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ASYMPTOTIC SPEED OF SPREADING IN A DELAY LATTICE DIFFERENTIAL EQUATION WITHOUT QUASIMONOTONICITY

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Abstract. This article concerns the asymptotic speed of spreading in a delay lattice differential equation without quasimonotonicity. We obtain the speed of spreading by constructing an auxiliary undelayed equation, whose speed of spreading is the same as that of the original equation. The minimal wave speed of bounded positive traveling wave solutions is obtained from the asymptotic spreading.

1. Introduction

In this article, we study the asymptotic speed of spreading of the delay lattice differential equation

$$\frac{du_n(t)}{dt} = [\mathcal{D}u]_n(x) + ru_n(t)[1 - u_n(t) - au_n(t-\tau)], \quad n \in \mathbb{Z},\ t > 0, \tag{1.1}$$

where $\tau \geq 0$ and all the other parameters are positive, and

$$[\mathcal{D}u]_n(x) = \sum_{i\in\mathbb{Z}\setminus\{0\}} d_i[u_{n+i}(t) - u_n(t)]$$

satisfying

(D1) $d_i = d_{-i} \geq 0$, $i \in \mathbb{N}$ and $\sum_{i\in\mathbb{Z}\setminus\{0\}} d_i > 0$;
(D2) there exists $\lambda_0 \in (0,\infty]$ such that for any $\lambda \in [0,\lambda_0)$, $\sum_{i\in\mathbb{Z}\setminus\{0\}} d_i e^{\lambda i} < \infty$.

The time delay in (1.1) leads to the deficiency of quasimonotonicity [29] in the reaction term

$$F =: ru_n(t)[1 - u_n(t) - au_n(t-\tau)].$$

First we recall some results of the asymptotic speed of spreading in delay lattice differential equations. For a reaction term of general form, namely

$$\frac{du_n(t)}{dt} = [\mathcal{D}u]_n(x) + f(u_n(t), u_n(t-\tau)), \quad n \in \mathbb{Z},\ t > 0, \tag{1.2}$$

in which f is a continuous function, some results on asymptotic spreading have been established. If f is nondecreasing with respect the second variable, the asymptotic speed of spreading of (1.2) has been studied by Liang and Zhao [15], Ma et al

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[21], Thieme and Zhao [31], and Weng et al [33]. If f is only monotone near the unstable steady state, then it is locally quasimonotone and its asymptotic spreading has been studie by Fang et al [10], Yi et al [35]. If $\tau = 0$, then the quasimonotone condition holds and its dynamical behavior has been widely studied by Anderson et al [1], Bates and Chmaj [3], Bell and Cosner [4], Chow [8], Keener [14], Mallet-Paret [24, 25].

Since (1.1) does not satisfy the monotone conditions in the works mentioned above, it does not admit proper comparison principle. Therefore, its study of asymptotic spreading cannot be answered by the known conclusions. At the same time, the reaction term F can be regarded as a special form of Logistic nonlinearity with distributed delay, of which the dynamics is an important topic in literature. The purpose of this paper is to estimate the asymptotic speed of spreading of $u_n(t)$ formulated by the initial value problem of (1.1), herein the asymptotic speed of spreading is given as follows.

Definition 1.1. Assume that $u_n(t)$ is well defined for $n \in \mathbb{Z}$, $t > 0$. Then a constant $c_1 > 0$ is the asymptotic speed of spreading of $u_n(t)$ if

(1) for any $c > c_1$, $\lim_{t\to\infty} \sup_{|n|>ct} u_n(t) = 0$;
(2) for any $c \in (0, c_1)$, $\liminf_{t\to\infty} \inf_{|n|<ct} u_n(t) > 0$.

In population dynamics, the speed formulates the evolutionary processes of individuals from the viewpoint of an observer. More precisely, if an observer were to move to the right or left at a fixed speed greater than c_1, the local population density would eventually look like naught, and if an observer were to move to the right or left at a fixed speed less than c_1, the local population density would eventually look like positivity, and the population spreads roughly at the speed c_1 [32]. In literature, the definition was first introduced by Aronson and Weinberger [2] for the Fisher equation from the viewpoint of population dynamics. Since then, this concept has been widely studied and some important results have been established for reaction-diffusion equations, lattice differential equations, discrete-time recursions and integral equations, see Berestycki et al [5], Berestycki et al [6], Diekmann [9], Hsu and Zhao [11], Liang and Zhao [15], Thieme [30], Thieme and Zhao [31], Weinberger et al. [32] and Zhao [36] for some important results.

However, these results only hold for (local) quasimonotone systems and cannot be applied to (1.1) if $a\tau > 0$. Very recently, Lin [16] and Pan [27] have investigated the asymptotic spreading of a delayed equation without (local) quasimonotonicity. The current paper is motivated by the studies of those in [16, 27]. More precisely, we shall first estimate the growth of unknown functions, then calculate the asymptotic speed of spreading of (1.1). Under proper conditions, we find that the speed of spreading of (1.1) with $\tau > 0$ is the same as that of (1.1) with $\tau = 0$. Note that the time delay leads to the failure of comparison principle, then our conclusions imply the persistence of asymptotic speed of spreading with respect to time delay leading to the deficiency of quasimonotonicity.

The minimal wave speed of traveling wave solutions in evolutionary systems is also an important threshold formulating the dynamical properties. For quasimonotone systems, the minimal wave speed has been widely studied, and one general method is to confirm the nonexistence of traveling wave solutions by the theory of asymptotic spreading. In this article, applying our conclusions of asymptotic spreading, we obtain the nonexistence of traveling wave solutions, and formulate the minimal wave speed of traveling wave solutions in (1.1), which is the same as the asymptotic speed of spreading.

2. Initial value problem

We first introduce some notation. Let

$$l^\infty = \{u_n : n \in \mathbb{Z} \text{ and } u_n \text{ is uniformly bounded for all } n \in \mathbb{Z}\}.$$

Then it is a Banach space equipped with the standard supremum norm. Consider the initial value problem

$$\begin{gathered} \frac{du_n(t)}{dt} = [\mathcal{D}u]_n(x) + ru_n(t)[1 - u_n(t)], \quad n \in \mathbb{Z},\ t > 0, \\ u_n(0) = \phi(n), \quad n \in \mathbb{Z}. \end{gathered} \tag{2.1}$$

Note that $[\mathcal{D}u]_n(x) : l^\infty \to l^\infty$ is a bounded linear operator, then it generates an analytic semigroup $T(t) : l^\infty \to l^\infty$. Moreover, the semigroup is also positive. By Fang et al [10], Ma et al [21] and Weng et al [33], we have the following two lemmas.

Lemma 2.1. *If $0 \le \phi(n) \le 1, n \in \mathbb{Z}$, then (2.1) has a solution $u_n(t)$ for all $n \in \mathbb{Z}, t > 0$. If $w_n(t)$ satisfies*

$$\begin{gathered} \frac{dw_n(t)}{dt} \ge (\le)[\mathcal{D}w]_n(t) + rw_n(t)[1 - w_n(t)], \quad n \in \mathbb{Z}, t > 0, \\ w_n(0) \ge (\le)\phi(n), \quad n \in \mathbb{Z}, \end{gathered} \tag{2.2}$$

then $w_n(t) \ge (\le)u_n(t)$ for all $n \in \mathbb{Z}, t > 0$. In particular, $w_n(x)$ is called an upper (a lower) solution *of (2.1). On the other hand, if $w(t) : [0, \infty) \to l^\infty$ such that*

$$w(t) \ge (\le)T(t - s)w(s) + \int_s^t T(t - \theta)[rw(\theta)[1 - w(\theta)]]d\theta$$

for any $0 \le s \le t < \infty$, then $w(t) \ge (\le)u(t)$ for all $t > 0$ in the sense of standard partial ordering of l^∞.

Lemma 2.2. *Let $c_2 =: \inf_{\lambda>0} \big(\sum_{i\in\mathbb{Z}\setminus\{0\}} d_i(e^{\lambda i} - 1) + r\big)/\lambda$.*

1. *$c_2 > 0$;*
2. *$\phi(n) \ge 0$ for all $n \in \mathbb{Z}$, if there exists $M > 0$ such that $\phi(n) = 0, |n| > M$ and $\phi(n) > 0$ holds for some $n \in \mathbb{Z}$, then c_2 is the asymptotic speed of spreading of $u_n(t)$ defined by (2.1);*
3. *$\phi(n) \ge 0$ for all $n \in \mathbb{Z}$ and $\phi(n) > 0$ holds for some $n \in \mathbb{Z}$, for any given $c \in (0, c_2)$, we have*
$$\liminf_{t\to\infty} \inf_{|n|<ct} u_n(t) = \limsup_{t\to\infty} \sup_{|n|<ct} u_n(t) = 1;$$
4. *c_2 is continuous in r;*
5. *c_2 is strictly decreasing in r.*

Consider the initial value problem

$$\begin{gathered} \frac{du_n(t)}{dt} = [\mathcal{D}u]_n(x) + ru_n(t)[1 - u_n(t) - au_n(t - \tau)], \quad n \in \mathbb{Z},\ t > 0, \\ u_n(s) = \psi(n, s), \quad n \in \mathbb{Z},\ s \in [-\tau, 0]. \end{gathered} \tag{2.3}$$

Applying the theory of abstract functional differential equations [26], we have the following conclusions.

Lemma 2.3. *Assume that $0 \le \psi(n,s) \le 1$ for all $n \in \mathbb{Z}, s \in [-\tau, 0]$ and for each $n \in \mathbb{Z}$, $\psi(n,s)$ is continuous in $s \in [-\tau, 0]$.*

(1) *(2.3) admits a mild solution $u_n(t), n \in \mathbb{Z}, t > 0$ satisfying $0 \le u_n(t) \le 1$, $n \in \mathbb{Z}$, $t > 0$. In particular, for $u(t) : [0, \infty) \to l^\infty$, it takes the form*
$$u(t) = T(t-s)u(s) + \int_s^t T(t-\theta)[ru(\theta)[1 - u(\theta) - au(\theta - \tau)]]d\theta\,,$$
for any $0 \le s \le t < \infty$;
(2) *if $t > \tau$, then $u_n(t)$ is a classical solution satisfying (1.1);*
(3) *if $\psi(n,0) > 0$ for some $n \in \mathbb{Z}$, then $u_n(t) > 0$ for all $n \in \mathbb{Z}$, $t > 0$.*

Moreover, $u_n(t)$ satisfies the following nice properties.

Lemma 2.4. *Assume that $0 \le \psi(n,s) \le 1$ for all $n \in \mathbb{Z}, s \in [-\tau, 0]$ and for each $n \in \mathbb{Z}$, $\psi(n,s)$ is continuous in $s \in [-\tau, 0]$. If $t > \tau$, then*
$$\Big|\frac{du_n(t)}{dt}\Big| \le \max\Big\{\sum_{i \in \mathbb{Z}\setminus\{0\}} d_i + \frac{r}{4}, \sum_{i \in \mathbb{Z}\setminus\{0\}} d_i + ra\Big\} =: L$$
for any $n \in \mathbb{Z}$ and $t > \tau$.

Proof. Since $u_n(t)$ is a classical solution when $t > \tau$, we have
$$\begin{aligned} \frac{du_n(t)}{dt} &= [\mathcal{D}u]_n(x) + ru_n(t)[1 - u_n(t) - au_n(t-\tau)] \\ &= \sum_{i \in \mathbb{Z}\setminus\{0\}} d_i u_{n+i}(t) + ru_n(t)[1 - u_n(t)] \\ &\quad - u_n(t) \sum_{i \in \mathbb{Z}\setminus\{0\}} d_i - rau_n(t)u_n(t-\tau). \end{aligned}$$
Note that
$$\begin{gathered} \sum_{i \in \mathbb{Z}\setminus\{0\}} d_i u_{n+i}(t) + ru_n(t)[1 - u_n(t)] \le \sum_{i \in \mathbb{Z}\setminus\{0\}} d_i + \frac{r}{4}, \\ u_n(t) \sum_{i \in \mathbb{Z}\setminus\{0\}} d_i + rau_n(t)u_n(t-\tau) \le \sum_{i \in \mathbb{Z}\setminus\{0\}} d_i + ra. \end{gathered}$$
Then
$$\Big|\frac{du_n(t)}{dt}\Big| \le \max\big\{\sum_{i \in \mathbb{Z}\setminus\{0\}} d_i + \frac{r}{4}, \sum_{i \in \mathbb{Z}\setminus\{0\}} d_i + ra\big\}$$
for any $n \in \mathbb{N}$, $t > \tau$. The proof is complete. $\square$

3. Asymptotic speed of spreading

In this section, we investigate the asymptotic speed of spreading of $u_n(t)$ defined by (2.3). When $a \ge 1$, the result is formulated as follows.

Theorem 3.1. *Assume that $a \ge 1$ such that $L = \sum_{i \in \mathbb{Z}\setminus\{0\}} d_i + ra$. Further suppose that the initial value satisfies*

(I1) *$0 \le \psi(n,s) \le 1$ for all $n \in \mathbb{Z}, s \in [-\tau, 0]$;*
(I2) *for each $n \in \mathbb{Z}$, $\psi(n,s)$ is continuous in $s \in [-\tau, 0]$;*
(I3) *there exists some $n \in \mathbb{Z}$ such that $\psi(n,0) > 0$;*
(I4) *there exists some $M \in \mathbb{N}$ such that $\psi(n,s) = 0$ for all $|n| > M$, $s \in [-\tau, 0]$.*

If $aL\tau < 1$, then c_2 is the asymptotic speed of spreading of $u_n(t)$ defined by (2.3).

We now prove the result by several lemmas, through which the conditions of Theorem 3.1 hold without further clarification.

Lemma 3.2. *If $c > c_2$, then $\lim_{t\to\infty} \sup_{|n|>ct} u_n(t) = 0$.*

Proof. By Lemma 2.3, we see that $u_n(t) \geq 0$ for $t > 0$, $n \in \mathbb{Z}$, and so $u(t) : [0,\infty) \to l^\infty$ satisfies

$$u(t) \leq T(t-s)u(s) + \int_s^t T(t-\theta)[ru(\theta)[1-u(\theta)]]d\theta$$

for any $0 \leq s \leq t < \infty$. Then

$$u(t) \leq w(t), \quad t > 0$$

where $w(t) : [0,\infty) \to l^\infty$ is defined by (2.1) with $\phi(n) = \psi(n,0)$. By the second item of Lemma 2.2, we have what we want. The proof is complete. $\square$

By Lemma 2.3, we have the following conclusion.

Lemma 3.3. *For any $\epsilon > 0$, consider the initial value problem*

$$\begin{gathered} \frac{du_n(t)}{dt} = [\mathcal{D}u]_n(x) + ru_n(t)[1 - aL\tau - (1+a)u_n(t)], \\ u_0(0) = \epsilon, \quad u_n(0) = 0, n \in \mathbb{Z} \setminus \{0\}. \end{gathered} \tag{3.1}$$

Then there exists $\delta = \delta(\epsilon) > 0$ such that $u_0(\tau) = \delta$.

Since the asymptotic speed of spreading is concerned with the long time behavior of the unknown function, we consider only $t \geq 2\tau + 1$, such that $u_n(t)$ is a classical solution satisfying the differential equation (1.1); we consider the differential equation.

Lemma 3.4. *For any $\epsilon > 0$, there exists $M = M(\epsilon) > 1$ such that*

$$\frac{du_n(t)}{dt} \geq [\mathcal{D}u]_n(x) + ru_n(t)[1 - \epsilon - Mu_n(t)]$$

for $n \in \mathbb{Z}$, $t > 2\tau + 2$.

Proof. If $u_n(t-\tau) < \epsilon/a$, then

$$1 - u_n(t) - au_n(t-\tau) < 1 - \epsilon - u_n(t).$$

If $u_n(t-\tau) \geq \epsilon/a$, then Lemma 3.3 implies

$$u_n(t) \geq \delta(\epsilon/a) > 0$$

by the comparison principle. Therefore, there exists $M > 1$ such that

$$(M-1)u_n(t) \geq (M-1)\delta(\epsilon/a) > a$$

and so

$$(M-1)u_n(t) \geq au_n(t-\tau).$$

The proof is complete. $\square$

Lemma 3.5. *For any fixed $c < c_2$, we have $\liminf_{t\to\infty} \inf_{|n|<ct} u_n(t) > 0$.*

Proof. Let $\epsilon > 0$ be such that
$$\inf_{\lambda>0} \frac{\sum_{i\in\mathbb{Z}\setminus\{0\}} d_i(e^{\lambda i}-1)+r(1-2\epsilon)}{\lambda} =: c_3 > c,$$
then ϵ is admissible by Lemma 2.2. From Lemma 3.4, we see that
$$\frac{du_n(t)}{dt} \geq [\mathcal{D}u]_n(x) + ru_n(t)[1-\epsilon - Mu_n(t)].$$
Therefore, if $t > 2\tau + 2$, then
$$\begin{gathered}\frac{du_n(t)}{dt} \geq [\mathcal{D}u]_n(x) + ru_n(t)[1-\epsilon - Mu_n(t)], \quad n\in\mathbb{Z},\ t>2\tau+2,\\ u_n(2\tau+2-s) > 0, \quad s\in[-\tau,0],\ n\in\mathbb{Z}.\end{gathered}$$
By Lemma 2.2, we see that
$$\liminf_{t\to\infty} \inf_{|n|<c_3 t} u_n(t) \geq \frac{1-\epsilon}{M} > 0,$$
which implies what we wanted. The proof is complete. $\square$

Remark 3.6. Lemma 3.5 remains valid if (I4) does not hold.

Summarizing Lemmas 3.2-3.5, we complete the proof of Theorem 3.1.

Note that in Theorem 3.1, $a \geq 1$ is assumed. If $a < 1$, then we have
$$\frac{du_n(t)}{dt} \geq [\mathcal{D}u]_n(x) + ru_n(t)[1-a-u_n(t)]$$
by Lemma 2.3. Replacing $aL\tau$ by a in (3.1), and we have the following conclusions after a discussion similar to the proof of Theorem 3.1.

Theorem 3.7. *Assume that $a < 1$ holds and* (I1)–(I4) *from Theorem 3.1 hold. Then c_2 is the asymptotic speed of spreading of $u_n(t)$ defined by* (2.3).

Theorem 3.7 was also proved by Pan [28].

4. Applications

In this part, we consider the traveling wave solutions. Hereafter, a traveling wave solution of (1.1) is a special solution with form $u_n(t) = \rho(n+ct)$, in which $c > 0$ is the wave speed and $\rho \in C^1(\mathbb{R},\mathbb{R})$ is the wave profile that propagates in $\mathbb{Z}$. Thus, ρ and c satisfy
$$c\frac{d\rho(\xi)}{d\xi} = \sum_{i\in\mathbb{Z}\setminus\{0\}} d_i[\rho(\xi+i)-\rho(\xi)] + r\rho(\xi)[1-\rho(\xi)-a\rho(\xi-c\tau)], \quad \xi\in\mathbb{R}. \tag{4.1}$$
To better reflect the evolutionary processes, we also require
$$\lim_{\xi\to-\infty}\rho(\xi) = 0, \quad \liminf_{\xi\to\infty}\rho(\xi) > 0. \tag{4.2}$$
In population dynamics, a positive solution satisfying (4.1)-(4.2) could formulate the successful invasion of individuals. The existence of traveling wave solutions in lattice differential equations with time delay have been widely studied, e.g. [10, 12, 13, 15, 17, 18, 20, 21, 22, 23, 31, 33, 34, 35]. We now present that c_2 is the minimal wave speed such that (4.1)-(4.2) does not have a bounded positive solution if $c < c_2$ while (4.1)-(4.2) has a bounded positive solution if $c \geq c_2$.

Lemma 4.1. *If $\rho(\xi)$ is a nonzero bounded positive solution of* (4.1), *then* $0 \leq \rho(\xi) \leq 1$ *for* $\xi \in \mathbb{R}$.

Proof. Denote
$$\overline{\rho} = \sup_{\xi \in \mathbb{R}} \rho(\xi), \quad \underline{\rho} = \inf_{\xi \in \mathbb{R}} \rho(\xi),$$
then both $\overline{\rho}$ and $\underline{\rho}$ are bounded and nonnegative. If there exists ξ_0 such that $\overline{\rho} = \rho(\xi_0)$, then
$$c\frac{d\rho(\xi)}{d\xi}\big|_{\xi=\xi_0} = 0, \qquad \sum_{i \in \mathbb{Z}\setminus\{0\}} d_i[\rho(\xi_0 + i) - \rho(\xi_0)] \leq 0,$$
which further implies that
$$1 - \rho(\xi_0) - a\rho(\xi_0 - c\tau) \geq 0,$$
and so $\overline{\rho} \leq 1$.

If $\overline{\rho} = \limsup_{\xi \to \infty} \rho(\xi)$, then there exists $\{\xi_m\}$ such that $\xi_m \to \infty$, $\rho(\xi_m) \to \overline{\rho}$, $m \to \infty$, and
$$\lim_{m \to \infty} \Big[c\frac{d\rho(\xi)}{d\xi}\big|_{\xi=\xi_m} - \sum_{i \in \mathbb{Z}\setminus\{0\}} d_i[\rho(\xi_m + i) - \rho(\xi_m)]\Big] \geq 0,$$
which indicates that $1 - \overline{\rho} \geq 0$ and so $\overline{\rho} \leq 1$.

Moreover, if $\overline{\rho} = \limsup_{\xi \to -\infty} \rho(\xi)$, then the discussion is similar to that of $\overline{\rho} = \limsup_{\xi \to \infty} \rho(\xi)$. The proof is complete. $\square$

Theorem 4.2. *Assume that either $a < 1$ or $a \geq 1$ with $aL\tau < 1$. If $c < c_2$, then* (4.1)-(4.2) *does not have a bounded positive solution.*

Proof. Were the statement false, then for some $c_4 \in (0, c_2)$, (4.1)–(4.2) has a bounded positive solution $\rho(\xi)$. Then Lemma 4.1 implies that $0 \leq \rho(\xi) \leq 1$, $\xi \in \mathbb{R}$, and $u_n(t) = \rho(n + c_4 t)$ is a solution of
$$\begin{gathered}\frac{du_n(t)}{dt} = [\mathcal{D}u]_n(x) + ru_n(t)[1 - u_n(t) - au_n(t-\tau)], \quad n \in \mathbb{Z},\ t > 0,\\ u_n(s) = \rho(n + cs), \quad n \in \mathbb{Z},\ s \in [-\tau, 0],\end{gathered}$$
where the initial value satisfies (I1)–(I3). By Theorems 3.1 and 3.7 (also see Remark 3.6), we see that
$$\liminf_{t \to \infty} \inf_{|2n| = (c_2 + c_4)t} u_n(t) = \liminf_{t \to \infty} \inf_{|2n| = (c_2 + c_4)t} \rho(n + ct) > 0.$$
Let $-2n = (c_2 + c_4)t$, then $t \to \infty$ leads to $\xi = n + c_4 t \to -\infty$ such that
$$\rho(n + c_4 t) \to 0, \quad t \to \infty,$$
which implies a contradiction. The proof is complete. $\square$

Moreover, we can also prove the existence of traveling wave solutions when $c \geq c_2$, of which the proof is similar to that in Pan [28].

Theorem 4.3. *Assume that $a \geq 1$ such that $aL\tau < 1$. If $c \geq c_2$, then* (4.1)–(4.2) *has a bounded positive solution.*

Proof. We shall prove the conclusion by the idea in Pan [28], in which the authors studied the problem if $a < 1$. For each $c > c_2$, define $\lambda_1(c)$ be the smaller root of

$$\sum_{i\in\mathbb{Z}\setminus\{0\}} d_i(e^{\lambda i}-1)-c\lambda+r=0$$

Define the continuous functions

$$\overline{\rho}(\xi)=\min\{e^{\lambda_1(c)\xi},1\},\quad \underline{\rho}(\xi)=\max\{e^{\lambda_1(c)\xi}-qe^{\eta\lambda_1(c)\xi},0\},$$

in which $\eta-1>0$ is small and $q>1$ is large. Similar to Pan [28, Lemma 3.3], we can prove that (4.1)–(4.2) has a bounded positive solution satisfying

$$\overline{\rho}(\xi)\geq\rho(\xi)\geq\underline{\rho}(\xi),\quad \xi\in\mathbb{R}$$

and

$$\liminf_{\xi\to\infty}\rho(\xi)>\frac{1-aL\tau}{2+a}. \tag{4.3}$$

We now prove the result for $c=c_2$ by passing to a limit function [19]. Let $c^i\to c^*, i\in\mathbb{N}$, be strictly decreasing, then for each fixed c^i, (4.1)-(4.2) has a positive fixed point $\rho_i(\xi)$ such that

$$0<\rho_i(\xi)<1,\quad \liminf_{\xi\to\infty}\rho_i(\xi)>\frac{1-aL\tau}{2+a},\quad \lim_{\xi\to-\infty}\rho_i(\xi)=0,\quad i\in\mathbb{N}.$$

Without loss of generality, we assume that

$$\rho_i(0)=\frac{1-aL\tau}{4+a},\quad \rho_i(\xi)<\frac{1-aL\tau}{4+a},\quad \xi<0.$$

It is clear that $\rho_i(\xi)$ and $\rho_i'(\xi)$ are equicontinuous and uniformly bounded. By Ascoli-Arzela and a nested subsequence argument [7], (4.1) with $c=c_2$ has a bounded solution $\widetilde{\rho}$ satisfying

$$0<\widetilde{\rho}(\xi)<1,\quad \liminf_{\xi\to\infty}\widetilde{\rho}(\xi)>\frac{1-aL\tau}{2+a},$$
$$\widetilde{\rho}(0)=\frac{1-aL\tau}{4+a},\widetilde{\rho}(\xi)\leq\frac{1-aL\tau}{4+a},\quad \xi<0.$$

If $\limsup_{\xi\to-\infty}\widetilde{\rho}(\xi)>0$, then the invariant form of traveling wave solutions implies that

$$\widetilde{\rho}(0)>\frac{1-aL\tau}{2+a}$$

by the asymptotic speed of spreading. This is a contradiction occurs; therefore, $\lim_{\xi\to-\infty}\widetilde{\rho}(\xi)=0$. The proof is complete. $\square$

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