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GROWTH OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS WITH ANALYTIC COEFFICIENTS OF $[p, q]$-ORDER IN THE UNIT DISC

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Abstract. In this article, we study the growth of solutions of homogeneous linear differential equation in which the coefficients are analytic functions of $[p, q]$-order in the unit disc. We obtain results about the (lower) $[p, q]$-order of the solutions, and the (lower) $[p, q]$-convergence exponent for the sequence of distinct zeros of $f(z) - \varphi(z)$.

1. Introduction

We study the growth of solutions of the following two equations, for $n \ge 2$,

$$f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{1.1}$$

and

$$f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z), \tag{1.2}$$

where $A_n(z), \dots, A_1(z), A_0(z)$ $(\not\equiv 0)$ and $F(z)(\not\equiv 0)$ are meromorphic functions in the complex plane $\mathbb{C}$ or in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

As is well known, the theory of meromorphic solutions of linear differential equations (1.1) and (1.2) in $\mathbb{C}$ becomes mature and the results are fruitful, but the theory of meromorphic solutions of equations (1.1) and (1.2) in Δ is not as developed as the one in $\mathbb{C}$. The reason may be that the properties of meromorphic functions in $\mathbb{C}$ or Δ are of some differences, which result in that some important tools in $\mathbb{C}$ are ineffective in Δ. However, we also discover that for meromorphic solutions of equations (1.1) and (1.2) in Δ, there are many similar properties as the ones of meromorphic solutions of equations (1.1) and (1.2) in $\mathbb{C}$. For example, it is well known that when the coefficients are entire functions, the solutions of equations (1.1) and (1.2) are entire functions. Similarly, when the coefficients are analytic functions in Δ, the solutions of equations (1.1) and (1.2) are analytic functions in Δ, and there are exactly n linearly independent solutions of equation (1.1) (see e.g. [8]). Hence, it is interesting to investigate meromorphic solutions of linear differential equations (1.1) and (1.2) in Δ.

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Typically, Heittokangas [8] investigated meromorphic solutions of linear differential equations (1.1) and (1.2) in Δ by introducing the definition of the function spaces and his results also gave some important tools for further investigations on the theory of meromorphic solutions of equations (1.1) and (1.2) in Δ. After that, many papers (see e.g. [2, 3, 4, 5, 9, 16]) focused on this topic. We proceed in this way in this paper, inspired by the relative case in $\mathbb{C}$; that is, we try to find some results similar to the one in Liu-Tu-Shi [15], which is stated as follows.

Theorem 1.1 ([15]). *Let $A_0(z), \dots, A_{n-1}(z)$ be entire functions satisfying*

$$\max\{\sigma_{[p,q]}(A_j) : j = 1, \dots, n-1\} \leq \sigma_{[p,q]}(A_0) < \infty,$$
$$\max\{\tau_{[p,q]}(A_j) : \sigma_{[p,q]}(A_j) = \sigma_{[p,q]}(A_0) > 0, j \neq 0\} < \tau_{[p,q]}(A_0).$$

Then every nontrivial solution $f(z)$ of (1.1) satisfies $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$.

Liu-Tu-Shi [15] used the $[p,q]$-type of $A_0(z)$ to dominate the $[p,q]$-types of other coefficients, and got the result about $\sigma_{[p+1,q]}(f)$. Thus, the following questions arise naturally: (1) Whether the results similar to Theorem 1.1 can be obtained in Δ? (2) If we use the lower $[p,q]$-type of $A_0(z)$ to dominate other coefficients, what can be said about $\mu_{[p+1,q]}(f)$? (3) Can we find some other conditions to dominate other coefficients? In this paper, we give some answers to the above questions.

Before we give our main results in the next section, it is necessary to introduce some notation. In this paper, we assume that the readers are familiar with the standard notation and the fundamental results of the Nevanlinna's theory in $\mathbb{C}$ and Δ (see e.g. [7, 8, 13, 14]). Moreover, in [11, 12], Juneja and his co-authors investigated some properties of entire functions of $[p,q]$-order, and obtained some results. In [15], in order to keep accordance with the general definition of an entire function $f(z)$ of iterated p-order, Liu-Tu-Shi gave a minor modification to the original definition of $[p,q]$-order given in [11, 12]. Further, in [2, 3], Belaïdi defined $[p,q]$-order of analytic and meromorphic functions in Δ. For conveniences, we list the following concepts (see e.g. [2, 3, 10]).

Definition 1.2. Let p, q be integers such that $p \geq q \geq 1$, and $f(z)$ be a meromorphic function in Δ. The $[p,q]$-order and the lower $[p,q]$-order of $f(z)$ are defined respectively by

$$\sigma_{[p,q]}(f) = \limsup_{r \to 1^-} \frac{\log_p^+ T(r,f)}{\log_q \frac{1}{1-r}}, \quad \mu_{[p,q]}(f) = \liminf_{r \to 1^-} \frac{\log_p^+ T(r,f)}{\log_q \frac{1}{1-r}}.$$

For an analytic function $f(z)$ in Δ, we also define

$$\sigma_{M,[p,q]}(f) = \limsup_{r \to 1^-} \frac{\log_{p+1}^+ M(r,f)}{\log_q \frac{1}{1-r}}, \quad \mu_{M,[p,q]}(f) = \liminf_{r \to 1^-} \frac{\log_{p+1}^+ M(r,f)}{\log_q \frac{1}{1-r}}.$$

Definition 1.3. Let p, q be integers such that $p \geq q \geq 1$, $a \in \mathbb{C} \bigcup \{\infty\}$, and $f(z)$ be a meromorphic function in Δ. The $[p,q]$-convergence exponents of the sequence of a-points and the sequence of distinct a-points of $f(z)$ are defined respectively by

$$\lambda_{[p,q]}(f-a) = \limsup_{r \to 1^-} \frac{\log_p^+ N(r, \frac{1}{f-a})}{\log_q \frac{1}{1-r}}, \quad \overline{\lambda}_{[p,q]}(f-a) = \limsup_{r \to 1^-} \frac{\log_p^+ \overline{N}(r, \frac{1}{f-a})}{\log_q \frac{1}{1-r}}.$$

The lower $[p,q]$-convergence exponents of the sequence of a-points and the sequence of distinct a-points of $f(z)$ are defined respectively by

$$\underline{\lambda}_{[p,q]}(f-a)=\liminf_{r\to 1^-}\frac{\log_p^+ N(r,\frac{1}{f-a})}{\log_q \frac{1}{1-r}},$$
$$\underline{\overline{\lambda}}_{[p,q]}(f-a)=\liminf_{r\to 1^-}\frac{\log_p^+ \overline{N}(r,\frac{1}{f-a})}{\log_q \frac{1}{1-r}}.$$

Furthermore, we obtain the definitions of $\lambda_{[p,q]}(f-\varphi)$, $\overline{\lambda}_{[p,q]}(f-\varphi)$, $\underline{\lambda}_{[p,q]}(f-\varphi)$ and $\underline{\overline{\lambda}}_{[p,q]}(f-\varphi)$ in Δ, when the constant a in Definiton 1.3 is replaced by a meromorphic function $\varphi(z)$ in Δ.

Definition 1.4. Let p,q be integers such that $p\geq q\geq 1$, and $f(z)$ be a meromorphic function of $[p,q]$-order $\sigma(0<\sigma<\infty)$ and lower $[p,q]$-order $\mu(0<\mu<\infty)$ in Δ. The $[p,q]$-type and the lower $[p,q]$-type of $f(z)$ are defined respectively by

$$\tau_{[p,q]}(f)=\limsup_{r\to 1^-}\frac{\log_{p-1}^+ T(r,f)}{(\log_{q-1}\frac{1}{1-r})^{\sigma}},\quad \underline{\tau}_{[p,q]}(f)=\liminf_{r\to 1^-}\frac{\log_{p-1}^+ T(r,f)}{(\log_{q-1}\frac{1}{1-r})^{\mu}}.$$

For an analytic function $f(z)$ in Δ, we also define

$$\tau_{M,[p,q]}(f)=\limsup_{r\to 1^-}\frac{\log_{p}^+ M(r,f)}{(\log_{q-1}\frac{1}{1-r})^{\sigma}},\quad \underline{\tau}_{M,[p,q]}(f)=\liminf_{r\to 1^-}\frac{\log_{p}^+ M(r,f)}{(\log_{q-1}\frac{1}{1-r})^{\mu}}.$$

Different from the case in $\mathbb{C}$, we have the following results for the case in Δ.

Proposition 1.5 ([2]). *Let p,q be integers such that $p\geq q\geq 1$, and $f(z)$ be an analytic function of $[p,q]$-order in Δ. The following two statements hold:*

(i) *If $p=q$, then $\sigma_{[p,q]}(f)\leq\sigma_{M,[p,q]}(f)\leq\sigma_{[p,q]}(f)+1$.*
(ii) *If $p>q$, then $\sigma_{[p,q]}(f)=\sigma_{M,[p,q]}(f)$.*

Similarly, we can get the following proposition.

Proposition 1.6. *Let p,q be integers such that $p\geq q\geq 1$, and $f(z)$ be an analytic function of lower $[p,q]$-order in Δ. The following two statements hold:*

(i) *If $p=q$, then $\mu_{[p,q]}(f)\leq\mu_{M,[p,q]}(f)\leq\mu_{[p,q]}(f)+1$;*
(ii) *If $p>q$, then $\mu_{[p,q]}(f)=\mu_{M,[p,q]}(f)$.*

2. Main results

In this paper, we consider the case that the coefficients are analytic functions in Δ, and obtain two main results on the growth of solutions of equation (1.1). Moreover, we get the results about the $[p,q]$-convergence exponent and the lower $[p,q]$-convergence exponent of the sequence of distinct zeros of $f(z)-\varphi(z)$.

Theorem 2.1. *Let p,q be integers such that $p>q\geq 2$, and $A_{n-1}(z),\dots,A_1(z)$, $A_0(z)$ $(\not\equiv 0)$ be analytic functions in Δ with $0<\mu=\mu_{[p,q]}(A_0)\leq\sigma_{[p,q]}(A_0)<\infty$. Assume that $\max\{\sigma_{[p,q]}(A_j)|j=1,\dots,n-1\}\leq\mu_{[p,q]}(A_0)$ and that $\max\{\tau_{[p,q]}(A_j)|\sigma_{[p,q]}(A_j)=\mu_{[p,q]}(A_0), j\neq 0\}<\underline{\tau}_{[p,q]}(A_0)=\tau<\infty$. If $f(z)$ $(\not\equiv 0)$ is a solution of (1.1), then we have*

$$\begin{aligned}\underline{\overline{\lambda}}_{[p+1,q]}(f-\varphi)&=\mu_{[p+1,q]}(f)=\mu_{[p,q]}(A_0)\\&\leq\sigma_{[p,q]}(A_0)=\sigma_{[p+1,q]}(f)=\overline{\lambda}_{[p+1,q]}(f-\varphi),\end{aligned}$$

where $\varphi(z)$ $(\not\equiv 0)$ *is an analytic function in* Δ *with* $\sigma_{[p+1,q]}(\varphi) < \mu_{[p,q]}(A_0)$.

Theorem 2.2. *Let* p, q *be integers such that* $p > q \geq 1$, *and* $A_{n-1}(z), \dots, A_1(z)$, $A_0(z)$ $(\not\equiv 0)$ *be analytic functions in* Δ *with* $0 < \mu = \mu_{[p,q]}(A_0) \leq \sigma_{[p,q]}(A_0) < \infty$. *Assume that* $\max\{\sigma_{[p,q]}(A_j) | j = 1, \dots, n-1\} \leq \mu_{[p,q]}(A_0)$ *and that* $\limsup_{r\to 1^-} \sum_{j=1}^{n-1} m(r, A_j)/m(r, A_0) < 1$. *If* $f(z)$ $(\not\equiv 0)$ *is a solution of* (1.1), *then we have*

$$\underline{\overline{\lambda}}_{[p+1,q]}(f-\varphi) = \mu_{[p+1,q]}(f) = \mu_{[p,q]}(A_0) \leq \sigma_{[p,q]}(A_0) = \sigma_{[p+1,q]}(f) = \overline{\lambda}_{[p+1,q]}(f-\varphi),$$

where $\varphi(z)$ $(\not\equiv 0)$ *is an analytic function in* Δ *with* $\sigma_{[p+1,q]}(\varphi) < \mu_{[p,q]}(A_0)$.

Remark 2.3. In Theorems 2.1 and 2.2, we just consider the case $p > q$ to make sure the Lemmas 3.8 and 3.11 hold. Moreover, $q \geq 2$ in Theorem 2.1 is necessary for using Lemma 3.10.

3. Preliminary lemmas

Lemma 3.1 ([9])**.** *Let* $A_j(z)$, $j = 0, \dots, n-1$ *be analytic functions in* D_R ($D_R = \{z \in \mathbb{C} \big| |z| < R\}$), *where* $0 < R \leq \infty$, *and* $f(z)$ *be a solution of* (1.1) *in* D_R, $1 \leq p < \infty$. *Then for all* $0 \leq r < R$,

$$m_p(r, f)^p \leq C\Big(\sum_{j=0}^{n-1} \int_0^{2\pi} \int_0^r |A_j(se^{i\theta})|^{\frac{p}{n-j}} ds d\theta + 1\Big),$$

where $C = C(n) > 0$ *is a constant depending on p, and on the initial values of* $f(z)$ *at the point* z_θ, *where* $A_j(z_\theta) \neq 0$ *for some* $j = 0, \dots, n-1$.

Lemma 3.2 ([8, 16])**.** *Let* $f(z)$ *be a meromorphic function in* Δ, *and* $k \in \mathbb{N}$. *Then*

$$m(r, \frac{f^{(k)}}{f}) = S(r, f),$$

where $S(r, f) = O(\log^+ T(r, f) + \log \frac{1}{1-r})$, *possibly outside a set* $E_1 \subset [0, 1)$ *with* $\int_{E_1} \frac{dr}{1-r} < \infty$. *If* $f(z)$ *is of finite order (namely, finite iterated 1-order), then*

$$m(r, \frac{f^{(k)}}{f}) = O(\log \frac{1}{1-r}).$$

Lemma 3.3 ([2])**.** *Let* p, q *be integers such that* $p \geq q \geq 1$, $k \geq 1$ *be an integer and* $f(z)$ *be a meromorphic function in* Δ *such that* $\sigma_{[p,q]}(f) = \sigma < \infty$. *Then*

$$m(r, \frac{f^{(k)}}{f}) = O\Big(\exp_{p-1}\{(\sigma + \varepsilon) \log_q \frac{1}{1-r}\}\Big)$$

holds for any $\varepsilon > 0$ *and all* $r \to 1^-$ *outside a set* $E_2 \subset [0, 1)$ *with* $\int_{E_2} \frac{dr}{1-r} < \infty$.

Lemma 3.4 ([1])**.** *Let* $g : (0, 1) \to \mathbb{R}$ *and* $h : (0, 1) \to \mathbb{R}$ *be monotone increasing functions such that* $g(r) \leq h(r)$ *holds outside an exceptional set* $E_3 \subset [0, 1)$ *with* $\int_{E_3} \frac{dr}{1-r} < \infty$. *Then there exists a constant* $d \in (0, 1)$ *such that if* $s(r) = 1-d(1-r)$, *then* $g(r) \leq h(s(r))$ *for all* $r \in [0, 1)$.

Lemma 3.5. *Let* p, q *be integers such that* $p \geq q \geq 1$ *and* $A_{n-1}(z), \dots, A_1(z)$, $A_0(z)$ $(\not\equiv 0)$, $F(z)$ $(\not\equiv 0)$ *be meromorphic functions in* Δ. *If* $f(z)$ *is a meromorphic solution of* (1.2) *satisfying*

$$\max\{\sigma_{[p,q]}(F), \sigma_{[p,q]}(A_j) | j = 0, \dots, n-1\} < \sigma_{[p,q]}(f) = \sigma < \infty,$$

then $\overline{\lambda}_{[p,q]}(f) = \lambda_{[p,q]}(f) = \sigma_{[p,q]}(f)$.

Proof. By (1.2), we have

$$\frac{1}{f} = \frac{1}{F}\left(\frac{f^{(n)}}{f} + A_{n-1}(z)\frac{f^{(n-1)}}{f} + \cdots + A_0(z)\right). \tag{3.1}$$

If $f(z)$ has a zero at $z_0 \in \Delta$ of order $\gamma(> n)$ and $A_{n-1}(z), \dots, A_1(z), A_0(z)$ are all analytic at z_0, then $F(z)$ has a zero at z_0 of order at least $\gamma - n$. Hence, we have

$$N(r, \frac{1}{f}) \le n\overline{N}(r, \frac{1}{f}) + N(r, \frac{1}{F}) + \sum_{j=0}^{n-1} N(r, A_j). \tag{3.2}$$

By (3.1), we have

$$m(r, \frac{1}{f}) \le m(r, \frac{1}{F}) + \sum_{j=0}^{n-1} m(r, A_j) + \sum_{j=1}^{n} m(r, \frac{f^{(j)}}{f}) + O(1). \tag{3.3}$$

Lemma 3.3 gives

$$m(r, \frac{f^{(j)}}{f}) = O\Big(\exp_{p-1}\{(\sigma + \varepsilon)\log_q \frac{1}{1-r}\}\Big), \quad j = 1, \dots, n, \tag{3.4}$$

holds for any $\varepsilon > 0$ and all $r \to 1^-$ outside a set $E_2 \subset [0, 1)$ with $\int_{E_2} \frac{dr}{1-r} < \infty$. Therefore, by (3.2)-(3.4) and the first fundamental theorem,

$$\begin{aligned} T(r, f) = T(r, \frac{1}{f}) + O(1) &\le n\overline{N}(r, \frac{1}{f}) + T(r, F) + \sum_{j=0}^{n-1} T(r, A_j) \\ &+ O\Big(\exp_{p-1}\{(\sigma + \varepsilon)\log_q \frac{1}{1-r}\}\Big) \end{aligned} \tag{3.5}$$

holds for all $r \to 1^-, r \notin E_2$. Set $\rho = \max\{\sigma_{[p,q]}(F), \sigma_{[p,q]}(A_j) | j = 0, \dots, n-1\}$, then for $r \to 1^-$, we have

$$\sum_{j=0}^{n-1} T(r, A_j) + T(r, F) \le (n + 1)\exp_p\{(\rho + \varepsilon)\log_q \frac{1}{1-r}\}. \tag{3.6}$$

Thus, by (3.5) and (3.6), for all $r \to 1^-$, $r \notin E_2$, we have

$$\begin{aligned} T(r, f) &\le n\overline{N}(r, \frac{1}{f}) + (n + 1)\exp_p\{(\rho + \varepsilon)\log_q \frac{1}{1-r}\} \\ &\quad + O\Big(\exp_{p-1}\{(\sigma + \varepsilon)\log_q \frac{1}{1-r}\}\Big) \\ &\le n\overline{N}(r, \frac{1}{f}) + \exp_p\{(\rho + 2\varepsilon)\log_q \frac{1}{1-r}\}. \end{aligned} \tag{3.7}$$

Hence, by Lemma 3.4 and (3.7), for all $r \to 1^-$, we have

$$T(r, f) \le n\overline{N}(s(r), \frac{1}{f}) + \exp_p\{(\rho + 2\varepsilon)\log_q \frac{1}{1-s(r)}\}, \tag{3.8}$$

where $s(r) = 1 - d(1 - r)$, $d \in (0, 1)$. If $\overline{\lambda}_{[p,q]}(f) < \sigma_{[p,q]}(f) = \sigma$, then for any ε $(0 < 3\varepsilon < \sigma - \max\{\overline{\lambda}_{[p,q]}(f), \rho\})$ and all $r \to 1^-$, we have

$$T(r, f) \le n\exp_p\{(\overline{\lambda}_{[p,q]}(f) + \varepsilon)\log_q \frac{1}{1-s(r)}\} + \exp_p\{(\rho + 2\varepsilon)\log_q \frac{1}{1-s(r)}\}$$

$$\leq (n+1)\exp_p\{(\sigma-\varepsilon)\log_q \frac{1}{1-s(r)}\},$$

which results in a contradiction that $\sigma = \sigma_{[p,q]}(f) < \sigma - \varepsilon$. Therefore, we have $\overline{\lambda}_{[p,q]}(f) \geq \sigma_{[p,q]}(f) = \sigma$. Since $\overline{\lambda}_{[p,q]}(f) \leq \lambda_{[p,q]}(f) \leq \sigma_{[p,q]}(f)$, the result holds. □

Lemma 3.6. *Let p, q be integers such that $p \geq q \geq 1$ and $A_{n-1}(z), \dots, A_1(z)$, $A_0(z)$ ($\not\equiv 0$), $F(z)$ ($\not\equiv 0$) be meromorphic functions in Δ. If $f(z)$ is a meromorphic solution of* (1.2) *satisfying*

$$\max\{\sigma_{[p,q]}(F), \sigma_{[p,q]}(A_j) | j = 0, \dots, n-1\} < \mu_{[p,q]}(f) \leq \sigma_{[p,q]}(f) < \infty,$$

then we have $\overline{\underline{\lambda}}_{[p,q]}(f) = \underline{\lambda}_{[p,q]}(f) = \mu_{[p,q]}(f)$.

Proof. Since $\max\{\sigma_{[p,q]}(F), \sigma_{[p,q]}(A_j)|j = 0, \dots, n-1\} < \mu_{[p,q]}(f)$, we have that for $r \to 1^-$,

$$T(r,F) = o(T(r,f)), \quad T(r,A_j) = o(T(r,f)), \quad j = 0, \dots, n-1. \tag{3.9}$$

By (3.5) and (3.9), we have

$$(1-o(1))T(r,f) \leq n\overline{N}(r,\frac{1}{f}) + O\Big(\exp_{p-1}\{(\sigma_{[p,q]}(f)+\varepsilon)\log_q \frac{1}{1-r}\}\Big), \tag{3.10}$$

for any $\varepsilon > 0$ and $r \to 1^-$, $r \notin E_2$, where $E_2 \subset [0,1)$ satisfies $\int_{E_2} \frac{dr}{1-r} < \infty$. Hence, by Lemma 3.4 and (3.10), for all $r \to 1^-$, we have

$$(1-o(1))T(r,f) \leq n\overline{N}(s(r),\frac{1}{f}) + O\Big(\exp_{p-1}\{(\sigma_{[p,q]}(f)+\varepsilon)\log_q \frac{1}{1-s(r)}\}\Big),$$

where $s(r) = 1 - d(1-r)$, $d \in (0,1)$. Hence, we have $\overline{\underline{\lambda}}_{[p,q]}(f) \geq \mu_{[p,q]}(f)$. Since $\overline{\underline{\lambda}}_{[p,q]}(f) \leq \underline{\lambda}_{[p,q]}(f) \leq \mu_{[p,q]}(f)$, the result holds. □

Lemma 3.7. *Let p, q be integers such that $p \geq q \geq 1$ and $f(z)$ be an analytic function in Δ with $\mu_{[p,q]}(f) = \mu < \infty$. Then for any given $\varepsilon > 0$, there exists a set $E_4 \subset [0,1)$ with $\int_{E_4} \frac{dr}{1-r} = \infty$, such that*

$$\mu = \mu_{[p,q]}(f) = \lim_{r\to 1^-,\, r\in E_4} \frac{\log_p^+ T(r,f)}{\log_q \frac{1}{1-r}},$$

and

$$T(r,f) < \exp_p\{(\mu+\varepsilon)\log_q \frac{1}{1-r}\}, r \in E_4,\ r \to 1^-.$$

Moreover, if $p > q \geq 1$, then we also have

$$M(r,f) < \exp_{p+1}\{(\mu+\varepsilon)\log_q \frac{1}{1-r}\}, \quad r \in E_4,\ r \to 1^-.$$

Proof. We use a similar proof as [17, Lemma 6]. By the definition of lower $[p,q]$-order, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to 1^- such that $1 - d(1-r_n) < r_{n+1}(0 < d < 1)$ (such a sequence $\{r_n\}_{n=1}^{\infty}$ is called an exponential sequence, see [6]), and

$$\lim_{r_n\to 1^-} \frac{\log_p^+ T(r_n,f)}{\log_q \frac{1}{1-r_n}} = \mu_{[p,q]}(f).$$

Then for any $r \in [1 - \frac{1-r_n}{d}, r_n]$, we have

$$\frac{\log_p^+ T(r,f)}{\log_q \frac{1}{1-r}} \leq \frac{\log_p^+ T(r_n,f)}{\log_q \frac{1}{1-r_n}} \frac{\log_q \frac{1}{1-r_n}}{\log_q \frac{1}{1-r}}.$$

When $q \geq 1$, we have $\frac{\log_q \frac{1}{1-r_n}}{\log_q \frac{1}{1-r}} \to 1$, $r_n \to 1^-$. Let $E_4 = \bigcup_{n=n_1}^{\infty} [1 - \frac{1-r_n}{d}, r_n]$, where n_1 is some sufficiently large positive integer, then for any given $\varepsilon > 0$, we have

$$\lim_{r \to 1^-,\, r \in E_4} \frac{\log_p^+ T(r,f)}{\log_q \frac{1}{1-r}} = \lim_{r_n \to 1^-} \frac{\log_p^+ T(r_n,f)}{\log_q \frac{1}{1-r_n}} = \mu_{[p,q]}(f),$$

$$T(r,f) < \exp_p\{(\mu + \varepsilon) \log_q \frac{1}{1-r}\},\ r \in E_4,\ r \to 1^-,$$

$$\int_{E_4} \frac{dr}{1-r} = \sum_{n=n_1}^{\infty} \int_{1-\frac{1-r_n}{d}}^{r_n} \frac{dt}{1-t} = \sum_{n=n_1}^{\infty} \log \frac{1}{d} = \infty.$$

If $p > q \geq 1$, then by the standard inequality

$$T(r,f) \leq \log^+ M(r,f) \leq \frac{1+3r}{1-r} T(\frac{1+r}{2}, f),$$

(see e.g. [13, p. 26]), we have

$$\lim_{r \to 1^-,\, r \in E_4} \frac{\log_p^+ T(r,f)}{\log_q \frac{1}{1-r}} = \lim_{r \to 1^-,\, r \in E_4} \frac{\log_{p+1}^+ M(r,f)}{\log_q \frac{1}{1-r}}.$$

Therefore,

$$M(r,f) < \exp_{p+1}\{(\mu + \varepsilon) \log_q \frac{1}{1-r}\}, \quad r \in E_4,\ r \to 1^-.$$

□

Lemma 3.8. *Let p, q be integers such that $p > q \geq 1$, and $A_{n-1}(z), \dots, A_1(z)$, $A_0(z)$ ($\not\equiv 0$) be analytic functions in Δ such that $\max\{\sigma_{[p,q]}(A_j) | j \neq s\} \leq \mu_{[p,q]}(A_s) < \infty$. If $f(z)(\not\equiv 0)$ is a solution of (1.1), then we have $\mu_{[p+1,q]}(f) \leq \mu_{[p,q]}(A_s)$.*

Proof. If $\mu_{[p,q]}(f) < \infty$, then $\mu_{[p+1,q]}(f) = 0 \leq \mu_{[p,q]}(A_s)$. So, we assume $\mu_{[p,q]}(f) = \infty$. By Lemma 3.1, we have

$$\begin{aligned} T(r,f) = m(r,f) &\leq C(\sum_{j=0}^{n-1} \int_0^{2\pi} \int_0^r |A_j(se^{i\theta})|^{\frac{1}{n-j}} ds d\theta + 1) \\ &\leq 2\pi C(\sum_{j=0}^{n-1} rM(r, A_j) + 1), \end{aligned} \tag{3.11}$$

where $C = C(n) > 0$ is a constant depending on the initial values of $f(z)$ at the point z_θ, where $A_j(z_\theta) \neq 0$ for some $j = 0, \dots, n-1$. Set $b = \max\{\sigma_{[p,q]}(A_j) | j \neq s\} = \max\{\sigma_{M,[p,q]}(A_j) | j \neq s\}$, then we have

$$M(r, A_j) \leq \exp_{p+1}\{(b + \varepsilon) \log_q \frac{1}{1-r}\}, \quad j \neq s, \tag{3.12}$$

for any $\varepsilon > 0$ and $r \to 1^-$. By Lemma 3.7, there exists a set $E_4 \subset [0,1)$ with $\int_{E_4} \frac{dr}{1-r} = \infty$ such that

$$M(r, A_s) \leq \exp_{p+1}\{(\mu_{[p,q]}(A_s) + \varepsilon) \log_q \frac{1}{1-r}\}, \quad r \in E_4,\ r \to 1^-. \tag{3.13}$$

By (3.11)-(3.13), for $r \in E_4$, $r \to 1^-$, we have

$$T(r,f) \leq O\Big(\exp_{p+1}\{(\mu_{[p,q]}(A_s) + 2\varepsilon) \log_q \frac{1}{1-r}\}\Big). \tag{3.14}$$

By (3.14) and Proposition 1.6, we have $\mu_{M,[p+1,q]}(f) = \mu_{[p+1,q]}(f) \leq \mu_{[p,q]}(A_s) = \mu_{M,[p,q]}(A_s)$. □

Lemma 3.9. *Let p, q be integers such that $p \geq q \geq 1$, and $A_{n-1}(z), \dots, A_1(z)$, $A_0(z)$ $(\not\equiv 0)$ be analytic functions in Δ. Assume that* $\max\{\sigma_{[p,q]}(A_j) | j = 1, \dots, n-1\} \leq \mu_{[p,q]}(A_0) = \mu$ $(0 < \mu < \infty)$ *and* $\max\{\tau_{[p,q]}(A_j) |\ \sigma_{[p,q]}(A_j) = \mu_{[p,q]}(A_0), j \neq 0\} < \underline{\tau}_{[p,q]}(A_0) = \tau$ $(0 < \tau < \infty)$*. If $f(z)(\not\equiv 0)$ is a solution of* (1.1)*, then we have* $\mu_{[p+1,q]}(f) \geq \mu_{[p,q]}(A_0)$.

Proof. Suppose that $f(z)$ is a nonzero solution of (1.1). By (1.1), we get

$$-A_0(z) = \frac{f^{(n)}(z)}{f(z)} + A_{n-1}(z)\frac{f^{(n-1)}(z)}{f(z)} + \dots + A_1(z)\frac{f'(z)}{f(z)}. \tag{3.15}$$

By (3.15), we have

$$T(r, A_0) = m(r, A_0) \leq \sum_{j=1}^{n-1} m(r, A_j) + \sum_{j=1}^{n} m(r, \frac{f^{(j)}}{f}).$$

Hence, by Lemma 3.2, we have

$$T(r, A_0) \leq \sum_{j=1}^{n-1} m(r, A_j) + O\Big(\log^+ T(r, f) + \log\frac{1}{1-r}\Big), \tag{3.16}$$

for $r \notin E_1$, where $E_1 \subset [0, 1)$ satisfies $\int_{E_1} \frac{dt}{1-t} < \infty$. Set

$$b = \max\{\sigma_{[p,q]}(A_j) | \sigma_{[p,q]}(A_j) < \mu_{[p,q]}(A_0) = \mu, j = 1, \dots, n-1\}.$$

If $\sigma_{[p,q]}(A_j) < \mu_{[p,q]}(A_0) = \mu$, then for any $\varepsilon(0 < 2\varepsilon < \min\{\mu - b, \tau - \tau_1\})$ and all $r \to 1^-$, we have

$$m(r, A_j) = T(r, A_j) \leq \exp_p\{(b+\varepsilon)\log_q \frac{1}{1-r}\} < \exp_p\{(\mu-\varepsilon)\log_q \frac{1}{1-r}\}. \tag{3.17}$$

Set $\tau_1 = \max\{\tau_{[p,q]}(A_j) | \sigma_{[p,q]}(A_j) = \mu_{[p,q]}(A_0), j \neq 0\}$, then $\tau_1 < \tau$. If $\sigma_{[p,q]}(A_j) = \mu_{[p,q]}(A_0) = \mu$, $\tau_{[p,q]}(A_j) \leq \tau_1 < \tau$, then for $r \to 1^-$ and the above ε, we have

$$m(r, A_j) = T(r, A_j) \leq \exp_{p-1}\big\{(\tau_1+\varepsilon)(\log_{q-1}\frac{1}{1-r})^{\mu}\big\}. \tag{3.18}$$

By the definition of lower $[p, q]$-type, for $r \to 1^-$, we have

$$T(r, A_0) > \exp_{p-1}\big\{(\tau-\varepsilon)(\log_{q-1}\frac{1}{1-r})^{\mu}\big\}. \tag{3.19}$$

By substituting (3.17)-(3.19) into (3.16), we have

$$\exp_{p-1}\big\{(\tau-2\varepsilon)(\log_{q-1}\frac{1}{1-r})^{\mu}\big\} \leq O(\log^+ T(r, f)), \quad r \notin E_1,\ r \to 1^-. \tag{3.20}$$

Then, by Lemma 3.4 and (3.20), for all $r \to 1^-$, we have

$$\exp_{p-1}\big\{(\tau-2\varepsilon)(\log_{q-1}\frac{1}{1-r})^{\mu}\big\} \leq O(\log^+ T(s(r), f)),$$

where $s(r) = 1 - d(1-r)$, $d \in (0, 1)$. Hence, we have $\mu_{[p+1,q]}(f) \geq \mu_{[p,q]}(A_0)$. □

Lemma 3.10. *Let p,q be integers such that $p \geq q \geq 2$ and $f(z)$ be an analytic function in Δ with $0 < \sigma_{[p,q]}(f) < \infty$. Then for any given $\varepsilon > 0$, there exists a set $E_5 \subset [0,1)$ with $\int_{E_5} \frac{dr}{1-r} = \infty$ such that*

$$\tau = \tau_{[p,q]}(f) = \lim_{r\to 1^-,\, r\in E_5} \frac{\log_{p-1}^+ T(r,f)}{(\log_{q-1} \frac{1}{1-r})^{\sigma_{[p,q]}(f)}}.$$

Proof. By the definition of $[p,q]$-type, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to 1^- satisfying $1-d(1-r_n) < r_{n+1} (0<d<1)$ such that

$$\tau_{[p,q]}(f) = \lim_{r_n\to 1^-} \frac{\log_{p-1}^+ T(r_n,f)}{(\log_{q-1} \frac{1}{1-r_n})^{\sigma_{[p,q]}(f)}}.$$

Then for any $r \in [r_n, 1-d(1-r_n)]$, we have

$$\frac{\log_{p-1}^+ T(r_n,f)}{(\log_{q-1} \frac{1}{1-r_n})^{\sigma_{[p,q]}(f)}} \Big(\frac{\log_{q-1}\frac{1}{1-r_n}}{\log_{q-1}\frac{1}{1-r}}\Big)^{\sigma_{[p,q]}(f)} \leq \frac{\log_{p-1}^+ T(r,f)}{(\log_{q-1} \frac{1}{1-r})^{\sigma_{[p,q]}(f)}}.$$

When $q \geq 2$, we have

$$\frac{\log_{q-1}\frac{1}{1-r_n}}{\log_{q-1}\frac{1}{1-r}} \to 1, \quad r_n \to 1^-.$$

Let $E_5 = \bigcup_{n=n_1}^{\infty} [r_n, 1-d(1-r_n)]$, where n_1 is some sufficiently large positive integer, then we have

$$\lim_{r\to 1^-,\, r\in E_5} \frac{\log_{p-1}^+ T(r,f)}{(\log_{q-1} \frac{1}{1-r})^{\sigma_{[p,q]}(f)}} = \lim_{r_n\to 1^-} \frac{\log_{p-1}^+ T(r_n,f)}{(\log_{q-1} \frac{1}{1-r_n})^{\sigma_{[p,q]}(f)}} = \tau_{[p,q]}(f),$$

and

$$\int_{E_5} \frac{dr}{1-r} = \sum_{n=n_1}^{\infty} \int_{r_n}^{1-d(1-r_n)} \frac{dt}{1-t} = \sum_{n=n_1}^{\infty} \log\frac{1}{d} = \infty.$$

□

Lemma 3.11. *Let p,q be integers such that $p > q \geq 1$. If $A_{n-1}(z), \dots, A_1(z)$, $A_0(z)$ $(\not\equiv 0)$ are analytic functions of $[p,q]$-order in Δ, then every solution $f(z) (\not\equiv 0)$ of (1.1) satisfies $\sigma_{[p+1,q]}(f) \leq \max\{\sigma_{[p,q]}(A_j) | j = 0,\dots,n-1\}$.*

Proof. Set $b = \max\{\sigma_{[p,q]}(A_j) | j = 0,\dots,n-1\} = \max\{\sigma_{M,[p,q]}(A_j) | j = 0,\dots,n-1\}$. Then we have

$$M(r,A_j) \leq \exp_{p+1}\{(b+\varepsilon)\log_q \frac{1}{1-r}\}, \tag{3.21}$$

for any given $\varepsilon > 0$ and $r \to 1^-$. By (3.11) and (3.21), for the above $\varepsilon > 0$ and $r \to 1^-$, we have

$$T(r,f) = m(r,f) \leq O(\exp_{p+1}\{(b+2\varepsilon)\log_q \frac{1}{1-r}\}). \tag{3.22}$$

Therefore, $\sigma_{[p+1,q]}(f) \leq \max\{\sigma_{[p,q]}(A_j) | j = 0,\dots,n-1\}$. □

4. Proofs of main theorems

Proof of Theorem 2.1. By Lemma 3.11, we have $\sigma_{[p+1,q]}(f) \leq \sigma_{[p,q]}(A_0)$. Set $b = \max\{\sigma_{[p,q]}(A_j) | \sigma_{[p,q]}(A_j) < \sigma_{[p,q]}(A_0)\}$. If $\sigma_{[p,q]}(A_j) < \mu_{[p,q]}(A_0) \leq \sigma_{[p,q]}(A_0)$ or $\sigma_{[p,q]}(A_j) \leq \mu_{[p,q]}(A_0) < \sigma_{[p,q]}(A_0)$, then for any given $\varepsilon (0 < 2\varepsilon < \sigma_{[p,q]}(A_0) - b)$ and $r \to 1^-$, we have

$$\begin{aligned} m(r, A_j) = T(r, A_j) &\leq \exp_p \big\{(b+\varepsilon)\log_q \frac{1}{1-r}\big\} \\ &< \exp_p \big\{(\sigma_{[p,q]}(A_0) - \varepsilon)\log_q \frac{1}{1-r}\big\}. \end{aligned} \tag{4.1}$$

Set $\tau_1 = \max\{\tau_{[p,q]}(A_j) |\ \sigma_{[p,q]}(A_j) = \mu_{[p,q]}(A_0), j \neq 0\}$. If $\sigma_{[p,q]}(A_j) = \mu_{[p,q]}(A_0) = \sigma_{[p,q]}(A_0)$, then we have $\tau_1 < \tau \leq \tau_{[p,q]}(A_0)$. Therefore,

$$m(r, A_j) = T(r, A_j) \leq \exp_{p-1} \big\{(\tau_1 + \varepsilon)(\log_{q-1} \frac{1}{1-r})^{\sigma_{[p,q]}(A_0)}\big\} \tag{4.2}$$

holds for $r \to 1^-$ and any given ε $(0 < 2\varepsilon < \tau_{[p,q]}(A_0) - \tau_1)$. By the definition of $[p,q]$-type and Lemma 3.10, for all $r \to 1^-$, $r \in E_5$, where $E_5 \subset [0,1)$ satisfies $\int_{E_5} \frac{dr}{1-r} = \infty$, we have

$$T(r, A_0) > \exp_{p-1} \big\{(\tau_{[p,q]}(A_0) - \varepsilon)(\log_{q-1} \frac{1}{1-r})^{\sigma_{[p,q]}(A_0)}\big\}. \tag{4.3}$$

Then by (3.16) and (4.1)-(4.3), for all $r \to 1^-$, $r \in E_5 \backslash E_1$ and the above ε, where $E_1 \subset [0,1)$ satisfies $\int_{E_1} \frac{dt}{1-t} < \infty$, we have

$$\exp_{p-1} \big\{(\tau_{[p,q]}(A_0) - 2\varepsilon)(\log_{q-1} \frac{1}{1-r})^{\sigma_{[p,q]}(A_0)}\big\} \leq O(\log^+ T(r,f)). \tag{4.4}$$

By (4.4), $\sigma_{[p+1,q]}(f) \geq \sigma_{[p,q]}(A_0)$. Thus, we have $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$.

By Lemmas 3.8 and 3.9, we know that every solution $f(z)$ $(\not\equiv 0)$ of (1.1) satisfies $\mu_{[p+1,q]}(f) = \mu_{[p,q]}(A_0)$.

Now, we need to prove $\underline{\overline{\lambda}}_{[p+1,q]}(f - \varphi) = \mu_{[p+1,q]}(f)$ and $\overline{\lambda}_{[p+1,q]}(f - \varphi) = \sigma_{[p+1,q]}(f)$. Setting $g = f - \varphi$, since $\sigma_{[p+1,q]}(\varphi) < \mu_{[p,q]}(A_0)$, we have $\sigma_{[p+1,q]}(g) = \sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$, $\mu_{[p+1,q]}(g) = \mu_{[p+1,q]}(f) = \mu_{[p,q]}(A_0)$, $\overline{\lambda}_{[p+1,q]}(g) = \overline{\lambda}_{[p+1,q]}(f - \varphi)$ and $\underline{\overline{\lambda}}_{[p+1,q]}(g) = \underline{\overline{\lambda}}_{[p+1,q]}(f - \varphi)$. By substituting $f = g + \varphi, f' = g' + \varphi', \dots, f^{(n)} = g^{(n)} + \varphi^{(n)}$ in (1.1), we obtain

$$g^{(n)} + A_{n-1}(z)g^{(n-1)} + \dots + A_0(z)g = -[\varphi^{(n)} + A_{n-1}(z)\varphi^{(n-1)} + \dots + A_0(z)\varphi]. \tag{4.5}$$

If $F(z) = \varphi^{(n)} + A_{n-1}(z)\varphi^{(n-1)} + \dots + A_0(z)\varphi \equiv 0$, then by Lemma 3.9, we have $\mu_{[p+1,q]}(\varphi) \geq \mu_{[p,q]}(A_0)$, which is a contradiction. Thus, $F(z) \not\equiv 0$. Since $\sigma_{[p+1,q]}(F) \leq \sigma_{[p+1,q]}(\varphi) < \mu_{[p,q]}(A_0) = \mu_{[p+1,q]}(f) = \mu_{[p+1,q]}(g) \leq \sigma_{[p+1,q]}(g) = \sigma_{[p+1,q]}(f)$, by Lemma 3.5 and (4.5), we have $\overline{\lambda}_{[p+1,q]}(g) = \lambda_{[p+1,q]}(g) = \sigma_{[p+1,q]}(g) = \sigma_{[p,q]}(A_0)$; i.e., $\overline{\lambda}_{[p+1,q]}(f - \varphi) = \lambda_{[p+1,q]}(f - \varphi) = \sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$. By Lemma 3.6 and (4.5), we have $\underline{\overline{\lambda}}_{[p+1,q]}(g) = \mu_{[p+1,q]}(g)$; i.e., $\underline{\overline{\lambda}}_{[p+1,q]}(f - \varphi) = \mu_{[p+1,q]}(f) = \mu_{[p,q]}(A_0)$. Therefore, $\underline{\overline{\lambda}}_{[p+1,q]}(f - \varphi) = \mu_{[p+1,q]}(f) = \mu_{[p,q]}(A_0) \leq \sigma_{[p,q]}(A_0) = \sigma_{[p+1,q]}(f) = \overline{\lambda}_{[p+1,q]}(f - \varphi) = \lambda_{[p+1,q]}(f - \varphi)$. The proof is complete. $\square$

Proof of Theorem 2.2. By Lemma 3.11, we obtain $\sigma_{[p+1,q]}(f) \leq \sigma_{[p,q]}(A_0)$. By

$$\limsup_{r\to 1^-} \sum_{j=1}^{n-1} m(r,A_j)/m(r,A_0) < 1, \tag{4.6}$$

for $r \to 1^-$, we have

$$\sum_{j=1}^{n-1} m(r,A_j) < \delta m(r,A_0) = \delta T(r,A_0), \tag{4.7}$$

where $\delta \in (0,1)$. By (3.16) and (4.7), for $r \to 1^-$, $r \notin E_1$, where $E_1 \subset [0,1)$ satisfies $\int_{E_1} \frac{dt}{1-t} < \infty$, we have

$$T(r,A_0) \leq O(\log^+ T(r,f) + \log \frac{1}{1-r}). \tag{4.8}$$

By Lemma 3.4 and (4.8), we have $\sigma_{[p+1,q]}(f) \geq \sigma_{[p,q]}(A_0)$. Thus, $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$.

By Lemma 3.4 and (4.8), we have $\mu_{[p+1,q]}(f) \geq \mu_{[p,q]}(A_0)$. By Lemma 3.8, we have $\mu_{[p+1,q]}(f) \leq \mu_{[p,q]}(A_0)$. Thus, $\mu_{[p+1,q]}(f) = \mu_{[p,q]}(A_0)$.

Using a proof similar to the one in Theorem 2.1, we obtain $\underline{\overline{\lambda}}_{[p+1,q]}(f-\varphi) = \mu_{[p+1,q]}(f) = \mu_{[p,q]}(A_0) \leq \sigma_{[p,q]}(A_0) = \sigma_{[p+1,q]}(f) = \overline{\lambda}_{[p+1,q]}(f-\varphi) = \lambda_{[p+1,q]}(f-\varphi)$. The proof is complete. $\square$

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