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LOWER BOUNDS FOR THE BLOWUP TIME OF SOLUTIONS TO A NONLINEAR PARABOLIC PROBLEM

HAIXIA LI, WENJIE GAO, YUZHU HAN

Abstract. In this short article, we study the blow-up properties of solutions to a parabolic problem with a gradient nonlinearity under homogeneous Dirichlet boundary conditions. By constructing an auxiliary function and by modifying the first order differential inequality technique introduced by Payne et al., we obtain a lower bound for the blow-up time of solutions in a bounded domain $\Omega \subset \mathbb{R}^n$ for any $n \geq 3$. This article generalizes a result in [16].

1. Introduction

When dealing with a parabolic problem there are several interesting features to analyze, one of which is the so called finite time blow-up. The question of blow-up of solutions to nonlinear parabolic equations and systems has received considerable attention since the elegant work of Fujita [6]. We refer to the interested readers the survey papers [2, 7, 10] and the book [17].

In practical situations, one would like to know, among other things, whether the solutions blow up, and if so, at what time T blow-up occurs. However, when the solution does blow up at some finite T, this time can seldom be determined explicitly, and much effort has been devoted to the calculation of bounds for T. Most of the methods used until recently can only yield upper bounds for T, which are of little value in particular situations when blow-up has to be avoided. By using the first-order differential inequality technique, lower bounds for the blow-up time of solutions to semilinear heat equations under different boundary conditions and suitable constraint on the data were obtained by Payne et al. [12, 13, 14, 16]. Thereafter, the differential inequality technique was successfully employed to derive lower bounds for the blow-up time of solutions to other parabolic problems, see [1, 3, 5, 11, 15].

In this article, we shall study a parabolic problem with a gradient nonlinearity of the following form

$$\begin{gathered} u_t = \Delta u + u^p - |\nabla u|^q, \quad (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \geq 0, \quad x \in \Omega, \end{gathered} \tag{1.1}$$

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where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, Δ and ∇ are the Laplace and gradient operator with respect to x, respectively, T is the possible blow-up time and $p, q > 1$ are fixed (finite) parameters. In [4, 9], conditions on p, q and $u_0(x)$ were given for which the solutions to (1.1) would blow up in finite time. In fact the restrictions on p and q were

$$1 < p < \frac{n+2}{n-2}, \quad 1 < q < \frac{2p}{p+1}, \quad \text{for } n \geq 2,$$

or

$$p \text{ is large enough and } q = \frac{2p}{p+1}, \text{ for } n = 1.$$

In a recent paper Payne et al. [16] obtained lower bounds of the blow-up time of solutions to (1.1) when $n = 3$. Naturally, we hope to obtain the lower bounds for blow-up time of solutions to (1.1) with any smooth bounds $\Omega \subset \mathbb{R}^n$ and any $n \geq 3$. That is what we will do in this article.

As indicated in [18] it is well known that if $p \leq q$ the solution will not blow up in finite time. Also it is well known that if the initial data are small enough the solution will actually decay exponentially as $t \to \infty$ (see e.g.[14, 19]). Since we are interested in a lower bound for the blow-up time T, only the case $p > q$ is considered.

2. A lower bound for the blow-up time

In this section we seek a lower bound for the blow-up time T of solutions to (1.1) in some appropriate measure. The idea of the proof of the following theorem is inspired by that in [1].

Theorem 2.1. *Let $u(x,t)$ be the nonnegative classical solution of problem* (1.1) *for $p > q > 1$ in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ with $n \geq 3$. Define*

$$\varphi(t) = \int_\Omega u^k \mathrm{d}x,$$

where k is a parameter restricted by the condition

$$k > \max\Big\{1, \frac{(7n-16)(p-1)}{2}, (q-1)(3n-8)\Big\}. \tag{2.1}$$

If $u(x,t)$ blows up in the measure φ at the finite time T, then T is bounded from below as

$$T \geq \int_{\varphi(0)}^{+\infty} \frac{1}{C_1 + C_2 \xi^{\frac{3n-6}{3n-8}}} d\xi, \tag{2.2}$$

where C_1 and C_2 are positive constants which will be determined in the proof.

Proof. Applying the divergence theorem to the first equation in (1.1), we have

$$\begin{aligned}\frac{\mathrm{d}\varphi}{\mathrm{d}t} &= k\int_\Omega u^{k-1}u_t \mathrm{d}x \\ &= k\int_\Omega u^{k-1}(\triangle u + u^p - |\nabla u|^q)\mathrm{d}x \\ &= k\int_\Omega u^{k-1}\triangle u \mathrm{d}x + k\int_\Omega u^{k+p-1}\mathrm{d}x - k\int_\Omega u^{k-1}|\nabla u|^2\mathrm{d}x \\ &= -\frac{4(k-1)}{k}\int_\Omega |\nabla u^{k/2}|^2\mathrm{d}x + k\int_\Omega u^{k+p-1}\mathrm{d}x \\ &\quad - \frac{kq^q}{(k+q-1)^q}\int_\Omega |\nabla u^{\frac{k+q-1}{q}}|^q\mathrm{d}x.\end{aligned} \tag{2.3}$$

Moreover, from [12, (2.10)] it follows that

$$\int_\Omega |\nabla u^{\frac{k+q-1}{q}}|^q\mathrm{d}x \geq (\frac{2\sqrt{\lambda}}{q})^q\int_\Omega u^{k+q-1}\mathrm{d}x, \tag{2.4}$$

where the positive constant λ is the first eigenvalue of the problem

$$\begin{gathered}\triangle w + \lambda w = 0 \quad \text{in } \Omega, \\ w = 0 \quad \text{on } \partial\Omega.\end{gathered} \tag{2.5}$$

Thus by combining (2.3) with (2.4) we obtain

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} \leq -\frac{4(k-1)}{k}\int_\Omega |\nabla u^{k/2}|^2\mathrm{d}x + k\int_\Omega u^{k+p-1}\mathrm{d}x - \frac{k(2\sqrt{\lambda})^q}{(k+q-1)^q}\int_\Omega u^{k+q-1}\mathrm{d}x. \tag{2.6}$$

Noticing (2.1), we can apply first Hölder's inequality and then Young's inequality to the second term on the right hand side of (2.3) to obtain

$$\begin{aligned}\int_\Omega u^{k+p-1}\mathrm{d}x &\leq |\Omega|^{m_1}\Big(\int_\Omega u^{\frac{k(7n-14)}{7n-16}}\mathrm{d}x\Big)^{m_2} \\ &\leq m_1|\Omega| + m_2\int_\Omega u^{\frac{k(7n-14)}{7n-16}}\mathrm{d}x,\end{aligned} \tag{2.7}$$

where

$$m_1 = 1 - \frac{(k+p-1)(7n-16)}{k(7n-14)} \in (0,1), \quad m_2 = \frac{(k+p-1)(7n-16)}{k(7n-14)} \in (0,1).$$

Combining (2.7) and (2.6) yields

$$\begin{aligned}\frac{\mathrm{d}\varphi}{\mathrm{d}t} &\leq -\frac{4(k-1)}{k}\int_\Omega |\nabla u^{k/2}|^2\mathrm{d}x + km_1|\Omega| + km_2\int_\Omega u^{\frac{k(7n-14)}{7n-16}}\mathrm{d}x \\ &\quad - \frac{k(2\sqrt{\lambda})^q}{(k+q-1)^q}\int_\Omega u^{k+p-1}\mathrm{d}x.\end{aligned} \tag{2.8}$$

We now use Hölder's inequality in the third term on the right hand side of (2.8):

$$\int_\Omega u^{\frac{k(7n-14)}{7n-16}}\mathrm{d}x \leq \Big(\int_\Omega u^k\mathrm{d}x\Big)^\alpha\Big(\int_\Omega u^{\frac{k}{2}\frac{2n}{n-2}}\mathrm{d}x\Big)^{1-\alpha}, \tag{2.9}$$

where $0 < \alpha = \frac{2(3n-7)}{7n-16} < 1$. Next, using the Sobolev inequality for $W_0^{1,2} \hookrightarrow L^{\frac{2n}{n-2}}$ ($n \geq 3$) [20]), we obtain

$$\|u^{k/2}\|_{L^{\frac{2n}{n-2}}}^{\frac{2n(1-\alpha)}{n-2}} \leq C_s^{\frac{2n(1-\alpha)}{n-2}} \|\nabla u^{k/2}\|_{L^2}^{\frac{2n(1-\alpha)}{n-2}}, \tag{2.10}$$

where $C_s = \big(\frac{1}{n(n-2)\pi}\big)^{1/2}\big(\frac{n!}{2\Gamma(\frac{n}{2}+1)}\big)^{1/n}$ is the best imbedding constant (see [8, Chap. 7]). By substituting (2.10) into (2.9), we arrive at

$$\int_\Omega u^{\frac{k(7n-14)}{7n-16}} \mathrm{d}x \leq C_s^{\frac{2n(1-\alpha)}{n-2}} \Big(\int_\Omega u^k \mathrm{d}x\Big)^\alpha \Big(\int_\Omega |\nabla u^{k/2}|^2 \mathrm{d}x\Big)^{\frac{n(1-\alpha)}{n-2}}, \tag{2.11}$$

which, with the help of Young's inequality, gives

$$\int_\Omega u^{\frac{k(7n-14)}{7n-16}} \mathrm{d}x \leq \frac{C_s^{\frac{n}{3n-8}}(6n-16)}{(7n-16)\varepsilon_1^{\frac{n}{2(3n-8)}}} \Big(\int_\Omega u^k \mathrm{d}x\Big)^{\frac{3n-7}{3n-8}} + \frac{n(1-\alpha)\varepsilon_1}{n-2} \int_\Omega |\nabla u^{k/2}|^2 \mathrm{d}x. \tag{2.12}$$

Here ε_1 is a positive constant to be determined later. By Hölder's inequality, we have

$$\int_\Omega u^{q+k-1} \mathrm{d}x \geq |\Omega|^{-\frac{q-1}{k}} \Big(\int_\Omega u^k \mathrm{d}x\Big)^{1+\frac{q-1}{k}}. \tag{2.13}$$

Combining (2.12) and (2.13) with (2.8) gives

$$\begin{aligned} \frac{\mathrm{d}\varphi}{\mathrm{d}t} &\leq km_1|\Omega| + \big[\frac{n(1-\alpha)\varepsilon_1 km_2}{n-2} - \frac{4(k-1)}{k}\big] \int_\Omega |\nabla u^{k/2}|^2 \mathrm{d}x \\ &\quad + \frac{km_2 C_s^{\frac{n}{3n-8}}(6n-16)}{(7n-16)\varepsilon_1^{\frac{n}{2(3n-8)}}} \varphi^{\frac{3n-7}{3n-8}} - \frac{k(2\sqrt{\lambda})^q}{(k+q-1)^q} |\Omega|^{-\frac{q-1}{k}} \varphi^{1+\frac{q-1}{k}}. \end{aligned} \tag{2.14}$$

Next, we apply Young's inequality to the third term on the right-hand side of (2.14) to conclude that

$$\varphi^{\frac{3n-7}{3n-8}} \leq \frac{\varepsilon_2}{m_3} \varphi^{1+\frac{q-1}{k}} + \frac{1}{m_4} \varepsilon_2^{-\frac{m_4}{m_3}} \varphi^{\frac{3n-6}{3n-8}}, \tag{2.15}$$

where

$$m_3 = \frac{2k-(q-1)(3n-8)}{k}, \quad m_4 = \frac{2k-(q-1)(3n-8)}{k-(q-1)(3n-8)},$$

and ε_2 is a positive constant to be fixed. Combining (2.15) and (2.14), we obtain

$$\begin{aligned} \frac{\mathrm{d}\varphi}{\mathrm{d}t} &\leq C_1 + \big[\frac{n(1-\alpha)\varepsilon_1 km_2}{n-2} - \frac{4(k-1)}{k}\big] \int_\Omega |\nabla u^{k/2}|^2 \mathrm{d}x + C_2 \varphi^{\frac{3n-6}{3n-8}} \\ &\quad + \big[\frac{\varepsilon_2 km_2 C_s^{\frac{n}{3n-8}}(6n-16)}{(7n-16)\varepsilon_1^{\frac{n}{2(3n-8)}} m_3} - \frac{k(2\sqrt{\lambda})^q |\Omega|^{-\frac{q-1}{k}}}{(k+q-1)^q}\big] \varphi^{1+\frac{q-1}{k}}, \end{aligned} \tag{2.16}$$

where

$$C_1 = km_1|\Omega|, \quad C_2 = \frac{km_2 C_s^{\frac{n}{3n-8}}(6n-16)\varepsilon_2^{-\frac{m_4}{m_3}}}{(7n-16)\varepsilon_1^{\frac{n}{2(3n-8)}} m_4}.$$

Therefore, by choosing

$$\varepsilon_1 = \frac{4(k-1)(n-2)}{nk^2 m_2(1-\alpha)}$$

first and

$$\varepsilon_2 = \frac{(7n-16)m_3k(2\sqrt{\lambda})^q|\Omega|^{-\frac{q-1}{k}}\varepsilon_1^{\frac{n}{2(3n-8)}}}{km_2(6n-16)C_s^{\frac{n}{3n-8}}(k+q-1)^q}$$

next, we obtain the differential inequality

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} \le C_1 + C_2\varphi^{\frac{3n-6}{3n-8}}, \tag{2.17}$$

or equivalently

$$\frac{\mathrm{d}\varphi}{C_1 + C_2\varphi^{\frac{3n-6}{3n-8}}} \le \mathrm{d}t. \tag{2.18}$$

Integrating of the differential inequality (2.18) from 0 to t leads to

$$\int_{\varphi(0)}^{\varphi(t)} \frac{1}{C_1 + C_2\xi^{\frac{3n-6}{3n-8}}} d\xi \le t. \tag{2.19}$$

Passing to the limit as $t \to T^-$, we obtain

$$\int_{\varphi(0)}^{+\infty} \frac{1}{C_1 + C_2\xi^{\frac{3n-6}{3n-8}}} d\xi \le T. \tag{2.20}$$

Thus, the proof is complete. □

Remark 2.2. It is easy to see that when $n = 3$, the lower bound for the blow-up time derived here is consistent with the one obtained by Payne et al. [16].

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Haixia Li
School of Mathematics, Jilin University, Changchun 130012, China
E-mail address: lihaixia0611@126.com

Wenjie Gao
School of Mathematics, Jilin University, Changchun 130012, China
E-mail address: wjgao@jlu.edu.cn

Yuzhu Han
School of Mathematics, Jilin University, Changchun 130012, China
E-mail address: yzhan@jlu.edu.cn