq \frac{1}{2}\int_{\Omega\setminus\Omega_{\delta/2}} M(x) v_\sigma^{\frac{(\theta_0-1)(p-1)}{\theta_0}}\gamma_0^{p-1}\Big[\frac{H_{\lambda,\gamma_0}(v_\sigma)}{v_\sigma}\Big]^{p-1}\tau\phi\\
&\geq \frac{1}{4}\int_{\Omega\setminus\Omega_{\delta/2}} M(x)\gamma_0^{(\theta_0-1)(p-1)}\gamma_0^{p-1}\Big[\frac{f(\gamma_0)}{\gamma_0^{p-1}}+\lambda\frac{g(\gamma_0)}{\gamma_0^{p-1}}\Big]\tau\phi\\
&\quad+\frac{1}{4}\int_{\Omega\setminus\Omega_{\delta/2}} M(x) v_\sigma^{\frac{(\theta_0-1)(p-1-q)}{\theta_0}}\gamma_0^{p-1-q}\Big[\frac{H_{\lambda,\gamma_0}(v_\sigma)}{v_\sigma}\Big]^{p-1}\frac{\theta_0^q}{\theta_0^q} v_\sigma^{\frac{(\theta_0-1)q}{\theta_0}}\gamma_0^q\tau\phi\\
&\geq \frac{\gamma_0^{(p-1)\theta_0}\big[\frac{f(\gamma_0)}{\gamma_0^{p-1}}+\lambda\frac{g(\gamma_0)}{\gamma_0^{p-1}}\big]}{4}\int_{\Omega\setminus\Omega_{\delta/2}} M(x)\tau\phi\\
&\quad+\frac{\gamma_0^{p-1-q}}{4C_2}\int_{\Omega\setminus\Omega_{\delta/2}} M(x)\theta_0^q v_\sigma^{\frac{(\theta_0-1)q}{\theta_0}}\frac{\gamma_0^q}{\theta_0^q}\Big[\frac{H_{\lambda,\gamma_0}(v_\sigma)}{v_\sigma}\Big]^q\tau\phi.
\end{aligned}
$$

Using (3.9), Lemma 2.4 (iv), the definition of M and (V1), we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega\setminus\Omega_{\delta/2}} |\nabla v_\sigma|^{p-2}\nabla v_\sigma \nabla\tau\phi\,dx\\
&\geq \frac{\gamma_0^{(p-1)(\theta_0-1)}[f(\gamma_0)+\lambda g(\gamma_0)]}{4}\int_{\Omega\setminus\Omega_{\delta/2}} M(x)\tau\phi\,dx\\
&\quad+\mu_\lambda^*(\gamma_0)\|\nabla\omega_M\|_{L^\infty(\Omega)}^q\int_{\Omega\setminus\Omega_{\delta/2}} M(x)[\theta_0\psi_\lambda(\omega_M+\sigma)^{\theta_0-1}\psi_\lambda'(\omega_M+\sigma)]^q\tau\phi\,dx\\
&\geq \frac{\gamma_0^{(p-1)(\theta_0-1)}[f(\gamma_0)+\lambda g(\gamma_0)]}{4}\int_{\Omega\setminus\Omega_{\delta/2}} \beta(x)\tau\phi\,dx\\
&\quad+\mu_\lambda^*(\gamma_0)\int_{\Omega\setminus\Omega_{\delta/2}} \alpha(x)[\theta_0\psi_\lambda(\omega_M+\sigma)^{\theta_0-1}\psi_\lambda'(\omega_M+\sigma)|\nabla\omega_M|]^q\tau\phi\,dx\\
&\geq \mu_\lambda^*(\gamma_0)\int_{\Omega\setminus\Omega_{\delta/2}} [\beta(x)+\alpha(x)|\nabla v_\sigma|^q]\tau\phi\,dx\\
&\geq \mu\int_{\Omega\setminus\Omega_{\delta/2}} V(x,\nabla v_\sigma)\tau\phi\,dx.
\end{aligned}
\tag{3.21}
$$

Going back to (3.17) and using (3.18) and (3.21), we obtain

$$
\begin{aligned}
&\int_{\Omega\setminus\Omega_{\delta/2}} |\nabla v_\sigma|^{p-2}\nabla v_\sigma \nabla\tau\phi\,dx\\
&\geq \int_{\Omega\setminus\Omega_{\delta/2}} [a(x)f(v_\sigma+\epsilon)+\lambda b(x)g(v_\sigma+\epsilon)+\mu V(x,\nabla v_\sigma)]\tau\phi,
\end{aligned}
\tag{3.22}
$$

for each $0<\lambda<\lambda^*$, $0\leq\mu<\mu_\lambda^*$, $\epsilon>0$.

Below we work on the ring Ω_δ. As before, using the definition of v_σ, it follows that

$$
\begin{aligned}
&\int_{\Omega_\delta} |\nabla v_\sigma|^{p-2}\nabla v_\sigma \nabla(1-\tau)\phi\,dx \\
&= \int_{\Omega_\delta} |\nabla \omega_M|^{p-2}\nabla\omega_M\nabla\big\{{\theta_0}^{p-1}[\psi_\lambda(\omega_M+\sigma)]^{(\theta_0-1)(p-1)} \\
&\quad \times [\psi_\lambda'(\omega_M+\sigma)]^{p-1}(1-\tau)\phi\big\} \\
&\quad - (\theta_0-1)(p-1)\int_{\Omega_\delta} |\nabla\omega_M|^p{\theta_0}^{p-1}[\psi_\lambda(\omega_M+\sigma)]^{(\theta_0-1)(p-1)-1} \\
&\quad \times [\psi_\lambda'(\omega_M+\sigma)]^{p}(1-\tau)\phi \\
&\quad - (p-1)\int_{\Omega_\delta} |\nabla\omega_M|^p{\theta_0}^{p-1}[\psi_\lambda(\omega_M+\sigma)]^{(\theta_0-1)(p-1)} \\
&\quad [\psi_\lambda'(\omega_M+\sigma)]^{p-2}\psi_\lambda''(\omega_M+\sigma)(1-\tau)\phi.
\end{aligned} \tag{3.23}
$$

In a way similar to the one for (3.18), we have

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega_\delta} |\nabla v_\sigma|^{p-2}\nabla v_\sigma \nabla(1-\tau)\phi\,dx \\
&\geq \frac{{\theta_0}^{p-1}}{2}\int_{\Omega_\delta} M(x)[\psi_\lambda(\omega_M+\sigma)]^{(\theta_0-1)(p-1)}[\psi_\lambda'(\omega_M+\sigma)]^{p-1}(1-\tau)\phi\,dx \\
&\geq \int_{\Omega_\delta} [a(x)f(v_\sigma+\epsilon)+\lambda b(x)g(v_\sigma+\epsilon)](1-\tau)\phi\,dx,
\end{aligned} \tag{3.24}
$$

for each $\lambda \in (0,\lambda^*)$, $\sigma\in(0,\tilde\sigma)$, $\epsilon>0$. Besides this, we will show that

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega_\delta} |\nabla v_\sigma|^{p-2}\nabla v_\sigma \nabla(1-\tau)\phi\,dx \\
&\geq \frac{(1-\theta_0)(p-1)[\gamma_0 H_{\lambda,\gamma_0}(1)]^{p-q}}{4\|\alpha\|_\infty}\int_{\Omega_\delta} \alpha(x)|\nabla v_\sigma|^q(1-\tau)\phi\,dx \\
&\quad + \frac{\gamma_0^{(p-1)(\theta_0-1)}[f(\gamma_0)+\lambda g(\gamma_0)]}{4}\int_{\Omega_\delta}\beta(x)(1-\tau)\phi\,dx
\end{aligned}
$$

and as a consequence of this, using (3.9), we obtain

$$
\frac{1}{2}\int_{\Omega_\delta} |\nabla v_\sigma|^{p-2}\nabla v_\sigma \nabla(1-\tau)\phi\,dx \geq \mu\int_{\Omega_\delta} V(x,\nabla v_\sigma)(1-\tau)\phi\,dx
$$

for each $0\leq\mu<\mu_\lambda^*$.

In fact, from the properties of the auxiliary functions and (M1), we have

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega_\delta} |\nabla v_\sigma|^{p-2}\nabla v_\sigma \nabla(1-\tau)\phi\,dx \\
&\geq -\frac{(\theta_0-1)(p-1)}{4}\int_{\Omega_\delta} |\nabla\omega_M|^p\theta_0^{p-1}[\psi_\lambda(\omega_M+\sigma)]^{(\theta_0-1)(p-1)-1} \\
&\quad \times [\psi_\lambda'(\omega_M+\sigma)]^{p}(1-\tau)\phi \\
&\quad + \frac{1}{4}\int_{\Omega_\delta} |\nabla \omega_M|^{p-2}\nabla\omega_M\nabla\{{\theta_0}^{p-1}[\psi_\lambda(\omega_M+\sigma)]^{(\theta_0-1)(p-1)} \\
&\quad \times [\psi_\lambda'(\omega_M+\sigma)]^{p-1}(1-\tau)\phi\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(1-\theta_0)(p-1)}{4}\int_{\Omega_\delta} |\nabla\omega_M|^p \theta_0^{p-1} v_\sigma^{\frac{p(\theta_0-1)-\theta_0}{\theta_0}} \frac{\gamma_0^p}{\theta_0^p}\Big[\frac{H_{\lambda,\gamma_0}(v_\sigma)}{v_\sigma}\Big]^p (1-\tau)\phi \\
&\quad + \frac{1}{4}\int_{\Omega_\delta} M(x){\theta_0}^{p-1}[\psi_\lambda(\omega_M+\sigma)]^{(\theta_0-1)(p-1)}[\psi'_\lambda(\omega_M+\sigma)]^{p-1}(1-\tau)\phi.
\end{aligned}
$$

That is, from (3.15) and Lemma 2.1, we have

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega_\delta} |\nabla v_\sigma|^{p-2}\nabla v_\sigma \nabla(1-\tau)\phi\,dx \\
&= \frac{(1-\theta_0)(p-1)}{4}\int_{\Omega_\delta} \frac{|\nabla\omega_M|^p}{v_\sigma}\frac{\theta_0^{p-1}}{\theta_0^{p-1}} v_\sigma^{\frac{p(\theta_0-1)}{\theta_0}} \gamma_0^p\Big[\frac{H_{\lambda,\gamma_0}(v_\sigma)}{v_\sigma}\Big]^p (1-\tau)\phi\,dx \\
&\quad + \frac{1}{4}\int_{\Omega_\delta} M(x){\theta_0}^{p-1} v_\sigma^{\frac{(\theta_0-1)(p-1)}{\theta_0}} \frac{\gamma_0^{p-1}}{\theta_0^{p-1}}\Big[\frac{H_{\lambda,\gamma_0}(v_\sigma)}{v_\sigma}\Big]^p (1-\tau)\phi\,dx \\
&\ge \frac{(1-\theta_0)(p-1)}{4}\int_{\Omega_\delta} |\nabla\omega_M|^q v_\sigma^{\frac{p(\theta_0-1)}{\theta_0}} \gamma_0^p\Big[\frac{H_{\lambda,\gamma_0}(v_\sigma)}{v_\sigma}\Big]^p (1-\tau)\phi\,dx \\
&\quad + \frac{1}{4}\int_{\Omega_\delta} M(x)\gamma_0^{(\theta_0-1)(p-1)}\gamma_0^{p-1}\Big[\frac{f(\gamma_0)}{\gamma_0^{p-1}}+\lambda\frac{g(\gamma_0)}{\gamma_0^{p-1}}\Big](1-\tau)\phi\,dx.
\end{aligned} \tag{3.25}
$$

Using that $v_\sigma < 1$ in Ω_δ (see (3.13)), $q < p$ and $\theta_0 < 1$, we obtain

$$
[v_\sigma(x)]^{\frac{(\theta_0-1)p}{\theta_0}} > [v_\sigma(x)]^{\frac{(\theta_0-1)q}{\theta_0}}, \quad \text{for each } x\in\Omega_\delta \tag{3.26}
$$

and from Lemma 2.1, we have

$$
[H_{\lambda,\gamma_0}(1)]^{p-q}\Big[\frac{H_{\lambda,\gamma_0}(v_\sigma)}{v_\sigma}\Big]^q \le \Big[\frac{H_{\lambda,\gamma_0}(v_\sigma)}{v_\sigma}\Big]^p,\ x\in\Omega_\delta. \tag{3.27}
$$

From (3.26) and (3.27), we rewrite (3.25) as

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega_\delta} |\nabla v_\sigma|^{p-2}\nabla v_\sigma \nabla(1-\tau)\phi\,dx \\
&\ge \frac{(1-\theta_0)(p-1)}{4}\int_{\Omega_\delta} |\nabla\omega_M|^q v_\sigma^{\frac{(\theta_0-1)q}{\theta_0}} \gamma_0^{p-q}\gamma_0^q\frac{\theta_0^q}{\theta_0^q}[H_{\lambda,\gamma_0}(1)]^{p-q} \\
&\quad \times \Big[\frac{H_{\lambda,\gamma_0}(v_\sigma)}{v_\sigma}\Big]^q (1-\tau)\phi\,dx + \frac{\gamma_0^{(p-1)\theta_0}[\frac{f(\gamma_0)}{\gamma_0^{p-1}}+\lambda\frac{g(\gamma_0)}{\gamma_0^{p-1}}]}{4}\int_{\Omega_\delta} M(x)(1-\tau)\phi\,dx \\
&= \frac{(1-\theta_0)(p-1)[\gamma_0 H_{\lambda,\gamma_0}(1)]^{p-q}}{4\|\alpha\|_\infty}\int_{\Omega_\delta} \|\alpha\|_\infty \theta_0^q v_\sigma^{\frac{(\theta_0-1)q}{\theta_0}}\Big[\frac{\gamma_0 H_{\lambda,\gamma_0}(v_\sigma)}{\theta_0 v_\sigma}\Big]^q \\
&\quad \times |\nabla\omega_M|^q(1-\tau)\phi + \frac{\gamma_0^{(p-1)(\theta_0-1)}[f(\gamma_0)+\lambda g(\gamma_0)]}{4}\int_{\Omega_\delta} M(x)(1-\tau)\phi\,dx.
\end{aligned}
$$

From Lemma 2.4 (iv), (3.9) and definitions of v_σ, M and (V1), we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega_\delta} |\nabla v_\sigma|^{p-2}\nabla v_\sigma \nabla(1-\tau)\phi\,dx \\
&\geq \mu_\lambda^*(\gamma_0)\|\alpha\|_{L^\infty(\Omega)}\int_{\Omega_\delta}|\nabla v_\sigma|^q(1-\tau)\phi\,dx + \mu_\lambda^*(\gamma_0)\int_{\Omega_\delta} M(x)(1-\tau)\phi \\
&\geq \mu_\lambda^*(\gamma_0)\int_{\Omega_\delta}[\alpha(x)|\nabla v_\sigma|^q + \beta(x)](1-\tau)\phi\,dx \\
&\geq \mu\int_{\Omega_\delta} V(x,\nabla v_\sigma)(1-\tau)\phi\,dx.
\end{aligned}
\tag{3.28}
$$

for each $0 \leq \mu < \mu_\lambda^*$.

Considering (3.23) and using (3.24) and (3.28), we have

$$
\begin{aligned}
&\int_{\Omega_\delta} |\nabla v_\sigma|^{p-2}\nabla v_\sigma \nabla(1-\tau)\phi \\
&\geq \int_{\Omega_\delta}[a(x)f(v_\sigma+\epsilon) + \lambda b(x)g(v_\sigma+\epsilon) + \mu V(x,\nabla v_\sigma)](1-\tau)\phi,
\end{aligned}
\tag{3.29}
$$

for each $0 < \lambda < \lambda^*$, $0 \leq \mu < \mu_\lambda^*$, $0 < \sigma < \tilde{\sigma}$ and $\epsilon > 0$.

Therefore, replacing (3.22) and (3.29) in (3.16), we conclude the proof of the existence of a upper solution for Proposition 3.1.

To finalize the proof of the proposition, we need to verify the estimate for μ^*. Assume (F_0). So, from Lemma 2.2 (ii), we have

$$
\lim_{\gamma\to 0}\Gamma_{0,1}(\gamma) = f_0^{\frac{-1}{p-1}} > \|\omega_M\|_{L^\infty(\Omega)}
$$

and a consequence of this there exists a $\tilde{\gamma} > 0$ sufficiently small such that $(0,\tilde{\gamma}) \subset \mathcal{A}$.

Given $0 < \lambda < \lambda^0$, where $\lambda^0 = \lambda^i$ with $i = 0$ (λ^i was defined in Theorem 1.2), we claim that there exists a $\gamma_0 < \tilde{\gamma}$ such that $\lambda < \Lambda^*(\gamma)$ for all $0 < \gamma < \gamma_0$. In fact, from $\lambda < \lambda^0$ and Lemma 2.2 (ii) we have

$$
\lim_{\gamma\to 0}\Gamma_{\lambda,1}(\gamma) = \frac{1}{(f_0+\lambda g_0)^{\frac{1}{p-1}}} > \|\omega_M\|_{L^\infty(\Omega)}.
$$

So, there exists a $\gamma_0 < \tilde{\gamma}$ such that $\Gamma_{\lambda,1}(\gamma) > \|\omega_M\|_{L^\infty(\Omega)} = \Gamma_{\Lambda^*(\gamma),1}(\gamma)$ for $0 < \gamma < \gamma_0$. Now, by Lemma 2.2 (iii), we obtain $\lambda < \Lambda^*(\gamma)$, for all $\gamma \in (0,\gamma_0)$. From (3.2) and (3.3) we have

$$
\begin{aligned}
\mu_\lambda^* &\geq \sup\{\mu_\lambda^*(\gamma) : \gamma \in (0,\gamma_0) \text{ and } \lambda < \Lambda^*(\gamma)\} \\
&\geq \liminf_{\gamma\to 0}\mu_\lambda^*(\gamma) = \min\Big\{\frac{[f^0+\lambda g^0]^{p-1-q}}{4\|\nabla\omega_M\|^q_{L^\infty(\Omega)}}, \frac{f^0+\lambda g^0}{4}\Big\}.
\end{aligned}
$$

If (F_∞) occurs, we proceed in a similar manner to the above case. We point out that, in this case, γ_0 is large. This completes the proof of Proposition 3.1. □

4. Conclusion of the proof of Theorem 1.2

We begin by constructing a lower solution for problem (3.1). It follows from the definition of λ_* that given $\lambda > \lambda_*$, there exists a $0 < \epsilon_1 \leq \min\{\gamma_0, \gamma_0^{\theta_0}\}$ such that

$$
f(s) + \lambda g(s) \geq \lambda_\Omega(\rho)s^{p-1}, \quad \text{for } 0 < s < \epsilon_1.
$$

Taking $C = C(\Omega, \epsilon_1) > 0$ such that $C\|\varphi_\Omega\|_{L^\infty(\Omega)} = \epsilon_1/2$, it follows that

$$C\|\varphi_\Omega\|_{L^\infty(\Omega)} + \epsilon < C\|\varphi_\Omega\|_{L^\infty(\Omega)} + \epsilon_1/2 = \epsilon_1 \tag{4.1}$$

for each $0 < \epsilon < \epsilon_1/2$, where $\varphi_\Omega > 0$ is the eigenfunction associated to the first eigenvalue $\lambda_\Omega > 0$ of problem (1.7).

Thus, given $\phi \in C_0^\infty(\Omega)$ with $\phi \geq 0$, we obtain

$$\int_\Omega |\nabla(C\varphi_\Omega)|^{p-2}\nabla(C\varphi_\Omega)\nabla\phi\,dx \leq \int_\Omega [\lambda b(x)g(C\varphi_\Omega + \epsilon) + a(x)f(C\varphi_\Omega + \epsilon)]\phi\,dx;$$

that is, $C\varphi_\Omega$ is a lower solution of (3.1) for each $0 < \epsilon < \epsilon_1/2$, $0 < \lambda < \lambda^*$ and $0 < \mu < \mu^*_\lambda$, because of the positivity of V.

Now, we claim that

$$C\varphi(x) \leq v_\sigma(x), \quad x \in \overline{\Omega}. \tag{4.2}$$

First, we consider $q \in [0, p-1]$. In this case, $v_\sigma = \psi_\lambda(\omega_M + \sigma)$ is defined in (3.4). So, from (4.1) and (2.1), for all $\phi \in C_0^\infty(\Omega)$, $\phi \geq 0$, we have

$$\begin{aligned}
&\int_\Omega |\nabla(C\varphi_\Omega)|^{p-2}\nabla(C\varphi_\Omega)\nabla\phi\,dx \\
&\leq \int_\Omega \Big[\gamma_0^{p-1}a(x)\frac{f(C\varphi_\Omega+\epsilon)}{(C\varphi_\Omega+\epsilon)^{p-1}} + \gamma_0^{p-1}\lambda b(x)\frac{g(C\varphi_\Omega+\epsilon)}{(C\varphi_\Omega+\epsilon)^{p-1}}\Big]\phi\,dx \\
&\leq \int_\Omega \Big[\gamma_0^{p-1}a(x)\frac{f_{\gamma_0}(C\varphi_\Omega+\epsilon)}{(C\varphi_\Omega+\epsilon)^{p-1}} + \gamma_0^{p-1}\lambda b(x)\frac{g_{\gamma_0}(C\varphi_\Omega+\epsilon)}{(C\varphi_\Omega+\epsilon)^{p-1}}\Big]\phi\,dx \\
&= \int_\Omega M(x)\gamma_0^{p-1}\frac{\zeta_{\lambda,\gamma_0}(C\varphi_\Omega+\epsilon)}{(C\varphi_\Omega+\epsilon)^{p-1}}\phi\,dx \\
&\leq \int_\Omega M(x)\gamma_0^{p-1}\frac{\zeta_{\lambda,\gamma_0}(C\varphi_\Omega)}{(C\varphi_\Omega)^{p-1}}\phi\,dx.
\end{aligned} \tag{4.3}$$

Moreover, from (3.6) and Lemma 2.1, we have

$$\int_\Omega |\nabla v_\sigma)|^{p-2}\nabla v_\sigma\nabla\phi\,dx \geq \int_\Omega M(x)\gamma_0^{p-1}\frac{\zeta_{\lambda,\gamma_0}(v_\sigma)}{v_\sigma^{p-1}}\phi\,dx, \tag{4.4}$$

for all $\phi \in C_0^\infty(\Omega), \phi \geq 0$. So, from (4.3), (4.4), $\zeta_{\lambda,\gamma_0}(s)/s^{p-1}$ non-increasing in $s > 0$ and $C\varphi_\Omega = 0 < \psi_\lambda(\sigma) = v_\sigma$ on $\partial\Omega$, we apply a comparison principle for weak solutions (see Tolksdorf [23]) to obtain (4.2).

In the second case, that is $q \in (p-1, p]$, we recall that $v_\sigma = [\psi_\lambda(\omega_M + \sigma)]^{\theta_0}$, where $\theta_0 \in (\|w_M\|_\infty f_i^{1/(p-1)}, 1)$, see (3.11). In a similar way to the first case (that is, $q \in [0, p-1]$), we obtain

$$\int_\Omega |\nabla v_\sigma|^{p-2}\nabla v_\sigma\nabla\phi\,dx \geq \int_\Omega M(x)\gamma_0^{(p-1)\theta_0}\frac{\zeta_{\lambda,\gamma_0}(v_\sigma)}{v_\sigma^{p-1}}\phi\,dx, \tag{4.5}$$

for all $\phi \in C_0^\infty(\Omega)$ with $\phi \geq 0$.

From (4.1), definitions and properties of auxiliary functions ξ_{f,γ_0}, ξ_{g,γ_0} and ξ_{λ,γ_0}, we have

$$\begin{aligned} &\int_\Omega |\nabla(C\varphi_\Omega)|^{p-2}\nabla(C\varphi_\Omega)\nabla\phi\,dx \\ &\leq \int_\Omega \Big[\gamma_0^{(p-1)\theta_0}a(x)\frac{f(C\varphi_\Omega+\epsilon)}{(C\varphi_\Omega+\epsilon)^{p-1}} + \gamma_0^{(p-1)\theta_0}\lambda b(x)\frac{g(C\varphi_\Omega+\epsilon)}{(C\varphi_\Omega+\epsilon)^{p-1}}\Big]\phi\,dx \\ &\leq \int_\Omega \Big[\gamma_0^{(p-1)\theta_0}a(x)\frac{f_{\gamma_0}(C\varphi_\Omega+\epsilon)}{(C\varphi_\Omega+\epsilon)^{p-1}} + \gamma_0^{(p-1)\theta_0}\lambda b(x)\frac{g_{\gamma_0}(C\varphi_\Omega+\epsilon)}{(C\varphi_\Omega+\epsilon)^{p-1}}\Big]\phi\,dx \\ &= \int_\Omega M(x)\gamma_0^{(p-1)\theta_0}\frac{\zeta_{\lambda,\gamma_0}(C\varphi_\Omega+\epsilon)}{(C\varphi_\Omega+\epsilon)^{p-1}}\phi\,dx \\ &\leq \int_\Omega M(x)\gamma_0^{(p-1)\theta_0}\frac{\zeta_{\lambda,\gamma_0}(C\varphi_\Omega)}{(C\varphi_\Omega)^{p-1}}\phi\,dx, \end{aligned} \tag{4.6}$$

for all $\phi \in C_0^\infty(\Omega), \phi \geq 0$.

Hence, from (4.5), (4.6), $\zeta_{\lambda,\gamma_0}(s)/s^{p-1}$ non-increasing in $s > 0$ and $C\varphi_\Omega = 0 < \psi_\lambda^{\theta_0}(\sigma) = v_\sigma$ on $\partial\Omega$, the claim follows. Here, again we used Tolksdorf [23].

Now, by taking $\sigma = 1/m$ and $\epsilon = 1/n$ with sufficiently large $m, n \in \mathbb{N}$, it follows from the lower upper solution Theorem (see Boccardo, Murat and Puel [2]) that there exists $u_{m,n} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $0 < C\varphi_\Omega \leq u_{m,n} \leq v_m$ satisfying

$$\int_\Omega |\nabla u_{m,n}|^{p-2}\nabla u_{m,n}\nabla\phi\,dx = \int_\Omega [a(x)f(u_{m,n}+\frac{1}{n})+\lambda b(x)g(u_{m,n}+\frac{1}{n})+\mu V(x,\nabla u_{m,n})]\phi\,dx$$

for all $\phi \in C_0^\infty(\Omega)$ and for each $\lambda_* < \lambda < \lambda^*$, $0 \leq \mu < \mu^*$.

Using a diagonal argument on n, for each fixed m, there exists $u_m \in C^1(\Omega)$ with $0 < C\varphi_\Omega \leq u_m \leq v_m < \gamma_0^{\theta_0}$ in Ω and

$$\begin{aligned} &\int_\Omega |\nabla u_m|^{p-2}\nabla u_m\nabla\phi\,dx \\ &= \int_\Omega [a(x)f(u_m) + \lambda b(x)g(u_m) + \mu V(x,u_m)]\phi\,dx,\ \ \phi \in C_0^\infty(\Omega). \end{aligned}$$

Again, by another diagonal argument on m, we obtain a $u \in C^1(\Omega) \cap C(\overline{\Omega})$ that satisfies $0 < C\varphi_\Omega \leq u \leq v < \gamma_0^{\theta_0}$ in $\overline{\Omega}$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\phi\,dx = \int_\Omega [a(x)f(u) + \lambda b(x)g(u) + \mu V(x,u)]\phi\,dx,$$

for $\phi \in C_0^\infty(\Omega)$, where v was defined in (3.19). This completes the proof of Theorem 1.2.

5. Problem (1.1) in $\mathbb{R}^N$

To prove Theorem 1.3, we consider the ϵ-family of problems

$$\begin{gathered} -\Delta_p u \geq a(x)f(u+\epsilon) + \lambda b(x)g(u+\epsilon) + \mu V(x,\nabla u) \quad \text{in } \mathbb{R}^N, \\ u > 0 \quad \text{in } \mathbb{R}^N \quad \text{and } u \to 0 \text{ as } |x| \to \infty, \end{gathered} \tag{5.1}$$

for $1 < p < N$. We show the following result.

Proposition 5.1. *Assume* (F1), (M1), (V1) *with* $q \in [0, p-1]$ *hold. Then, there exists a* $\lambda^* > 0$ *such that for each* $0 < \lambda < \lambda^*$ *and* $\epsilon > 0$*, there exist a* $\mu^* = \mu_\lambda^* > 0$ *and a function* $v = v_{\lambda,\mu} \in C^1(\mathbb{R}^N)$*, both independent of* ϵ*, with* v *being a solution of* (5.1) *for each* $0 \leq \mu < \mu^*$*. Additionally:*

(i) *if* (F_i) *occurs for* $i \in \{0, \infty\}$, *then*

$$\lambda^* \geq \frac{1}{g_i}\Big(\frac{1}{\|\omega_M\|^{p-1}_{L^\infty(\mathbb{R}^N)}} - f_i\Big);$$

(ii) *there is* $d > 0$ *such that*

$$\mu^*_\lambda \geq d \min\big\{[f^i + \lambda g^i]^{\frac{p-1-q}{p-1}}, f^i + \lambda g^i\big\}.$$

Proof. The proof of this result is analogous to the proof of part one of Proposition 3.1. Considering $\Omega = \mathbb{R}^N$ and $\theta_0 = 1$, we define the set $\mathcal{A} = \mathcal{A}_{\mathbb{R}^N} = \{\gamma \in (0, \infty) : \Gamma_{0,1}(\gamma) > \|\omega_M\|_{L^\infty(\mathbb{R}^N)}\}$. So, we obtain (2.3) and the positive number $\lambda^* = \lambda^*(\mathbb{R}^N) = \sup\{\Lambda^*(\gamma) : \gamma \in \mathcal{A}_{\mathbb{R}^N}\}$.

Moreover, we define the positive number

$$\mu^*_\lambda(\gamma) = \mu^*_{\lambda,\mathbb{R}^N}(\gamma) = \min\Big\{\frac{[f(\gamma) + \lambda g(\gamma)]^{\frac{p-1-q}{p-1}}}{4\|\nabla\omega_M\|^q_{L^\infty(\mathbb{R}^N)}}, \frac{f(\gamma) + \lambda g(\gamma)}{4}\Big\}. \tag{5.2}$$

for each $\gamma, \lambda > 0$.

Now, for $0 < \lambda < \lambda^*$, we take the number $\mu^*_\lambda = \mu^*_{\lambda,\mathbb{R}^N} > 0$ as defined in (3.3). Thus, for $0 \leq \mu < \mu^*_\lambda$ given, we have that there exists a $\gamma_0 \in \mathcal{A}$ such that $\lambda < \lambda^*(\gamma_0)$ and $\mu < \mu^*_\lambda(\gamma_0)$. Now, we fix this γ_0.

So, given $0 < \lambda < \lambda^*$, we define

$$v(x) = v_\lambda(x) = \psi_\lambda(\omega_M(x)), \quad x \in \mathbb{R}^N$$

and, as a consequence of the properties of ψ_λ, we obtain that $v \in C^1(\mathbb{R}^N)$, $v(x) \to 0$ as $\|x\| \to \infty$ and $0 < v(x) \leq \|v\|_{L^\infty(\mathbb{R}^N)} < \gamma_0$, $x \in \mathbb{R}^N$, because of $\omega_M(x) \leq \|\omega_M\|_{L^\infty(\mathbb{R}^N)} < \eta_\lambda(\gamma_0)$. That is, taking sufficiently small $\epsilon > 0$, we have

$$\|v\|_{L^\infty(\mathbb{R}^N)} < \gamma_0 - \epsilon. \tag{5.3}$$

So, for each $\phi \in C_0^\infty(\mathbb{R}^N)$ with $\phi \geq 0$ given, we have (in a similar way to (3.6)) that

$$\int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \nabla\phi\, dx \geq \int_{\mathbb{R}^N} M(x)\gamma_0^{p-1}\Big[\frac{H_{\lambda,\gamma_0}(\psi_\lambda(\omega_M))}{\psi_\lambda(\omega_M))}\Big]^{p-1}\phi\, dx. \tag{5.4}$$

Below, we analyze the previous integral in two parts. First, we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N} M(x)\gamma_0^{p-1}\Big[\frac{H_{\lambda,\gamma_0}(\psi_\lambda(\omega_M))}{\psi_\lambda(\omega_M)}\Big]^{p-1}\phi\, dx\\
&\geq \frac{1}{2}\int_{\mathbb{R}^N} M(x)\gamma_0^{p-1}\frac{\zeta_{\lambda,\gamma_0}(v)}{v^{p-1}}\phi\, dx\\
&\geq \frac{1}{2}\int_{\mathbb{R}^N} M(x)\gamma_0^{p-1}\frac{\zeta_{\lambda,\gamma_0}(v+\epsilon)}{(v+\epsilon)^{p-1}}\phi\, dx\\
&\geq \frac{1}{2}\int_{\mathbb{R}^N} M(x)[f_{\gamma_0}(v+\epsilon) + \lambda g_{\gamma_0}(v+\epsilon)]\phi\, dx.
\end{aligned}$$

As a consequence of this, (5.3), definitions of ζ_{f,γ_0}, ζ_{g,γ_0} and M, we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N} M(x)\gamma_0^{p-1}\Big[\frac{H_{\lambda,\gamma_0}(\psi_\lambda(\omega_M))}{\psi_\lambda(\omega_M)}\Big]^{p-1}\phi\, dx\\
&\geq \int_{\mathbb{R}^N}[a(x)f(v+\epsilon) + \lambda b(x)g(v+\epsilon)]\phi\, dx.
\end{aligned} \tag{5.5}$$

For the other part, using the properties of the auxiliary functions and (5.2), we have

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N} M(x)\gamma_0^{p-1}\Big[\frac{H_{\lambda,\gamma_0}(\psi_\lambda(\omega_M))}{\psi_\lambda(\omega_M)}\Big]^{p-1}\phi\,dx\\
&\geq \frac{1}{4}\int_{\mathbb{R}^N} M(x)\gamma_0^{p-1}\Big[\frac{f(\gamma_0)}{\gamma_0^{p-1}}+\lambda\frac{g(\gamma_0)}{\gamma_0^{p-1}}\Big]\phi\,dx\\
&\quad+\frac{1}{4}\int_{\mathbb{R}^N} M(x)\gamma_0^{p-1-q}\Big[\frac{H_{\lambda,\gamma_0}(\psi_\lambda(\omega_M))}{\psi_\lambda(\omega_M)}\Big]^{p-1-q}\Big[\frac{\gamma_0 H_{\lambda,\gamma_0}(\psi_\lambda(\omega_M))}{\psi_\lambda(\omega_M)}\Big]^{q}\phi\,dx\\
&\geq \frac{(f(\gamma_0)+\lambda g(\gamma_0))}{4}\int_{\mathbb{R}^N} M(x)\phi\,dx \qquad (5.6)\\
&\quad+\frac{1}{4}\int_{\mathbb{R}^N} M(x)\gamma_0^{p-1-q}\Big[\frac{f(\gamma_0)}{\gamma_0^{p-1}}+\lambda\frac{g(\gamma_0)}{\gamma_0^{p-1}}\Big]^{\frac{p-1-q}{p-1}}[\psi_\lambda'(\omega_M)]^q\phi\,dx\\
&\geq \mu_\lambda^*\int_{\mathbb{R}^N}\beta(x)\phi\,dx+\mu_\lambda^*\int_{\mathbb{R}^N} M(x)[\psi_\lambda'(\omega_M)]^q|\nabla\omega_M|^q\phi\,dx\\
&\geq \mu_\lambda^*\int_{\mathbb{R}^N}[\beta(x)+\alpha(x)|\nabla v|^q]\phi\,dx\geq \mu\int_{\mathbb{R}^N} V(x,\nabla v)\phi\,dx,
\end{aligned}
$$

for each $0 \leq \mu < \mu_\lambda^*$.

Hence, replacing (5.5) and (5.6) in (5.4), we get that v satisfies (5.1), for each $0 < \lambda < \lambda^*$ and $0 \leq \mu < \mu_\lambda^*$.

The estimates given for λ^* and μ_λ^* are obtained in a similar way as those of Proposition 3.1. This proves Proposition 5.1.

6. Conclusion of the proof of Theorem 1.3

First, we note that $\mathcal{A}_{\mathbb{R}^N} \subset \mathcal{A}_{B_R}$ for all $R \geq 1$. In fact, if $\gamma \in \mathcal{A}_{\mathbb{R}^N}$, then (using Lemma 2.2 (iv)) we have

$$\Gamma_{0,\theta}(\gamma) > \|\omega_M\|_{L^\infty(\mathbb{R}^N)} \geq \|(\omega_M)_{|B_R}\|_{L^\infty(B_R)}, \quad \text{for all } R \geq 1;$$

that is, $\gamma \in \mathcal{A}_{B_R}$. So, we obtain

$$\lambda^*(\mathbb{R}^N) = \sup\{\lambda^*(\gamma) : \gamma \in \mathcal{A}_{\mathbb{R}^N}\} \leq \sup\{\lambda^*(\gamma) : \gamma \in \mathcal{A}_{B_R}\} = \lambda^*(B_R)$$

for all $R \geq 1$.

Concerning μ_λ^*. As a direct consequence of (3.2) and (5.2), we obtain that $\mu_\lambda^*(\mathbb{R}^N) \leq \mu_\lambda^*(B_R)$, for all $R \geq 1$. So, given $\lambda_* < \lambda < \lambda^*(\mathbb{R}^N)$, $0 \leq \mu < \mu_\lambda^*(\mathbb{R}^N)$ and taking $v_R = v_{|B_R}$ as an upper solution, there exists (Theorem 1.2 and its demonstration) a $u_R \in W_0^{1,p}(B_R) \cap C(\overline{B}_R)$ with $0 < C_R\varphi_{B_R} \leq u_R \leq v_{\mathbb{R}^N} < \gamma_0$ in B_R satisfying

$$
\begin{gathered}
-\Delta_p u_R = a(x)f(u_R) + \lambda b(x)g(u_R) + \mu V(x,\nabla u_R) \quad \text{in } B_R\\
u_R > 0 \quad \text{in } B_R, \quad u_R = 0 \quad \text{on } \partial B_R,
\end{gathered} \tag{6.1}
$$

for each $R > 1$, where v is given by Proposition 5.1.

Besides this, from the definition of λ_*, $0 < \lambda < \lambda_*$ and $\lambda_1(\rho) = \lim_{R\to\infty} \lambda_{B_R}(\rho)$, it follows that there exists a $L_0 > 1$ such that $\lambda_{B_{L_0}}(\rho) < \lambda g_0 + f_0$. That is, from the monotonicity of the first eigenvalue concerning the domain, there exists one $\delta = \delta(L_0) > 0$ such that

$$f(s) + \lambda g(s) > \lambda_{B_R}(\rho)s^{p-1}, \quad \text{for all } s \in (0,\delta) \text{ and } R \geq L_0. \tag{6.2}$$

Now, considering C_{L_0} the constant of the lower solution of (6.1) with $R = L_0$ defined in (4.1), we take a sufficiently small $C = C(\delta) \in (0, C_{L_0})$ such that

$$0 < C\|\varphi_{B_{L_0}}\|_{L^\infty(B_{L_0})} < \delta. \tag{6.3}$$

With this choice and noting that $C\varphi_{B_{L_0}}$ and u_R satisfy (1.7) and (6.1), respectively, it follows from (6.2), (6.3) and the classical Díaz and Saá's inequality [7], that

$$C\varphi_{B_{L_0}}(x) \le u_R(x), \quad x \in B_{L_0}, \text{ for all } R > L_0.$$

Now, proceeding as in the end of proof of Theorem 1.2, we finish the proof of Theorem 1.3. □

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Manuela C. Rezende
Departamento de Matemática, Universidade de Brasília, 70910-900 Brasília, DF - Brasil
E-mail address: manuela@mat.unb.br

Carlos A. Santos
Departamento de Matemática, Universidade de Brasília , 70910-900 Brasília, DF - Brasil
E-mail address: csantos@unb.br