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NONEXISTENCE OF GLOBAL SOLUTIONS FOR DIFFERENTIAL INEQUALITIES OF SOBOLEV TYPE

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Abstract. In this article we study three differential inequalities of Sobolev type. Using Pokhozhaev's nonlinear capacity method, we provide sufficient conditions for the nonexistence of global nontrivial weak solutions to these problems.

1. Introduction

Pokhozhaev [9] introduced a new method for studying the blow-up of solutions. This method is based on the the notion of nonlinear capacity generated by a differential operator. Using this approach, Pokhozhaev and Mitidieri [8] studied for the first time the blow-up of solutions of nonlinear partial differential equations of arbitrary order. After these works, many other results on the nonexistence of time-global solutions of nonlinear evolution equations were obtained on the basis of this method. For more details, we refer the reader to [2, 3, 4, 5, 6, 7] and references therein.

Basing on Pokhozhaev's nonlinear capacity method, Korpusov and Sveshnikov [7] determined the Fujita critical exponents for some differential inequalities of Sobolev type in the entire space $\mathbb{R}^N$. Recently, Alsaedi et al [1] extended the results in [7] by considering time fractional derivatives and fractional power of the Laplacian instead of the classical time derivatives and the usual Laplacian.

In this paper, we study the question of nonexistence of nontrivial solutions to three classes of differential inequalities of Sobolev type. This work aims to extend the recent results of Korpusov and Sveshnikov [7].

2. Results and proofs

Problem 1. We start with the differential inequality

$$-\Delta\frac{\partial^k u}{\partial t^k} \geq |u|^q, \tag{2.1}$$

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subject to the initial conditions

$$u(x,0)=u_0(x),\quad \frac{\partial u}{\partial t}(x,0)=u_1(x),\ \dots,\ \frac{\partial^{k-1}u}{\partial t^{k-1}}(x,0)=u_{k-1}(x),\quad x\in\mathbb{R}^N, \tag{2.2}$$

where $k\in\mathbb{N}^*$ and $q>1$. We set $\mathbb{R}_+^{N+1}=[0,\infty)\times\mathbb{R}^N$.

Let us make precise the mean of a weak solution to (2.1)-(2.2).

Definition 2.1. A weak solution to (2.1)-(2.2) is a function

$$u\in\mathbb{L}^q_{\rm loc}(\mathbb{R}_+^{N+1}),\quad (u_0,\dots,u_{k-2},u_{k-1})\in\big(\mathbb{L}^1_{\rm loc}(\mathbb{R}^N)\big)^{k-1}\times\mathbb{L}^1(\mathbb{R}^N)$$

satisfying the inequality

$$-\sum_{i=1}^k I_i+(-1)^{k+1}\int_{\mathbb{R}_+^{N+1}}u\Delta\varphi^{(k)}\,dx\,dt\ge\int_{\mathbb{R}_+^{N+1}}|u|^q\varphi\,dx\,dt, \tag{2.3}$$

for any nonnegative regular function φ, $\varphi(\cdot,t)=0$, $t\ge T$, where

$$I_i=(-1)^i\int_{\mathbb{R}^N}u_{k-i}(x)\Delta\varphi^{(i-1)}(x,0)\,dx,\quad 1\le i\le k.$$

Our first main result is the following.

Theorem 2.2. *For any functions $(u_0,\dots,u_{k-2},u_{k-1})\in\big(\mathbb{L}^1_{\rm loc}(\mathbb{R}^N)\big)^{k-1}\times\mathbb{L}^1(\mathbb{R}^N)$ and any $q>1$, problem (2.1)-(2.2) has no global nontrivial weak solutions.*

Proof. Assume that the solution is nontrivial and global. Using the ε-Young inequality

$$ab\le\varepsilon a^r+C_\varepsilon b^s,$$

with $r=q$ and $s=q/(q-1)$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}_+^{N+1}}|u||\Delta\varphi^{(k)}|\,dx\,dt\\&\le\varepsilon\int_{\mathbb{R}_+^{N+1}}|u|^q\varphi\,dx\,dt+C_\varepsilon\int_{\mathbb{R}_+^{N+1}}\varphi^{-1/(q-1)}|\Delta\varphi^{(k)}|^{q/(q-1)}\,dx\,dt.\end{aligned}\tag{2.4}$$

Using (2.3), (2.4), and taking $\varepsilon=1/2$, we obtain

$$\begin{aligned}\int_{\mathbb{R}_+^{N+1}}|u|^q\varphi\,dx\,dt\le C\Big(&\int_{\mathbb{R}_+^{N+1}}\varphi^{-1/(q-1)}|\Delta\varphi^{(k)}|^{q/(q-1)}\,dx\,dt\\&+\sum_{i=1}^k\int_{\mathbb{R}^N}|u_{k-i}||\Delta\varphi^{(i-1)}(x,0)|\,dx\Big).\end{aligned}\tag{2.5}$$

At this stage, we take the test function $\varphi(x,t)$ in the form

$$\varphi(x,t)=\varphi_1^\ell(t)\varphi_2^\ell(x),\quad \varphi_1(t)=\psi\big(\frac{t}{R^\alpha}\big),\quad \varphi_2(x)=\psi\big(\frac{|x|^2}{R^{2\beta}}\big), \tag{2.6}$$

with $\alpha>0$, $\beta>0$, and $R>0$; where

$$\psi(\xi)=\begin{cases}1 & \text{if } 0\le\xi\le1,\\ \text{decreasing} & \text{if } 1\le\xi\le2,\\ 0 & \text{if } \xi\ge2,\end{cases}$$

with ℓ chosen sufficiently large such that

$$\int_1^2 \frac{|(\psi^\ell)^{(k)}(\sigma)|^{q/(q-1)}}{\psi^{\ell/(q-1)}(\sigma)}\,d\sigma<\infty \quad\text{and}\quad \int_{1\le|y|\le\sqrt{2}} \frac{|\Delta\psi^\ell(|y|^2)|^{q/(q-1)}}{\psi(|y|^2)^{\ell/(q-1)}}\,dy<\infty.$$

Observe that

$$\int_{\mathbb{R}^N} |u_{k-i}||\Delta\varphi^{(i-1)}(x,0)|\,dx=0,\quad 2\le i\le k. \tag{2.7}$$

On the other hand, since $u_{k-1}\in L^1(\mathbb{R}^N)$, we obtain

$$\lim_{R\to\infty}\int_{\mathbb{R}^N}|u_{k-1}||\Delta\varphi(x,0)|\,dx=\lim_{R\to\infty}\int_{R^\beta\le|x|\le\sqrt{2}R^\beta}|u_{k-1}||\Delta\varphi_2(x)|\,dx=0. \tag{2.8}$$

Now, Let us consider the change of variables $\sigma=R^{-\alpha}t$ and $y=R^{-\beta}x$. We obtain

$$\int_{\mathbb{R}_+^{N+1}}\varphi^{-1/(q-1)}|\Delta\varphi^{(k)}|^{q/(q-1)}\,dx\,dt=C_1R^{N\beta+\alpha-(2\beta+k\alpha)q/(q-1)}, \tag{2.9}$$

where

$$C_1=\Big(\int_1^2 \frac{|(\psi^\ell)^{(k)}(\sigma)|^{q/(q-1)}}{\psi^{\ell/(q-1)}(\sigma)}\,d\sigma\Big)\Big(\int_{1\le|y|\le\sqrt{2}} \frac{|\Delta\psi^\ell(|y|^2)|^{q/(q-1)}}{\psi(|y|^2)^{\ell/(q-1)}}\,dy\Big)<\infty.$$

Now, from (2.5), (2.7) and (2.9), we have

$$\int_{\mathbb{R}_+^{N+1}}|u|^q\varphi\,dx\,dt\le C\Big(C_1R^{N\beta+\alpha-(2\beta+k\alpha)q/(q-1)}+\int_{\mathbb{R}^N}|u_{k-1}||\Delta\varphi(x,0)|\,dx\Big). \tag{2.10}$$

If we choose

$$\frac{\alpha}{\beta}>\frac{N(q-1)-2q}{(k-1)q+1},$$

then we obtain $N\beta+\alpha-(2\beta+k\alpha)q/(q-1)<0$, which implies that

$$\lim_{R\to\infty}R^{N\beta+\alpha-(2\beta+k\alpha)q/(q-1)}=0. \tag{2.11}$$

Moreover, by the monotone convergence theorem, we have

$$\lim_{R\to\infty}\int_{\mathbb{R}_+^{N+1}}|u|^q\varphi\,dx\,dt=\int_{\mathbb{R}_+^{N+1}}|u|^q\,dx\,dt\,. \tag{2.12}$$

Finally, using (2.8), (2.10), (2.11) and (2.12), we obtain

$$\int_{\mathbb{R}_+^{N+1}}|u|^q\,dx\,dt=0,$$

which is contradiction. □

Remark 2.3. If $k=1$ in Theorem 2.2, then we recover the constrain imposed in [7].

Problem 2. We consider the differential inequality

$$-\Delta\frac{\partial^k u}{\partial t^k}-\Delta(|u|^{p-1}u)\geq |u|^q, \tag{2.13}$$

subject to the initial conditions

$$u(x,0)=u_0(x),\quad \frac{\partial u}{\partial t}(x,0)=u_1(x),\ \dots,\ \frac{\partial^{k-1}u}{\partial t^{k-1}}(x,0)=u_{k-1}(x),\quad x\in\mathbb{R}^N, \tag{2.14}$$

where $k\in\mathbb{N}^*$, $p>1$ and $q>1$.

Definition 2.4. A weak solution to (2.13)-(2.14) is a function

$$u\in\mathbb{L}^{\max\{p,q\}}_{\rm loc}(\mathbb{R}^{N+1}_+),\quad (u_0,\dots,u_{k-2},u_{k-1})\in\big(\mathbb{L}^1_{\rm loc}(\mathbb{R}^N)\big)^{k-1}\times\mathbb{L}^1(\mathbb{R}^N)$$

satisfying the inequality

$$\begin{aligned}&-\sum_{i=1}^k I_i+(-1)^{k+1}\int_{\mathbb{R}^{N+1}_+}u\Delta\varphi^{(k)}\,dx\,dt-\int_{\mathbb{R}^{N+1}_+}|u|^{p-1}u\Delta\varphi\,dx\,dt\\ &\geq\int_{\mathbb{R}^{N+1}_+}|u|^q\varphi\,dx\,dt,\end{aligned} \tag{2.15}$$

for any nonnegative regular function φ, $\varphi(\cdot,t)=0$, $t\geq T$, where

$$I_i=(-1)^i\int_{\mathbb{R}^N}u_{k-i}(x)\Delta\varphi^{(i-1)}(x,0)\,dx,\quad 1\leq i\leq k.$$

Theorem 2.5. *Let* $(u_0,\dots,u_{k-2},u_{k-1})\in\big(\mathbb{L}^1_{\rm loc}(\mathbb{R}^N)\big)^{k-1}\times\mathbb{L}^1(\mathbb{R}^N)$. *If*

$$q>p>1,\quad N<\frac{2(qk+1-p)}{k(q-p)},$$

then problem (2.13)-(2.14) *has no global nontrivial weak solutions.*

Proof. Assume that the solution is nontrivial and global. We start by writing

$$\int_{\mathbb{R}^{N+1}_+}|u|^p|\Delta\varphi|\,dx\,dt=\int_{\mathbb{R}^{N+1}_+}|u|^p\varphi^{p/q}\varphi^{-p/q}|\Delta\varphi|\,dx\,dt.$$

Using the ε-Young inequality with parameters $r=q/p$ and $s=q/(q-p)$, we obtain

$$\int_{\mathbb{R}^{N+1}_+}|u|^p|\Delta\varphi|\,dx\,dt\leq\varepsilon\int_{\mathbb{R}^{N+1}_+}|u|^q\varphi\,dx\,dt+C_\varepsilon\int_{\mathbb{R}^{N+1}_+}\varphi^{-p/(q-p)}|\Delta\varphi|^{q/(q-p)}\,dx\,dt. \tag{2.16}$$

Again, by the ε-Young inequality with parameters $r=q$ and $s=q/(q-1)$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^{N+1}_+}|u||\Delta\varphi^{(k)}|\,dx\,dt\\ &\leq\varepsilon\int_{\mathbb{R}^{N+1}_+}|u|^q\varphi\,dx\,dt+C_\varepsilon\int_{\mathbb{R}^{N+1}_+}\varphi^{-1/(q-1)}|\Delta\varphi^{(k)}|^{q/(q-1)}\,dx\,dt.\end{aligned} \tag{2.17}$$

Using (2.15), (2.16), (2.17) and taking $\varepsilon = 1/4$, we obtain

$$\begin{aligned} \int_{\mathbb{R}_+^{N+1}} |u|^q \varphi \,dx\,dt \le C\Big(&\int_{\mathbb{R}_+^{N+1}} \varphi^{-p/(q-p)} |\Delta\varphi|^{q/(q-p)} \,dx\,dt \\ &+ \int_{\mathbb{R}_+^{N+1}} \varphi^{-1/(q-1)} |\Delta\varphi^{(k)}|^{q/(q-1)} \,dx\,dt \\ &+ \sum_{i=1}^{k} \int_{\mathbb{R}^N} |u_{k-i}||\Delta\varphi^{(i-1)}(x,0)|\,dx\Big). \end{aligned} \tag{2.18}$$

We take the test function $\varphi(x,t)$ in the form (2.6) such that

$$\int_1^2 \frac{|(\psi^\ell)^{(k)}(\sigma)|^{q/(q-1)}}{\psi^{\ell/(q-1)}(\sigma)}\,d\sigma < \infty, \quad \int_{1\le|y|\le\sqrt{2}} \frac{|\Delta\psi^\ell(|y|^2)|^{q/(q-1)}}{\psi(|y|^2)^{\ell/(q-1)}}\,dy < \infty$$

and

$$\int_{1\le|y|\le\sqrt{2}} \frac{|\Delta\psi^\ell(|y|^2)|^{q/(q-p)}}{\psi(|y|^2)^{\ell p/q-p}}\,dy < \infty.$$

As in the proof of Theorem 2.2, we observe that

$$\int_{\mathbb{R}^N} |u_{k-i}||\Delta\varphi^{(i-1)}(x,0)|\,dx = 0, \quad 2 \le i \le k \tag{2.19}$$

and

$$\lim_{R\to\infty} \int_{\mathbb{R}^N} |u_{k-1}||\Delta\varphi(x,0)|\,dx = \lim_{R\to\infty} \int_{R^\beta\le|x|\le\sqrt{2}R^\beta} |u_{k-1}||\Delta\varphi_2(x)|\,dx = 0. \tag{2.20}$$

Now, Let us consider the change of variables $\sigma = R^{-\alpha}t$ and $y = R^{-\beta}x$. We obtain

$$\int_{\mathbb{R}_+^{N+1}} \varphi^{-1/(q-1)} |\Delta\varphi^{(k)}|^{q/(q-1)} \,dx\,dt = C_1 R^{N\beta+\alpha-(2\beta+k\alpha)q/(q-1)}, \tag{2.21}$$

where

$$C_1 = \Big(\int_1^2 \frac{|(\psi^\ell)^{(k)}(\sigma)|^{q/(q-1)}}{\psi^{\ell/(q-1)}(\sigma)}\,d\sigma\Big)\Big(\int_{1\le|y|\le\sqrt{2}} \frac{|\Delta\psi^\ell(|y|^2)|^{q/(q-1)}}{\psi(|y|^2)^{\ell/(q-1)}}\,dy\Big) < \infty.$$

Similarly, we have

$$\int_{\mathbb{R}_+^{N+1}} \varphi^{-p/(q-p)} |\Delta\varphi|^{q/(q-p)} \,dx\,dt = C_2 R^{N\beta+\alpha-2\beta q/(q-p)}, \tag{2.22}$$

where

$$C_2 = \Big(\int_0^2 \psi^\ell(\sigma)\,d\sigma\Big)\Big(\int_{1\le|y|\le\sqrt{2}} \frac{|\Delta\psi^\ell(|y|^2)|^{q/(q-p)}}{\psi(|y|^2)^{\ell p/q-p}}\,dy\Big) < \infty.$$

Now, using (2.18), (2.19), (2.21) and (2.22), we obtain

$$\begin{aligned} \int_{\mathbb{R}_+^{N+1}} |u|^q \varphi \,dx\,dt \le C\Big(&C_1 R^{N\beta+\alpha-(2\beta+k\alpha)q/(q-1)} \\ &+ C_2 R^{N\beta+\alpha-2\beta q/(q-p)} + \int_{\mathbb{R}^N} |u_{k-1}||\Delta\varphi(x,0)|\,dx\Big). \end{aligned} \tag{2.23}$$

In the case

$$N < \frac{2(qk+1-p)}{k(q-p)},$$

we have
$$\frac{N(q-1)-2q}{1+(k-1)q} < \frac{2q}{q-p} - N.$$
We choose the pair (α,β) such that
$$\frac{N(q-1)-2q}{1+(k-1)q} < \frac{\alpha}{\beta} < \frac{2q}{q-p} - N.$$
We obtain
$$N\beta+\alpha-(2\beta+k\alpha)q/(q-1) < 0 \quad \text{and} \quad N\beta+\alpha-2\beta q/(q-p) < 0. \tag{2.24}$$
Finally, letting $R \to \infty$ in (2.23), using (2.20) and (2.24), we obtain
$$\int_{\mathbb{R}_+^{N+1}} |u|^q \, dx \, dt = 0,$$
which is contradiction. □

Remark 2.6. If $k = p = 1$ in Theorem 2.5, then we recover the constrain imposed in [7].

Problem 3. We consider the differential inequality
$$-\frac{\partial}{\partial t}(\Delta(|u|^m u) + \lambda |u|^p u) - \Delta(|u|^n u) \geq |u|^q, \tag{2.25}$$
subject to the initial condition
$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^N, \tag{2.26}$$
where $m > 0$, $p > 0$, $n > 0$, $q > 0$, $\lambda \in \mathbb{R}$.

Definition 2.7. A weak solution to (2.25)-(2.26) is a function
$$u \in \mathbb{L}^{\max\{m+1,p+1,n+1,q\}}_{\rm loc}(\mathbb{R}_+^{N+1})$$
such that
$$u_0|u_0|^m \in \mathbb{L}^1(\mathbb{R}^N), \quad u_0|u_0|^p \in \mathbb{L}^1(\mathbb{R}^N)$$
and the condition
$$\begin{aligned} &\int_{\mathbb{R}_+^{N+1}} [|u|^m u \Delta\varphi' + \lambda |u|^p u \varphi'] \, dx \, dt - \int_{\mathbb{R}_+^{N+1}} |u|^n u \Delta\varphi \, dx \, dt \\ &+ \int_{\mathbb{R}^N} [|u_0|^m u_0 \Delta\varphi(x,0) + \lambda |u_0|^p u_0 \varphi(x,0)] \, dx \\ &\geq \int_{\mathbb{R}_+^{N+1}} |u|^q \varphi \, dx \, dt \end{aligned} \tag{2.27}$$
is satisfied for any nonnegative regular function φ, $\varphi(\cdot,t) = 0$, $t \geq T$.

Theorem 2.8. *Let u_0 be such that $u_0|u_0|^m \in \mathbb{L}^1(\mathbb{R}^N)$ and $u_0|u_0|^p \in \mathbb{L}^1(\mathbb{R}^N)$. Suppose that $q > \min\{m,p,n\} + 1$,*
$$\max\Big\{\frac{N(q-m-1)-2q}{m+1}, \frac{N(q-p-1)}{p+1}\Big\} < \frac{2q-N(q-n-1)}{q-n-1},$$
$$\lambda \int_{\mathbb{R}^N} |u_0|^p u_0 \, dx \leq 0.$$
Then problem (2.25)-(2.26) has no global nontrivial weak solutions.

Proof. Assume that the solution is nontrivial and global. We start by writing

$$\int_{\mathbb{R}_+^{N+1}} |u|^{m+1}|\Delta\varphi'|\,dx\,dt = \int_{\mathbb{R}_+^{N+1}} |u|^{m+1}\varphi^{(m+1)/q}\varphi^{-(m+1)/q}|\Delta\varphi'|\,dx\,dt.$$

Using the ε-Young inequality with parameters $r = q/(m+1)$ and $s = q/[q-(m+1)]$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}_+^{N+1}} |u|^{m+1}|\Delta\varphi'|\,dx\,dt\\ &\le \varepsilon\int_{\mathbb{R}_+^{N+1}} |u|^q\varphi\,dx\,dt + C_\varepsilon\int_{\mathbb{R}_+^{N+1}} \varphi^{-(m+1)/[q-(m+1)]}|\Delta\varphi'|^{q/[q-(m+1)]}\,dx\,dt.\end{aligned}\tag{2.28}$$

Similarly, we have

$$\begin{aligned}&\int_{\mathbb{R}_+^{N+1}} |u|^{p+1}|\varphi'|\,dx\,dt\\ &\le \varepsilon\int_{\mathbb{R}_+^{N+1}} |u|^q\varphi\,dx\,dt + C_\varepsilon\int_{\mathbb{R}_+^{N+1}} \varphi^{-(p+1)/[q-(p+1)]}|\varphi'|^{q/[q-(p+1)]}\,dx\,dt\end{aligned}\tag{2.29}$$

and

$$\begin{aligned}&\int_{\mathbb{R}_+^{N+1}} |u|^{n+1}|\Delta\varphi|\,dx\,dt\\ &\le \varepsilon\int_{\mathbb{R}_+^{N+1}} |u|^q\varphi\,dx\,dt + C_\varepsilon\int_{\mathbb{R}_+^{N+1}} \varphi^{-(n+1)/[q-(n+1)]}|\Delta\varphi|^{q/[q-(n+1)]}\,dx\,dt.\end{aligned}\tag{2.30}$$

Now, for $\varepsilon = 1/[2(2+|\lambda)]$, using (2.27), (2.28), (2.29) and (2.30), we obtain

$$\begin{aligned}\int_{\mathbb{R}_+^{N+1}} |u|^q\varphi\,dx\,dt \le C\Big(&\int_{\mathbb{R}_+^{N+1}} \varphi^{-(m+1)/[q-(m+1)]}|\Delta\varphi'|^{q/[q-(m+1)]}\,dx\,dt\\ &+\int_{\mathbb{R}_+^{N+1}} \varphi^{-(p+1)/[q-(p+1)]}|\varphi'|^{q/[q-(p+1)]}\,dx\,dt\\ &+\int_{\mathbb{R}_+^{N+1}} \varphi^{-(n+1)/[q-(n+1)]}|\Delta\varphi|^{q/[q-(n+1)]}\,dx\,dt\\ &+\int_{\mathbb{R}^N} [|u_0|^m u_0\Delta\varphi(x,0) + \lambda|u_0|^p u_0\varphi(x,0)]\,dx\Big).\end{aligned}\tag{2.31}$$

We take the test function $\varphi(x,t)$ defined by (2.6) such that

$$\int_1^2 \frac{|\psi'(\sigma)|^{q/[q-(r+1)]}}{\psi(\sigma)^{q/[q-(r+1)]-\ell}}\,d\sigma < \infty, \qquad \int_{1\le|y|\le\sqrt{2}} \frac{|\Delta\psi^\ell(|y|^2)|^{q/[q-(s+1)]}}{\psi(|y|^2)^{\ell(s+1)/[q-(s+1)]}}\,dy < \infty,$$

for $r \in \{m,p\}$ and $s \in \{m,n\}$. Observe that

$$\lim_{R\to\infty}\int_{\mathbb{R}^N} [|u_0|^m u_0\Delta\varphi(x,0) + \lambda|u_0|^p u_0\varphi(x,0)]\,dx = \lambda\int_{\mathbb{R}^N} |u_0|^p u_0\,dx.\tag{2.32}$$

Let us now consider the change of variables $\sigma = R^{-\alpha}t$ and $y = R^{-\beta}x$. We obtain

$$\begin{aligned}&\int_{\mathbb{R}_+^{N+1}} \varphi^{-(m+1)/[q-(m+1)]}|\Delta\varphi'|^{q/[q-(m+1)]}\,dx\,dt\\ &= C_1 R^{\alpha+\beta N - q(2\beta+\alpha)/[q-(m+1)]},\end{aligned}\tag{2.33}$$

where

$$C_1 = \ell^{q/[q-(m+1)]}\Big(\int_1^2 \frac{|\psi'(\sigma)|^{q/[q-(m+1)]}}{\psi(\sigma)^{q/[q-(m+1)]-\ell}}\,d\sigma\Big) \times \Big(\int_{1\le|y|\le\sqrt{2}} \frac{|\Delta\psi^\ell(|y|^2)|^{q/[q-(m+1)]}}{\psi(|y|^2)^{\ell(m+1)/[q-(m+1)]}}\,dy\Big) < \infty.$$

Similarly,

$$\int_{\mathbb{R}_+^{N+1}} \varphi^{-(p+1)/[q-(p+1)]}|\varphi'|^{q/[q-(p+1)]}\,dx\,dt = C_2 R^{\alpha+N\beta-\alpha q/[q-(p+1)]}, \tag{2.34}$$

where

$$C_2 = \ell^{q/[q-(p+1)]}\Big(\int_1^2 \frac{|\psi'(\sigma)|^{q/[q-(p+1)]}}{\psi(\sigma)^{q/[q-(p+1)]-\ell}}\,d\sigma\Big)\Big(\int_{\mathbb{R}^N} \psi^\ell(|y|^2)\,dy\Big) < \infty.$$

Again, we have

$$\int_{\mathbb{R}_+^{N+1}} \varphi^{-(n+1)/[q-(n+1)]}|\Delta\varphi|^{q/[q-(n+1)]}\,dx\,dt = C_3 R^{\alpha+N\beta-2\beta q/[q-(n+1)]}, \tag{2.35}$$

where

$$C_3 = \Big(\int_0^2 \psi^\ell(\sigma)\,d\sigma\Big)\Big(\int_{1\le|y|\le\sqrt{2}} \frac{|\Delta\psi^\ell(|y|^2)|^{q/[q-(n+1)]}}{\psi(|y|^2)^{\ell(n+1)/[q-(n+1)]}}\,dy\Big) < \infty.$$

Now, using (2.31), (2.33), (2.34) and (2.35), we obtain

$$\begin{aligned} &\int_{\mathbb{R}_+^{N+1}} |u|^q\varphi\,dx\,dt \\ &\le C\Big(R^{\alpha+\beta N-q(2\beta+\alpha)/[q-(m+1)]} + R^{\alpha+N\beta-\alpha q/[q-(p+1)]} \\ &\quad + R^{\alpha+N\beta-2\beta q/[q-(n+1)]}\int_{\mathbb{R}^N}[|u_0|^m u_0\Delta\varphi(x,0) + \lambda|u_0|^p u_0\varphi(x,0)]\,dx\Big). \end{aligned} \tag{2.36}$$

In the case

$$\max\Big\{\frac{N(q-m-1)-2q}{m+1}, \frac{N(q-p-1)}{p+1}\Big\} < \frac{2q-N(q-n-1)}{q-n-1},$$

we can choose the pair (α,β) such that

$$\max\Big\{\frac{N(q-m-1)-2q}{m+1}, \frac{N(q-p-1)}{p+1}\Big\} < \frac{\alpha}{\beta} < \frac{2q-N(q-n-1)}{q-n-1}.$$

Under this choice, we obtain

$$\begin{gathered} \alpha+N\beta-\frac{q(2\beta+\alpha)}{q-(m+1)} < 0, \\ \alpha+N\beta-\frac{\alpha q}{q-(p+1)} < 0, \\ \alpha+N\beta-\frac{2\beta q}{q-(n+1)} < 0. \end{gathered}$$

Finally, letting $R\to\infty$ in (2.36), using the above inequalities and (2.32), we obtain

$$\int_{\mathbb{R}_+^{N+1}} |u|^q\,dx\,dt \le \lambda\int_{\mathbb{R}^N} |u_0|^p u_0\,dx.$$

By assumption, we have

$$\lambda \int_{\mathbb{R}^N} |u_0|^p u_0 \, dx \leq 0.$$

Then we obtained a contradiction. □

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