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STURM-PICONE TYPE THEOREMS FOR SECOND-ORDER NONLINEAR ELLIPTIC DIFFERENTIAL EQUATIONS

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Abstract. The aim of this article is to give Sturm-Picone type theorems for the pair of second order nonlinear elliptic differential equations

$$\operatorname{div}(p_1(x)|\nabla u|^{\alpha-1}\nabla u) + q_1(x)f_1(u) + r_1(x)g_1(u) = 0,$$
$$\operatorname{div}(p_2(x)|\nabla v|^{\alpha-1}\nabla v) + q_2(x)f_2(v) + r_2(x)g_2(v) = 0,$$

where $|\cdot|$ denotes the Euclidean length and $\nabla = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})^T$ (the superscript T denotes the transpose). Our results include some earlier results and generalize to n-dimensions well-known comparison theorems given by Sturm, Picone and Leighton [26, 37] which play a key role in the qualitative behavior of solutions. By using generalization of n dimensional Leigton's comparison theorem, an oscillation result is given as an application.

1. Introduction

In the qualitative theory of ordinary differential equations, the celebrated Sturm-Picone theorem plays a crucial role. In 1836, the first important comparison theorem was established by Sturm [36]. In 1909, Picone [33] modified Sturm's theorem. For a detailed study and earlier developments of this subject, we refer the reader to the books [26, 37]. Sturm-Picone theorem is extended in several directions, see [2] and [3] for linear systems, [30] for nonself adjoint differential equations, [40] for implicit differential equations, [20, 29] for half linear equations, [7] for degenerate elliptic equations,[48] for linear equations on time scales and [39, 41] for a pair of nonlinear differential equations. On the other hand, we emphasize that the classical proof of Sturm-Picone theorem heavily depends on the Leighton's variational lemma [28] (see [37] also). Since when it was proved, it has been extended in different contexts, see, for instance [16, 21, 25].

There is also a good amount of interest in the qualitative theory of differential equations to determine whether the given equation is oscillatory or not and Sturm-Picone theorem also plays an important role in this direction. For earlier developments, we refer to [26, 33, 36, 37] and for recent developments, we refer to Yoshida's book [44]. Sturm comparison theorems for half linear elliptic equations

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and Picone type identities have been studied in, for example [5, 6, 7, 8, 13, 15, 19, 22, 23, 24, 27, 38, 42, 43, 44, 45, 46, 47].

Recently, Tyagi [41] studied a pair of second order nonlinear elliptic partial differential equations

$$-\Delta u = q_1(x)f_1(u) + b_1(x)r_1(u), \tag{1.1}$$

$$-\Delta v = q_2(x)f_2(v) + b_2(x)r_2(v), \tag{1.2}$$

under suitable conditions. By establishing a nonlinear version of Leighton's variational lemma, he gave the generalization of Sturm-Picone theorem for (1.1) and (1.2). But it is obvious that this result does not work for the half linear elliptic case. A natural question now arises: Is it possible to generalize the Sturm comparison results to the nonlinear elliptic partial differential equations that contain the half linear case by using a nonlinear version of Leighton's variational lemma?

Motivated by the ideas in [27, 39, 41], extending Tyagi's results, we prove a nonlinear analogue for n-dimensional Leighton's theorem and we give a generalization of n-dimensional Sturm-Picone theorem by establishing a suitable nonlinear version of Leighton's variational lemma which contain the half linear and also linear elliptic equations.

2. Main results

Let us consider a pair of second-order nonlinear elliptic type partial differential operators:

$$\ell u := \nabla\cdot(p_1(x)|\nabla u|^{\alpha-1}\nabla u) + q_1(x)f_1(u) + r_1(x)g_1(u), \tag{2.1}$$

$$Lv := \nabla\cdot(p_2(x)|\nabla v|^{\alpha-1}\nabla v) + q_2(x)f_2(u) + r_2(x)g_2(u), \tag{2.2}$$

where $|\cdot|$ denotes the Euclidean length and $\nabla = \left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right)^T$ (the superscript T denotes the transpose). In this section, by establishing a nonlinear version of Leighton's variational Lemma, we focused on obtaining a generalization of n-dimensional Sturm-Picone theorem for (2.1) and (2.2).

Let G be a bounded domain in $\mathbb{R}^n$ with boundary ∂G having a piecewise continuous unit normal. Let also $p_i \in C(\bar{G},\mathbb{R})$, q_i, $r_i \in C^\mu(\bar{G},\mathbb{R})$, $f_1 \in C^1(\mathbb{R},\mathbb{R})$, $f_2 \in C(\mathbb{R},\mathbb{R})$, $g_i \in C(\mathbb{R},\mathbb{R})$, for $i = 1,2$ where $0 < \mu \le 1$, q_i's are of indefinite sign for $i = 1,2$ and $p_i(x) > 0$, $r_i(x) \ge 0$ for all $x \in \bar{G}$ and α is a positive real constant.

The domain $D_\ell(G)$ of ℓ is defined to be the set of all functions u of class $C^1(\bar{G},\mathbb{R})$ with the property that $p_1(x)|\nabla u|^{\alpha-1}\nabla u \in C^1(G;\mathbb{R})\cap C(\bar{G},\mathbb{R})$. The domain $D_L(G)$ of L is defined similarly. Note that such a function $u \in D_\ell(G)$ (and $v \in D_L(G)$) exists for (2.1) (and (2.2)) [12, 34]. The principal part of (2.1) (and (2.2)) is reduced to the p-Laplacian $\Delta_p u := \nabla\cdot(|\nabla u|^{p-2}\nabla u)$ ($p = \alpha+1$, $p_1(x) \equiv 1$). We know that a variety of physical phenomena are modelled by equations involving the p-Laplacian [4, 9, 10, 14, 31, 32, 35].

In what follows, we make the following hypotheses on f_i and g_i.

- (H1) Let $f_1 \in C^1(\mathbb{R},\mathbb{R})$ and there exist α_0, $\alpha_1 \in (0,\infty)$ such that $\alpha_0|u|^{\alpha-1} \le f_1'(u)$ and $\alpha_1|u|^{\alpha-1}u \ge f_1(u) \ne 0$ for all $0 \ne u \in R$.
- (H1*) Let $f_1 \in C^1(\mathbb{R},\mathbb{R})$ and there exists a $k > 0$ such that $\frac{f_1'(u)}{|f_1(u)|^{\frac{\alpha-1}{\alpha}}} \ge k$ for all $0 \ne u \in R$.
- (H2) Let $g_1 \in C(\mathbb{R},\mathbb{R})$ and there exists a $\beta \ge 0$ such that $\frac{g_1(u)}{f_1(u)} \ge \beta$ for all $0 \ne u \in R$.

(H3) Let $f_2, g_2 \in C(\mathbb{R}, \mathbb{R})$ and there exists $\alpha_2, \alpha_3, \alpha_4 \in (0, \infty)$ such that $\alpha_3|v|^{\alpha+1} \le f_2(v)v \le \alpha_2|v|^{\alpha+1}$ and $g_2(v)v \le \alpha_4|v|^{\alpha+1}$

Remark 2.1. Assumption (H1) motivates us to study nonlinearities of the form

$$f_1(u) = |u|^{\alpha-1}u(1 \mp \text{ a nonlinear part})$$

where nonlinear part is decays at ∞.

Remark 2.2. Assumption (H3) simply says that $\frac{f_2(v)}{|v|^{\alpha-1}v}$ is bounded for all $0 \ne v \in R$.

Remark 2.3. Assumption (H1*) is a very common condition in the literature for half linear equations.

We begin with a lemma and the definition of some concepts needed in this article.

Lemma 2.4 ([27]). *Define $\Phi(\xi) = |\xi|^{\alpha-1}\xi$, $\xi \in \mathbb{R}^n$, $\alpha > 0$. If $X, Y in \mathbb{R}^n$, then*

$$X\Phi(X) + \alpha Y\Phi(Y) - (\alpha+1)X \cdot \Phi(Y) \ge 0. \tag{2.3}$$

where the equality holds if and only if $X = Y$.

Let U be the set of all real valued continuous functions defined on $\bar{G}$ which vanish on ∂G and have uniformly continuous firs partial derivatives on G. Also define the functions j, j^* and J: $U \to \mathbb{R}$ by

$$\begin{gathered} j(\eta) = \int_G \{p_1(x)|\nabla\eta|^{\alpha+1} - C_1\big(q_1(x) + \beta r_1(x)\big)|\eta|^{\alpha+1}\}dx, \\ j^*(\eta) = \int_G \{p_1(x)|\nabla\eta|^{\alpha+1} - C_2\big(q_1(x) + \beta r_1(x)\big)|\eta|^{\alpha+1}\}dx, \\ J(\eta) = \int_G \{p_2(x)|\nabla\eta|^{\alpha+1} - (\alpha_2 q_2^+(x) - \alpha_3 q_2^-(x) + \alpha_4 r_2(x))|\eta|^{\alpha+1}\}dx \end{gathered} \tag{2.4}$$

where $C_1 = (\frac{\alpha_0}{\alpha_1\alpha})^\alpha\alpha_1$, $C_2 = (\frac{k}{\alpha})^\alpha$, $q_2^+ = max\{q_2, 0\}$ and $q_2^- = max\{-q_2, 0\}$. The variation $V(\eta)$ and $V^*(\eta)$ are defined as

$$\begin{gathered} V(\eta) = J(\eta) - j(\eta), \\ V^*(\eta) = J(\eta) - j^*(\eta) \end{gathered} \tag{2.5}$$

with domain $D := D_j \cap D_J = D_{j^*} \cap D_J$.

To prove a nonlinear analogue of Leighton's theorem we first establish a nonlinear version of Leighton's variational lemma (Generalization of n-dimensional Leighton's variational type lemma).

Lemma 2.5. *Assume that there exists a nontrivial function $\eta \in U$ such that $j(\eta) \le 0$ (or $j^*(\eta) \le 0$). Then under the hypotheses* (H1) *(or* (H1*)*) and* (H2)*, every solution $u \in D_j$ of $\ell(u) = 0$ vanishes at some points of $\bar{G}$.*

Proof. Let us give the proof under the conditions $j(\eta) \le 0$, (H1) and (H2). Similarly proof holds for $j^*(\eta) \le 0$, (H1*) and (H2). Assume on the contrary that the statement is false. Suppose that there exists a solution $u \in D_\ell(G)$ of $\ell(u) = 0$ satisfying $u \ne 0$ on $\bar{G}$. By (H1), we have $f_1(u(x)) \ne 0$, $\forall x \in \bar{G}$. Then for $\eta \in U$, the following equality is valid in G:

$$\nabla \cdot \Big(\frac{\alpha\eta\Phi(\eta)}{f_1(u(x))}p_1(x)|\nabla u|^{\alpha-1}\nabla u\Big)$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\{\frac{\partial}{\partial x_i}(\alpha\eta\Phi(\eta))\frac{p_1(x)|\nabla u|^{\alpha-1}\nabla u}{f_1(u(x))} \\
&\quad + \alpha\eta\Phi(\eta)(\frac{\partial}{\partial x_i}\frac{1}{f_1(u(x))})p_1(x)|\nabla u|^{\alpha-1}\nabla u \\
&\quad + \frac{\alpha\eta\Phi(\eta)}{f_1(u(x))}\frac{\partial}{\partial x_i}(p_1(x)|\nabla u|^{\alpha-1}\nabla u)\} \\
&= p_1(x)\frac{|f_1(u(x))|^{\alpha-1}}{(f_1'(u(x)))^{\alpha}}|\alpha\nabla\eta|^{\alpha+1} - \alpha q_1(x)|\eta|^{\alpha+1} - \alpha r_1(x)\frac{g_1(u(x))}{f_1(u(x))}|u|^{\alpha+1} \\
&\quad - p_1(x)\frac{|f_1(u(x))|^{\alpha-1}}{(f_1'(u(x)))^{\alpha}}F\Big(\frac{\eta\nabla u f_1'(u(x))}{f_1(u(x))},\alpha\nabla\eta\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
&F(\frac{\eta\nabla u f_1'(u(x))}{f_1(u(x))},\alpha\nabla\eta) \\
&= |\alpha\nabla\eta|^{\alpha+1} + \alpha\Big|\frac{\eta\nabla u f_1'(u(x))}{f_1(u(x))}\Big|^{\alpha+1} - (\alpha+1)\alpha\nabla\eta\cdot\Phi\Big(\frac{\eta\nabla u f_1'(u(x))}{f_1(u(x))}\Big)
\end{aligned}
$$

By (H1) and (H2), we obtain

$$
\begin{aligned}
&p_1(x)|\nabla\eta|^{\alpha+1} - C_1(q_1(x)+\beta r_1(x))|\eta|^{\alpha+1} \\
&\ge C_1\nabla\cdot\Big(\frac{\eta\Phi(\eta)}{f_1(u(x))}p_1(x)|\nabla u|^{\alpha-1}\nabla u\Big) \\
&\quad + \frac{C_1}{\alpha}p_1(x)\frac{|f_1(u(x))|^{\alpha-1}}{f_1'(u(x))^{\alpha}}F\Big(\frac{\eta\nabla u f_1'(u(x))}{f_1(u(x))},\alpha\nabla\eta\Big).
\end{aligned} \tag{2.6}
$$

We integrate (2.6) over G and then apply the divergence theorem to obtain

$$
j(\eta) \ge \frac{C_1}{\alpha}\int_G p_1(x)\frac{|f_1(u(x))|^{\alpha-1}}{f_1'(u(x))^{\alpha}}F\Big(\frac{\alpha\nabla u f_1'(u(x))}{f_1(u(x))},\alpha\nabla\eta\Big)dx \ge 0\,.
$$

Therefore,

$$
\int_G p_1(x)\frac{|f_1(u(x))|^{\alpha-1}}{f_1'(u(x))^{\alpha}}F\Big(\frac{\eta\nabla u f_1'(u(x))}{f_1(u(x))},\alpha\nabla\eta\Big) = 0.
$$

From Lemma 2.4, we see that

$$
\frac{\eta\nabla u f_1'(u(x))}{f_1(u(x))} = \alpha\nabla\eta \quad\text{or}\quad \nabla(\frac{|\eta(x)|^{\alpha}}{|f_1(u(x))|}) = 0 \quad \text{in } G.
$$

Since $\eta\in U$, there exists a nonzero constant K such that

$$
|\eta(x)|^{\alpha} = |Kf_1(u(x))|
$$

in G and hence on $\bar G$ by continuity. This is not possible because $\eta(x)=0$ on ∂G but $f_1(u(x))\neq 0$ on ∂G ($u(x)\neq 0$ on ∂G). This implies that $j(\eta)>0$, which is a contradiction and hence every solution u of $\ell u=0$ vanishes at some point of $\bar G$. This completes the proof. □

Lemma 2.5 plays a crucial role to establish the following Generalization of n-dimensional Leighton's theorem.

Theorem 2.6. *Let* (H1) *(or* (H1*)*),* (H2) *and* (H3) *hold. If there exists a nontrivial solution $v\in D$ of $Lv=0$ in $\bar G$ such that $v=0$ on ∂G and $V(v)\ge 0$ (or $V^*(v)\ge 0$), then every solution u of $\ell u=0$ vanishes at some point of $\bar G$.*

Proof. As in the proof of Lemma 2.5 , let us give the proof under the conditions (H1), (H2) and $V(v) \geq 0$. Since v is a solution of $Lv = 0$ and $v = 0$ on ∂G so by an application of Green's theorem we have

$$\begin{aligned} &\int_G \Big(q_2(x) f_2(v) v + r_2(x) g_2(v) v \Big) dx \\ &= - \int_G v \nabla \cdot (p_2(x) |\nabla v|^{\alpha-1} \nabla v) dx \\ &= -v (p_2(x) |\nabla v|^{\alpha-1} \nabla v) \mid_{\partial G} + \int_G p_2(x) |\nabla v|^{\alpha+1} dx \\ &= \int_G p_2(x) |\nabla v|^{\alpha+1} dx. \end{aligned} \tag{2.7}$$

In view of (H3), one can see that

$$\int_G \Big(q_2(x) f_2(v) v + r_2(x) g_2(v) v \Big) dx \leq \int_G [(\alpha_2 q_2^+(x) - \alpha_3 q_2^-(x)) + \alpha_4 r_2(x)] |v|^{\alpha+1} dx. \tag{2.8}$$

By (2.7) and (2.8), we have $J(v) \leq 0$. Since $V(u) \geq 0$, this implies

$$j(v) \leq J(v) \leq 0$$

and hence by application of Lemma 2.5 every nontrivial solution u of $\ell u = 0$ vanishes at some point of $\bar{G}$. This completes the proof. □

Remark 2.7. If the condition $V(v) \geq 0$ (or $V^*(v) \geq 0$) is strengthened to $V(v) > 0$ (or $V^*(v) > 0$), the conclusion of Theorem 2.6 holds also in the domain G.

From Theorem 2.6. we immediately have the following Corollary which is an n-dimensional extension of Sturm-Picone comparison theorem for the operators (2.1) and (2.2).

Corollary 2.8. *Let* (H1) *(or* (H1*)*),* (H2) *and* (H3) *hold. Suppose there exists a nontrivial solution v of $Lv = 0$ in $\bar{G}$ such that $v = 0$ on ∂G. If $p_2(x) \geq p_1(x)$ and*

$$C_1(q_1(x) + \beta r_1(x)) \geq [\alpha_2 q_2(x) - (\alpha_3 - \alpha_2) q_2^-(x) + \alpha_4 r_2(x)],$$
$$\Big(\text{or } C_2(q_1(x) + \beta r_1(x)) \geq [\alpha_2 q_2(x) - (\alpha_3 - \alpha_2) q_2^-(x) + \alpha_4 r_2(x)] \Big),$$

for every $x \in \bar{G}$. Then every nontrivial solution u of $\ell u = 0$ vanishes at some point of $\bar{G}$.

From Lemma 2.5, Theorem 2.6 and Corollary 2.8 we easily obtain the following results which are straightforward extensions of the variational Lemma, Leighton's theorem and the celebrated Sturm-Picone theorem from [26, 37] valid for linear second order ordinary differential equations to half linear elliptic partial differential equations that contain linear case.

Corollary 2.9. *Let $f_1(u) = |u|^{\alpha-1} u$ and either $r_1(x) \equiv 0$ or $g_1(u) \equiv 0$ in* (2.1). *If there exists a nontrivial function $\eta \in U$ such that*

$$\int_G \{ p_1(x) |\nabla \eta|^{\alpha+1} - q_1(x) |\eta|^{\alpha+1} \} dx \leq 0 \tag{2.9}$$

then every nontrivial solution u of half linear elliptic equation

$$\nabla \cdot (p_1(x) |\nabla u|^{\alpha-1} \nabla u) + q_1(x) |u|^{\alpha-1} u = 0 \tag{2.10}$$

vanishes at some point in $\bar{G}$.

Corollary 2.10. *Suppose that there exists a nontrivial solution* v *of*

$$\nabla \cdot (p_2(x)|\nabla v|^{\alpha-1}\nabla v) + q_2(x)|v|^{\alpha-1}v = 0 \tag{2.11}$$

in $\bar{G}$ *such that* $v = 0$ *on* ∂G. *If*

$$\int_G \{(p_2(x) - p_1(x))|\nabla v|^{\alpha+1} + (q_1(x) - q_2(x))|v|^{\alpha+1}\}dx \geq 0 \tag{2.12}$$

then every nontrivial solution u *of* (2.10) *vanishes at some point of* $\bar{G}$.

Corollary 2.11. *Let* $p_2(x) \geq p_1(x)$ *and* $q_1(x) \geq q_2(x)$ *for every* $x \in \bar{G}$. *If there exists a nontrivial solution* v *of* (2.11) *in* $\bar{G}$ *such that* $v = 0$ *on* ∂G, *then any nontrivial solution* u *of* (2.10) *vanishes at some point of* $\bar{G}$.

Note that the Corollaries 2.9–2.11 were also obtained in [15, 18, 23, 27, 44]. But their proofs depend on the Picone-type and Wirtinger type inequalities.

Recently Bal [11] gave a nonlinear version of the Sturmian comparison principle for a special case of (2.1) and (2.2) as the follows.

Theorem 2.12 ([11]). *Let* q_1 *and* q_2 *be the two weight functions such that* $q_2 < q_1$ *and* f_1 *satisfies* $f_1'(u) \geq (p-1)\big(f_1(u)^{\frac{p-2}{p-1}}\big)$. *If there is a positive solution* v *satisfying*

$$-\Delta_p v = q_2(x)|v|^{p-2}v \text{ for } \Omega^*, \quad v = 0 \text{ on } \partial\Omega^*, \tag{2.13}$$

then any nontrivial solution u *of*

$$-\Delta_p u = q_1(x)f_1(u) \quad \text{for } x \in \Omega^* \tag{2.14}$$

must change sign, where Ω^* *denotes any domain in* $\mathbb{R}^n$, $1 < p < \infty$ *and* $f_1 : (0,\infty) \to (0,\infty)$ *is a* C^1 *function.*

From the hypothesis of f_1 this conclusion is not true. Because for $u \in (0,\infty)$, $f_1(u) > 0$ but for $u < 0$ $f_1(u)$ is not defined.

This result can be corrected by using Corollary 2.8, for the bounded domain $\bar{G}$ in $\mathbb{R}^n$ and we can give the following Sturmian comparison result for the equations (2.13) and (2.14) as follows:

Corollary 2.13. *Let* (H1*) *hold with* $k = \alpha = p - 1$. *If there exists a nontrivial solution* $v \in D$ *of* (2.13) *in* $\bar{G}$ *such that* $v = 0$ *on* ∂G *and* $q_1(x) \geq q_2(x)$, *then every solution* u *of* (2.14) *vanishes at some point of* $\bar{G}$.

3. An application

This section deals with an application of Theorem 2.6. This theorem enables us to develop some oscillation criteria for the equation $\ell u = 0$.

Let Ω be an exterior domain in $\mathbb{R}^n$, that is, a domain such that $\Omega \supset \{x \in \mathbb{R}^n : |x| \geq r_0\}$ for some $r_0 > 0$, and consider the nonlinear elliptic equation

$$\nabla \cdot (p_1(x)|\nabla u|^{\alpha-1}\nabla u) + q_1(x)f_1(u) = 0 \tag{3.1}$$

in Ω where $\alpha > 0$ is a constant, $p_1 \in C(\Omega, \mathbb{R}^+)$, $q_1 \in C(\Omega, \mathbb{R})$ and f_1 satisfy the hypothesis (H1) (or (H1*)).

A nontrivial solution of (3.1) is said to be oscillatory if it has a zero in $\Omega \cap \{x \in \mathbb{R}^n : |x| > r^*\}$ for any $r^* > r_0$. For brevity, (3.1) is called oscillatory if all of its nontrivial solutions are oscillatory.

We will show that an explicit oscillation criterion for (3.1) can be obtained via the comparison principle proven in the preceding section. Our main idea is to compare (3.1) with suitably chosen equations with radial symmetry of the type

$$\nabla \cdot (\tilde{p_1}(|x|)|\nabla v|^{\alpha-1}\nabla v) + \bar{q_1}(|x|)f_1(v) = 0 \tag{3.2}$$

in $\{x \in \mathbb{R}^n : |x| \geq r_0\}$ and employ information about the oscillatory behavior of radially symmetric solutions of (3.2).

It is easily verified that if $v = y(|x|)$ is radially symmetric solution of (3.2), then the function $y(r)$ satisfies the differential equation

$$(r^{n-1}\tilde{p_1}(r)|y'|^{\alpha-1}y')' + r^{n-1}\bar{q_1}(r)f_1(y) = 0, \quad r \geq r_0 \tag{3.3}$$

We note that (3.3) is a special case of the equation

$$(p(r)|y'|^{\alpha-1}y')' + q(r)f_1(y) = 0, \;\; r \geq r_0 \tag{3.4}$$

the oscillatory behavior of which has been intensively investigated in recent years by numerous authors [1, 17].

Suppose that $p(r)$ and $q(r)$ are continuous functions defined on $[r_0, \infty)$ such that $p(r) > 0$ on $[r_0, \infty)$. A solution of (3.4) is a function $y : [r_0, \infty) \to \mathbb{R}$ which is continuously differentiable on $[r_0, \infty)$ together with $p|y'|^{\alpha-1}y'$ and satisfies (3.4) at every point of $[r_0, \infty)$. A nontrivial solution is said to be oscillatory if it has a sequence of zeros clustering ar $r = \infty$, and nonoscillatory otherwise. Now we give an oscillation criterion for (3.4). Its proof can be found, for example, in [1].

Lemma 3.1. *Let* (H1) *(or* (H1*)*) hold. Suppose that* $p \in C([r_0, \infty), \mathbb{R}^+)$ *and* $q \in C([r_0, \infty), \mathbb{R})$ *satisfies*

$$\int_{r_1}^{r} \Big(\int_{r_0}^{s} p(u)du \Big)^{-1/\alpha} ds = \infty$$

and

$$\lim_{r\to\infty} \frac{1}{r} \int_{r_0}^{r} \Big(\int_{r_0}^{s} q(u)du \Big) ds = \infty.$$

Then (3.4) *is oscillatory.*

We first establish a principle which enables us to deduce the oscillation of (3.1) from the one-dimensional oscillation of (3.3).

Theorem 3.2. *If there exist functions* $\tilde{p_1} \in C([r_0, \infty), \mathbb{R}^+)$ *and* $\tilde{q_1} \in C([r_0, \infty), \mathbb{R})$ *such that*

$$\tilde{p_1}(r) \geq \max_{|x|=r} p_1(x)$$

and

$$\alpha_2 \tilde{q_1}^+(r) - \alpha_3 \tilde{q_1}(r) \leq C_1 \min_{|x|=r} q_1(x) \tag{3.5}$$

$$\Big(or \; \alpha_2 \tilde{q_1}^+(r) - \alpha_3 \tilde{q_1}(r) \leq C_2 \min_{|x|=r} q_1(x) \Big)$$

where α_2*,* α_3*,* C_1 *and* C_2 *are defined as before, and the ordinary differential equation* (3.3) *is oscillatory, then* (3.1) *is oscillatory in* Ω*.*

Proof. By hypothesis there exists an oscillatory solution $y(r)$ of (3.3) on $[r_0, \infty)$. Let $\{r_i\}$ be the set of all zeros of $y(r)$ such that $r_0 \leq r_1 < r_2 < \dots < r_i < \dots$, $\lim_{i\to\infty} r_i = \infty$. Then the function $v(x) = y(|x|)$ is a radially symmetric solution of (3.2) which is defined in $\{x \in \mathbb{R}^n : |x| \geq r_0\}$ and has the spherical nodes

$|x| = r_i$, $i = 1, 2, \dots$. Let us compare (3.1) with (3.2) in the annular domains $G_i = \{x \in \mathbb{R}^n : r_i < |x| < r_{i+1}\}$, $i = 1, 2, \dots$. For each i, v is a solution of (3.2) in G_i such that $v \neq 0$ in G_i and $v = 0$ on ∂G_i. Since (3.5) implies

$$\tilde{p_1}(|x|) \geq p_1(x)$$

and

$$\alpha_2 \tilde{q_1}^+(|x|) - \alpha_3 \tilde{q_1}^-(|x|) \leq C_1 q_1(x)$$
$$\Big(\text{or } \alpha_2 \tilde{q_1}^+(|x|) - \alpha_3 \tilde{q_1}^-(|x|) \leq C_2 q_1(x)\Big)$$

in $\{x \in \mathbb{R}^n : |x| \geq r_0\}$, we obtain

$$\begin{aligned} V(v) \equiv \int_{G_i} &\{(\tilde{p_1}(|x|) - p_1(x))|\nabla v|^{\alpha+1} + [C_1 q_1(x) \\ &- (\alpha_2 \tilde{q_1}^+(|x|) - \alpha_3 \tilde{q_1}^-(|x|))]|v|^{\alpha+1}\}dx \geq 0 \\ \Big(\text{or } V^*(v) \equiv \int_{G_i} &\{(\tilde{p_1}(|x|) - p_1(x))|\nabla v|^{\alpha+1} + [C_2 q_1(x) \\ &- (\alpha_2 \tilde{q_1}^+(|x|) - \alpha_3 \tilde{q_1}^-(|x|))]|v|^{\alpha+1}\}dx \geq 0\Big). \end{aligned}$$

Consequently from Theorem 2.6, it follows that every solution u of (3.1) has a zero in G_i, $i = 1, 2, \dots$, which shows that u is oscillatory in Ω. This completes the proof. □

Remark 3.3. An immediate consequence of Theorem 3.2 is that (3.2) with $\tilde{p_1} \in C([r_0, \infty), \mathbb{R}^+)$ and $\tilde{a_1} \in C([r_0, \infty), \mathbb{R})$ is oscillatory in $\{x \in \mathbb{R}^n : |x| \geq r_0\}$ if it has one radially symmetric solution which is oscillatory there.

Combining Theorem 3.2 with Lemma 3.1 applied to (3.3) gives the following oscillation criteria for (3.1).

Theorem 3.4. *Let $\tilde{p_1} \in C([r_0, \infty), \mathbb{R}^+)$ and $\tilde{q_1} \in C([r_0, \infty), \mathbb{R})$ be functions satisfying* (3.5)*. Let also* (H1) *(or* (H1*)*) hold. If the functions $\tilde{p_1}$ and $\tilde{q_1}$ satisfy*

$$\int_{r_1}^{r} (\int_{r_0}^{s} u^{n-1} \tilde{p_1}(u) du)^{-1/\alpha} ds = \infty$$

and

$$\lim_{r \to \infty} \frac{1}{r} \int_{r_0}^{r} \Big(\int_{r_0}^{s} u^{n-1} \tilde{q_1}(u) du \Big) ds = \infty,$$

then (3.1) *is oscillatory in* Ω.

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