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SECOND-ORDER BOUNDARY-VALUE PROBLEMS WITH INTEGRAL BOUNDARY CONDITIONS ON THE REAL LINE

FULYA YORUK DEREN, NUKET AYKUT HAMAL

Abstract. This article shows the existence and multiplicity of nonnegative solutions for nonlinear boundary-value problems with integral boundary conditions on the whole line. The arguments are based upon the Krasnoselskii' s fixed point theorem of cone expansion-compression type. An example is given to demonstrate our results.

1. Introduction

The theory of boundary-value problems on an infinite interval for differential equations has become an important area of investigation in recent years. There are many results about the existence of positive solutions on an infinite interval for boundary value problems. We refer the reader to [1, 5, 6, 7, 9, 10, 11, 12, 15] and the references therein.

At the same time, boundary value problems with integral boundary conditions for ordinary differential equations represent a very interesting and important class of problems. They constitute two, three, multi-point and nonlocal boundary value problems as special cases. The existence results of positive solutions for such problems have received a great deal of attention. To identify a few, we refer the reader to [2, 4, 14] and the references therein.

To the author' s knowledge, there are relatively few papers available for the boundary-value problems with integral boundary conditions on the half line and the real line. (See [7, 10, 12, 13]). Yoruk and Hamal [10] considered the following boundary-value problem with integral boundary conditions on an infinite interval,

$$\frac{1}{p(t)}(p(t)x'(t))' + f(t, x(t), x'(t)) = 0, \quad t \in (0, \infty), \tag{1.1}$$

$$\begin{gathered} a_1 x(0) - b_1 \lim_{t\to 0^+} p(t)x'(t) = \int_0^\infty g_1(x(s))\psi(s)ds, \\ a_2 \lim_{t\to+\infty} x(t) + b_2 \lim_{t\to+\infty} p(t)x'(t) = \int_0^\infty g_2(x(s))\psi(s)ds. \end{gathered} \tag{1.2}$$

The authors showed the existence results of solutions by means of the Shauder fixed point theorem and the Leggett-Williams fixed point theorem.

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In this article, we are interested in the existence and multiplicity of nonnegative solutions for the following integral boundary-value problem on the whole line

$$(p(t)x'(t))' + \lambda q(t)f(t, x(t), x'(t)) = 0, \quad t \in \mathbb{R}, \tag{1.3}$$

$$\begin{aligned} a_1 \lim_{t\to-\infty} x(t) - b_1 \lim_{t\to-\infty} p(t)x'(t) &= \int_{-\infty}^{\infty} g_1(s, x(s), x'(s))\psi(s)ds, \\ a_2 \lim_{t\to+\infty} x(t) + b_2 \lim_{t\to+\infty} p(t)x'(t) &= \int_{-\infty}^{\infty} g_2(s, x(s), x'(s))\psi(s)ds, \end{aligned} \tag{1.4}$$

where $\lambda > 0$ is a parameter, $f, g_1, g_2 \in \mathcal{C}(\mathbb{R}\times[0,\infty)\times\mathbb{R}, [0,\infty))$, $q, \psi \in \mathcal{C}(\mathbb{R}, (0,\infty))$ and $p \in \mathcal{C}(\mathbb{R}, (0,\infty)) \cap \mathcal{C}^1(\mathbb{R})$. Here, the values of $\int_{-\infty}^{+\infty} g_i(s, x(s), x'(s))ds$ $(i = 1, 2)$, $\int_{-\infty}^{+\infty} \frac{ds}{p(s)}$ and $\sup_{s\in\mathbb{R}} \psi(s)$ are finite and $a_1 + a_2 > 0$, $b_i > 0$ $(i = 1, 2)$ satisfying $D = a_2b_1 + a_1b_2 + a_1a_2 \int_{-\infty}^{+\infty} \frac{ds}{p(s)} > 0$.

The main features of our paper are as follows. Firstly, compared with [10], we establish the existence results of solutions on $\mathbb{R}$ which expands the domain of definition of t from a half line to the real line. Secondly, we investigate the existence of solutions for the case $\lambda > 0$, not $\lambda = 1$ as in [10].

The rest of this article is organized as follows. In Section 2, we represent some necessary lemmas that will be used to prove our main results. In Section 3, we apply the Krasnoselskii' s fixed point theorem to obtain the existence and multiplicity of nonnegative solutions for (1.3)-(1.4). Finally, an example is given to illustrate the main results.

To the best of our knowledge, only a few papers deal with the existence results of solutions for the boundary-value problem whose nonlinear term f involves x and the first order derivative x' explicitly, especially by means of the Krasnoselskii's fixed point theorem. (See [8] and the references therein.) So the main aim of this work is to fill this gap.

2. Preliminaries

In this section, we present some preliminary results and lemmas that will be used in the proof of our main results. For convenience, we denote $\theta(t)$ and $\varphi(t)$ by

$$\theta(t) = b_1 + a_1 \int_{-\infty}^{t} \frac{d\tau}{p(\tau)}, \quad \varphi(t) = b_2 + a_2 \int_{t}^{\infty} \frac{d\tau}{p(\tau)}\,. \tag{2.1}$$

Lemma 2.1. *Under the conditions $D > 0$ and $\int_{-\infty}^{\infty} \frac{ds}{p(s)} < +\infty$, the boundary-value problem*

$$(p(t)x'(t))' + h(t) = 0, \quad t \in \mathbb{R}, \tag{2.2}$$

$$\begin{aligned} a_1 \lim_{t\to-\infty} x(t) - b_1 \lim_{t\to-\infty} p(t)x'(t) &= \int_{-\infty}^{\infty} \sigma_1(s)ds, \\ a_2 \lim_{t\to+\infty} x(t) + b_2 \lim_{t\to+\infty} p(t)x'(t) &= \int_{-\infty}^{\infty} \sigma_2(s)ds \end{aligned} \tag{2.3}$$

has a unique solution for any $h, \sigma_1, \sigma_2 \in L(\mathbb{R})$. Furthermore, this unique solution can be expressed as

$$x(t) = \int_{-\infty}^{\infty} G(t, s)h(s)ds + \frac{\varphi(t)}{D} \int_{-\infty}^{\infty} \sigma_1(s)ds + \frac{\theta(t)}{D} \int_{-\infty}^{\infty} \sigma_2(s)ds. \tag{2.4}$$

Here,

$$G(t,s)=\frac{1}{D}\begin{cases}\theta(t)\varphi(s), & -\infty<t\le s<+\infty;\\ \theta(s)\varphi(t), & -\infty<s\le t<+\infty,\end{cases} \tag{2.5}$$

where $\theta(t)$ *and* $\varphi(t)$ *are given by* (2.1).

Remark 2.2. From (2.5), we can get the following properties of $G(t,s)$:

(1) $G(t,s)$ is continuous on $\mathbb{R}\times\mathbb{R}$.
(2) For any $s\in\mathbb{R}$, $G(t,s)$ is continuous differentiable on $\mathbb{R}$, except $t=s$.
(3) $\frac{\partial G(t,s)}{\partial t}\mid_{t=s^+}-\frac{\partial G(t,s)}{\partial t}\mid_{t=s^-}=-\frac{1}{p(s)}$.
(4) For any $t,s\in\mathbb{R}$, $G(t,s)\le G(s,s)$,

$$\overline{G}(s):=\lim_{t\to+\infty}G(t,s)=\frac{b_2}{D}\theta(s)\le G(s,s)<+\infty,$$
$$\underline{G}(s):=\lim_{t\to-\infty}G(t,s)=\frac{b_1}{D}\varphi(s)\le G(s,s)<+\infty.$$

(5) For any $k>0$ real number, $t\in[-k,k]$ and $s\in\mathbb{R}$, we have

$$G(t,s)\ge wG(s,s),\quad \text{where } w=\frac{\min\{\varphi(k),\theta(-k)\}}{\max\{\varphi(-\infty),\theta(\infty)\}}. \tag{2.6}$$

It is obvious that $0<w<1$.

We define the Banach space

$$B=\{x\in\mathcal{C}'(\mathbb{R}):\lim_{t\to\mp\infty}x(t)<+\infty,\lim_{t\to\mp\infty}x'(t)<+\infty\}$$

equipped with the norm $\|x\|=\sup_{t\in\mathbb{R}}[|x(t)|+|x'(t)|]$ and the cone $P\subset B$ by

$$P=\big\{x\in B: x(t)\ge 0\ \forall t\in\mathbb{R},\ \min_{t\in[-k,k]}x(t)\ge w\sup_{t\in\mathbb{R}}|x(t)|,\ k>0,\ [-k,k]\subset\mathbb{R}\big\},$$

in which w is given by (2.6). In this article, we need the following assumptions:

(H1) $f,g_1,g_2\in\mathcal{C}(\mathbb{R}\times[0,\infty)\times\mathbb{R},[0,\infty))$ and for any $t\in\mathbb{R}$ and $i=1,2$, we have

$$u_2(t)h_3(x,y)\le f(t,x,y)\le u_1(t)h_3(x,y),\quad g_i(t,x,y)\le v_i(t)h_i(x,y),$$

where $h_i\in\mathcal{C}([0,\infty)\times\mathbb{R},[0,\infty))$ $(i=1,2,3)$, $u_i,v_i\in L(\mathbb{R},(0,\infty))$ $(i=1,2)$; also there exists $0<\gamma_0<1$ such that $u_2(t)\ge\gamma_0u_1(t)$.
(H2) $\int_{-\infty}^{\infty}G(s,s)q(s)u_i(s)ds<+\infty$, $(i=1,2)$.
(H3) $\psi:\mathbb{R}\to(0,\infty)$ is a continuous function with $\sup_{s\in\mathbb{R}}\psi(s)<+\infty$.

Using the above assumptions, we define the operator A on P by

$$\begin{aligned}Ax(t)=\lambda\int_{-\infty}^{\infty}G(t,s)q(s)f(s,x(s),x'(s))ds+\frac{\varphi(t)}{D}\int_{-\infty}^{\infty}g_1(s,x(s),x'(s))\psi(s)ds\\ +\frac{\theta(t)}{D}\int_{-\infty}^{\infty}g_2(s,x(s),x'(s))\psi(s)ds,\end{aligned}$$

where $G(t,s)$ is given by (2.5). Obviously, x is a solution of (1.3)-(1.4) if and only if x is a fixed point of the operator A.

3. Main Results

In this section, we will apply the following Krasnoselskii's fixed point theorem to establish the existence and multiplicity of nonnegative solutions for (1.3)-(1.4).

Lemma 3.1 ([3]). *Let $\mathcal{B}$ be a real Banach space and $P \subset B$ be a cone in B. Assume that Ω_1, Ω_2 are open subsets of B with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$ and let $A : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ be a completely continuous operator such that, either*

(i) $\|Ax\| \le \|x\|$ *for* $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \ge \|x\|$ *for* $x \in P \cap \partial\Omega_2$*;*
(ii) $\|Ax\| \ge \|x\|$ *for* $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \le \|x\|$ *for* $x \in P \cap \partial\Omega_2$*.*

Then A has at least one fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Lemma 3.2. *Assume that* (H1)–(H3) *are satisfied. Then the operator $A : P \to P$ is completely continuous.*

Proof. We assert that A is a completely continuous operator. To justify this, we first show that $A : P \to B$ is well defined. Let $x \in P$, then there exists $r > 0$ such that $\|x\| \le r$. From condition (H1), for any $t \in \mathbb{R}$, we have

$$
\begin{gathered}
N_r := \sup\{h_1(x, y) : |x| + |y| \le r\} < +\infty, \\
N'_r := \sup\{h_2(x, y) : |x| + |y| \le r\} < +\infty, \\
M_r := \sup\{h_3(x, y) : |x| + |y| \le r\} < +\infty.
\end{gathered}
$$

Let $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$, then it follows from (H2) and (H3) that

$$
\begin{aligned}
&\lambda \int_{-\infty}^{\infty} |G(t_1, s) - G(t_2, s)| q(s) u_1(s) h_3(x(s), x'(s)) ds \\
&\le 2\lambda M_r \int_{-\infty}^{\infty} G(s, s) q(s) u_1(s) ds < +\infty
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{-\infty}^{\infty} [v_1(s) h_1(x(s), x'(s)) + v_2(s) h_2(x(s), x'(s))] \psi(s) ds \\
&\le \int_{-\infty}^{\infty} [N_r v_1(s) + N'_r v_2(s)] \psi(s) ds < +\infty.
\end{aligned}
$$

Hence by the Lebesgue dominated convergence theorem and the fact that $G(t, s)$ is continuous on t, we have

$$
\begin{aligned}
&|(Ax)(t_1) - (Ax)(t_2)| \\
&\le \lambda \int_{-\infty}^{\infty} |G(t_2, s) - G(t_1, s)| q(s) f(s, x(s), x'(s)) ds \\
&\quad + \frac{|\varphi(t_2) - \varphi(t_1)|}{D} \int_{-\infty}^{\infty} g_1(s, x(s), x'(s)) \psi(s) ds \\
&\quad + \frac{|\theta(t_2) - \theta(t_1)|}{D} \int_{-\infty}^{\infty} g_2(s, x(s), x'(s)) \psi(s) ds \qquad (3.1) \\
&\le 2\lambda M_r \int_{-\infty}^{\infty} |G(t_2, s) - G(t_1, s)| q(s) u_1(s) ds \\
&\quad + \frac{1}{D} \int_{-\infty}^{\infty} [N_r |\varphi(t_2) - \varphi(t_1)| v_1(s) + N'_r |\theta(t_2) - \theta(t_1)| v_2(s)] \psi(s) ds \\
&\to 0, \quad \text{as } t_1 \to t_2
\end{aligned}
$$

and

$$
\begin{aligned}
&|(Ax)'(t_1)-(Ax)'(t_2)|\\
&\le \frac{\lambda a_2}{D}|\frac{1}{p(t_1)}-\frac{1}{p(t_2)}|\int_{-\infty}^{t_1}\theta(s)q(s)f(s,x(s),x'(s))ds\\
&\quad+\frac{\lambda a_1}{Dp(t_1)}\int_{t_1}^{t_2}\varphi(s)q(s)f(s,x(s),x'(s))ds\\
&\quad+\frac{\lambda a_1}{D}|\frac{1}{p(t_1)}-\frac{1}{p(t_2)}|\int_{t_2}^{\infty}\varphi(s)q(s)f(s,x(s),x'(s))ds\\
&\quad+\frac{\lambda a_2}{Dp(t_2)}\int_{t_1}^{t_2}\theta(s)q(s)f(s,x(s),x'(s))ds\\
&\quad+\frac{a_2}{D}|\frac{1}{p(t_1)}-\frac{1}{p(t_2)}|\int_{-\infty}^{\infty}g_1(s,x(s),x'(s))\psi(s)ds\\
&\quad+\frac{a_1}{D}|\frac{1}{p(t_1)}-\frac{1}{p(t_2)}|\int_{-\infty}^{\infty}g_2(s,x(s),x'(s))\psi(s)ds\\
&\le \frac{\lambda a_2 M_r}{D}|\frac{1}{p(t_1)}-\frac{1}{p(t_2)}|\int_{-\infty}^{t_1}\theta(s)q(s)u_1(s)ds\\
&\quad+\frac{\lambda a_1 M_r}{Dp(t_1)}\int_{t_1}^{t_2}\varphi(s)q(s)u_1(s)ds+\frac{\lambda a_1 M_r}{D}|\frac{1}{p(t_1)}-\frac{1}{p(t_2)}|\int_{t_2}^{\infty}\varphi(s)q(s)u_1(s)ds\\
&\quad+\frac{\lambda a_2 M_r}{Dp(t_2)}\int_{t_1}^{t_2}\theta(s)q(s)u_1(s)ds+\frac{a_2N_r}{D}|\frac{1}{p(t_1)}-\frac{1}{p(t_2)}|\int_{-\infty}^{\infty}v_1(s)\psi(s)ds\\
&\quad+\frac{a_1N_r'}{D}|\frac{1}{p(t_1)}-\frac{1}{p(t_2)}|\int_{-\infty}^{\infty}v_2(s)\psi(s)ds\to 0\quad\text{as } t_1\to t_2.
\end{aligned}
\tag{3.2}
$$

Thus, $Ax\in\mathcal{C}^1(\mathbb{R})$.

We can show that $Ax\in B$. Notice that

$$
\begin{aligned}
&\lim_{t\to+\infty}(Ax)(t)\\
&=\lambda\int_{-\infty}^{\infty}\overline{G}(s)q(s)f(s,x(s),x'(s))ds+\frac{\varphi(+\infty)}{D}\int_{-\infty}^{\infty}g_1(s,x(s),x'(s))\psi(s)ds\\
&\quad+\frac{\theta(+\infty)}{D}\int_{-\infty}^{\infty}g_2(s,x(s),x'(s))\psi(s)ds<+\infty
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{t\to-\infty}(Ax)(t)\\
&=\lambda\int_{-\infty}^{\infty}\underline{G}(s)q(s)f(s,x(s),x'(s))ds+\frac{\varphi(-\infty)}{D}\int_{-\infty}^{\infty}g_1(s,x(s),x'(s))\psi(s)ds\\
&\quad+\frac{\theta(-\infty)}{D}\int_{-\infty}^{\infty}g_2(s,x(s),x'(s))\psi(s)ds<+\infty.
\end{aligned}
$$

In addition, we have

$$
\begin{aligned}
&|(Ax)'(t)|\\
&\le\frac{1}{D}[\lambda\int_{-\infty}^{t}|\varphi'(t)|\theta(s)q(s)f(s,x(s),x'(s))ds
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda \int_t^{\infty} |\theta'(t)|\varphi(s)q(s)f(s,x(s),x'(s))ds \\
&+|\varphi'(t)| \int_{-\infty}^{\infty} g_1(s,x(s),x'(s))\psi(s)ds + \theta'(t)\int_{-\infty}^{\infty} g_2(s,x(s),x'(s))\psi(s)ds] \\
&\leq \frac{\max\{a_1,a_2\}}{\min\{b_1,b_2\}}[\frac{\lambda}{p(t)}\int_{-\infty}^{\infty} G(s,s)q(s)f(s,x(s),x'(s))ds \\
&+\frac{\max\{\theta(+\infty),\varphi(-\infty)\}}{Dp(t)}\int_{-\infty}^{\infty}[g_1(s,x(s),x'(s))+g_2(s,x(s),x'(s))]\psi(s)ds] \\
&\leq \frac{\max\{a_1,a_2\}}{\min\{b_1,b_2\}}\sup_{t\in\mathbb{R}}\frac{1}{p(t)}[\lambda M_r\int_{-\infty}^{\infty} G(s,s)q(s)u_1(s)ds \\
&+\frac{\max\{\theta(+\infty),\varphi(-\infty)\}}{D}\int_{-\infty}^{\infty}[N_r v_1(s)+N_r' v_2(s)]\psi(s)ds] < +\infty,
\end{aligned}
$$

so, we have $\lim_{t\to\mp\infty}(Ax)'(t) < +\infty$. Hence, $A: P \to B$ is well defined.

Now, we prove that $A: P\to P$. It is obvious that $Ax(t)\geq 0$ for any $t\in\mathbb{R}$. Let $x\in P$, then for all $t\in\mathbb{R}$, we have

$$
\begin{aligned}
&|Ax(t)| \\
&\leq \lambda\int_{-\infty}^{\infty} G(s,s)q(s)f(s,x(s),x'(s))ds \\
&+\frac{\max\{\theta(+\infty),\varphi(-\infty)\}}{D}\int_{-\infty}^{\infty}[g_1(s,x(s),x'(s))+g_2(s,x(s),x'(s))]\psi(s)ds.
\end{aligned}
$$

On the other hand, for any $k>0$, $t\in[-k,k]\subset\mathbb{R}$, we obtain

$$
\begin{aligned}
&|Ax(t)| \\
&\geq w\lambda\int_{-\infty}^{\infty} G(s,s)q(s)f(s,x(s),x'(s))ds \\
&+\frac{\min\{\varphi(k),\theta(-k)\}}{D}\int_{-\infty}^{\infty}[g_1(s,x(s),x'(s))+g_2(s,x(s),x'(s))]\psi(s)ds \\
&= w\Big[\lambda\int_{-\infty}^{\infty} G(s,s)q(s)f(s,x(s),x'(s))ds \\
&+\frac{\max\{\theta(+\infty),\varphi(-\infty)\}}{D}\int_{-\infty}^{\infty}[g_1(s,x(s),x'(s))+g_2(s,x(s),x'(s))]\psi(s)ds\Big] \\
&\geq w\sup_{t\in\mathbb{R}}|Ax(t)|.
\end{aligned}
$$

Therefore, $A: P\to P$ is well defined.

Next we prove that $A: P\to P$ is continuous. Let $x_n, x\in P$ with $\|x_n - x\|\to 0$ as $n\to\infty$. We will show that $\|Ax_n - Ax\|\to 0$ as $n\to\infty$ in P.

From (H1)–(H3) we obtain

$$
\begin{aligned}
&\lambda\int_{-\infty}^{\infty} G(s,s)q(s)|f(s,x_n(s),x_n'(s)) - f(s,x(s),x'(s))|ds \\
&\leq 2\lambda M_{r_0}\int_{-\infty}^{\infty} G(s,s)q(s)u_1(s)ds < +\infty, \\
&\int_{-\infty}^{\infty}|g_1(s,x_n(s),x_n'(s)) - g_1(s,x(s),x'(s))|\psi(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&\le 2N_{r_0}\int_{-\infty}^{\infty} v_1(s)\psi(s)ds < +\infty,\\
&\int_{-\infty}^{\infty} |g_2(s,x_n(s),x_n'(s)) - g_2(s,x(s),x'(s))|\psi(s)ds\\
&\le 2N'_{r_0}\int_{-\infty}^{\infty} v_2(s)\psi(s)ds < +\infty,
\end{aligned}
$$

where $r_0 > 0$ is a real number such that $r_0 \ge \max_{n\in\mathbb{N}-\{0\}}\{\|x_n\|,\|x\|\}$. Therefore,

$$
\begin{aligned}
&|(Ax_n)(t) - (Ax)(t)|\\
&\le \lambda\int_{-\infty}^{\infty} G(s,s)q(s)|f(s,x_n(s),x_n'(s)) - f(s,x(s),x'(s))|ds\\
&\quad + \frac{\varphi(-\infty)}{D}\int_{-\infty}^{\infty} |g_1(s,x_n(s),x_n'(s)) - g_1(s,x(s),x'(s))|\psi(s)ds\\
&\quad + \frac{\theta(+\infty)}{D}\int_{-\infty}^{\infty} |g_2(s,x_n(s),x_n'(s)) - g_2(s,x(s),x'(s))|\psi(s)ds \to 0 \quad \text{as } n\to\infty.
\end{aligned}
$$

Similarly, we can see that when $\|x_n - x\| \to 0$ as $n \to +\infty$,

$$
\lim_{n\to\infty}\sup_{t\in\mathbb{R}}|(Ax_n)'(t) - (Ax)'(t)| = 0.
$$

This implies that $A : P \to P$ is a continuous operator.

Now, we show that A maps bounded subsets into bounded subsets. Let $D \subset P$ be bounded and $x \in D$, then there exists $R > 0$ such that $\|x\| \le R$, for any $x \in D$. Furthermore, for $t \in \mathbb{R}$, we obtain

$$
\begin{aligned}
|(Ax)(t)| &\le \lambda\int_{-\infty}^{\infty} G(s,s)q(s)f(s,x(s),x'(s))ds\\
&\quad + \frac{\max\{\theta(+\infty),\varphi(-\infty)\}}{D}\int_{-\infty}^{\infty}\big[g_1(s,x(s),x'(s))\\
&\quad + g_2(s,x(s),x'(s))\big]\psi(s)ds \qquad (3.3)\\
&\le \lambda M_R\int_{-\infty}^{\infty} G(s,s)q(s)u_1(s)ds\\
&\quad + \frac{\max\{\theta(+\infty),\varphi(-\infty)\}}{D}\int_{-\infty}^{\infty}[N_R v_1(s) + N'_R v_2(s)]\psi(s)ds
\end{aligned}
$$

and

$$
\begin{aligned}
|(Ax)'(t)| &\le \frac{\max\{a_1,a_2\}}{\min\{b_1,b_2\}}\sup_{t\in\mathbb{R}}\frac{1}{p(t)}\Big[\lambda\int_{-\infty}^{\infty} G(s,s)q(s)f(s,x(s),x'(s))ds\\
&\quad + \frac{\max\{\theta(+\infty),\varphi(-\infty)\}}{D}\int_{-\infty}^{\infty}\big[g_1(s,x(s),x'(s)) \qquad (3.4)\\
&\quad + g_2(s,x(s),x'(s))\big]\psi(s)ds\Big].
\end{aligned}
$$

Inequalities (3.3) and (3.4) imply

$$
\begin{aligned}
&|Ax(t)| + |(Ax)'(t)|\\
&\le (1 + \frac{\max\{a_1,a_2\}}{\min\{b_1,b_2\}}\sup_{t\in\mathbb{R}}\frac{1}{p(t)})\Big[\lambda\int_{-\infty}^{\infty} G(s,s)q(s)f(s,x(s),x'(s))ds
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \frac{\max\{\theta(+\infty),\varphi(-\infty)\}}{D}\int_{-\infty}^{\infty}[g_1(s,x(s),x'(s))+g_2(s,x(s),x'(s))]\psi(s)ds\Big]\\
&\le (1+\frac{\max\{a_1,a_2\}}{\min\{b_1,b_2\}}\sup_{t\in\mathbb{R}}\frac{1}{p(t)})\Big[\lambda M_R\int_{-\infty}^{\infty}G(s,s)q(s)u_1(s)ds\\
&\quad + \frac{\max\{\theta(+\infty),\varphi(-\infty)\}}{D}\int_{-\infty}^{\infty}[N_Rv_1(s)+N_R'v_2(s)]\psi(s)ds\Big].
\end{aligned}
$$

Hence, we obtain $\sup_{t\in\mathbb{R}}[|Ax(t)|+|(Ax)'(t)|]<+\infty$; that is, A is uniformly bounded.

Using the similar proof as the one for (3.1) and (3.2), for any $N\in(0,\infty)$, $t,t_1\in[-N,N]$ and $x\in D$, we have $\|Ax(t)-Ax(t_1)\|\to 0$ as $t\to t_1$. Thus, AD is equicontinuous on any compact interval of $\mathbb{R}$.

By (H2), (H3) and the Lebesgue dominated convergence theorem, we have

$$
\begin{aligned}
&|(Ax)(t)-(Ax)(+\infty)|\\
&\le \lambda\int_{-\infty}^{\infty}|G(t,s)-\overline{G}(s)|q(s)f(s,x(s),x'(s))ds\\
&\quad + \frac{|\varphi(t)-\varphi(+\infty)|}{D}\int_{-\infty}^{\infty}g_1(s,x(s),x'(s))\psi(s)ds\\
&\quad + \frac{|\theta(t)-\theta(+\infty)|}{D}\int_{-\infty}^{\infty}g_2(s,x(s),x'(s))\psi(s)ds\\
&\le \lambda M_R\int_{-\infty}^{\infty}|G(t,s)-\overline{G}(s)|q(s)u_1(s)ds \quad + \frac{1}{D}\int_{-\infty}^{\infty}[N_R|\varphi(t)-\varphi(+\infty)|v_1(s)\\
&\quad + N_R'|\theta(t)-\theta(+\infty)|v_2(s)]\psi(s)ds \to 0 \quad \text{as } t\to\infty
\end{aligned}
$$

and

$$
\begin{aligned}
&|(Ax)'(t)-(Ax)'(\infty)|\\
&\le \frac{1}{D}[|\frac{1}{p(t)}-\frac{1}{p(\infty)}|\int_{-\infty}^{t}a_2\theta(s)q(s)f(s,x(s),x'(s))ds\\
&\quad + |\frac{1}{p(t)}-\frac{1}{p(\infty)}|\int_{t}^{\infty}a_1\varphi(s)q(s)f(s,x(s),x'(s))ds\\
&\quad + \max\{a_1,a_2\}|\frac{1}{p(t)}-\frac{1}{p(\infty)}|\int_{-\infty}^{\infty}[g_1(s,x(s),x'(s))+g_2(s,x(s),x'(s))]\psi(s)ds]\\
&\le \frac{1}{D}[|\frac{1}{p(t)}-\frac{1}{p(\infty)}|M_R\int_{-\infty}^{t}a_2\theta(s)q(s)u_1(s)ds\\
&\quad + |\frac{1}{p(t)}-\frac{1}{p(\infty)}|M_R\int_{t}^{\infty}a_1\varphi(s)q(s)u_1(s)ds\\
&\quad + \max\{a_1,a_2\}|\frac{1}{p(t)}-\frac{1}{p(\infty)}|\int_{-\infty}^{\infty}(N_Rv_1(s)+N_R'v_2(s))\psi(s)ds] \to 0 \quad \text{as } t\to\infty.
\end{aligned}
$$

Therefore, $\{Ax : x\in D\}$ and $\{(Ax)' : x\in D\}$ are equiconvergent at $+\infty$. Similarly, we can show that AD is equiconvergent at $-\infty$. Hence, we conclude that $A : P\to P$ is completely continuous. Therefore, Lemma 3.2 is proved. □

For convenience, we denote

$$
A = w^2\gamma_0\int_{-k}^{k}G(s,s)q(s)u_1(s)ds,\quad B = 2(1+\sup_{t\in\mathbb{R}}\frac{c}{p(t)})\int_{-\infty}^{\infty}G(s,s)q(s)u_1(s)ds,
$$

$$C = \frac{2(1+\sup_{t\in\mathbb{R}} \frac{c}{p(t)})\max\{\theta(+\infty),\varphi(-\infty)\}}{D}\int_{-\infty}^{\infty}(v_1(s)+v_2(s))\psi(s)ds,$$

where

$$c = \frac{\max\{a_1,a_2\}}{\min\{b_1,b_2\}}, \tag{3.5}$$

$k>0$ is a real number and w is defined by (2.6).

In the next theorem, we also assume the following conditions on $h_i(x,y)$ ($i = 1,2,3$).

(H4) There exist numbers $0<r<R<+\infty$ such that for all $t\in\mathbb{R}$,
$$h_3(x,y) \ge \frac{|x|+|y|}{\lambda A} \quad \text{for } R\le |x|+|y|<+\infty,\ 0\le |x|+|y|\le r.$$

(H5) There exist numbers $0<r<p_1<R<+\infty$ ($r<\frac{Ap_1}{B}$) such that for all $t\in\mathbb{R}$,
$$h_3(x,y)\le \frac{p_1}{\lambda B},\quad h_i(x,y)\le \frac{p_1}{C},\quad (i=1,2)\ 0\le |x|+|y|\le p_1.$$

Theorem 3.3. *Assume that* (H1)-(H5) *are satisfied. Then* (1.3)-(1.4) *has at least two nonnegative solutions x_1, x_2, $t\in\mathbb{R}$ such that*
$$0<\|x_1\|\le p_1\le \|x_2\|.$$

Proof. Let $x\in P$ with $\|x\|=r$, then by (H4), we have

$$\begin{aligned}
\|Ax\| &\ge |Ax(t)|\\
&\ge \lambda w\int_{-k}^{k} G(s,s)q(s)f(s,x(s),x'(s))ds\\
&\ge \lambda w\int_{-k}^{k} G(s,s)q(s)u_2(s)h_3(x(s),x'(s))ds\\
&\ge \frac{1}{\lambda A}\lambda w\int_{-k}^{k} G(s,s)q(s)u_2(s)[x(s)+|x'(s)|]ds\\
&\ge \frac{w^2\gamma_0}{A}\|x\|\int_{-k}^{k} G(s,s)q(s)u_1(s)ds = \|x\|.
\end{aligned}$$

If we let $\Omega_1=\{x\in B: \|x\|<r\}$, then
$$\|Ax\|\ge \|x\| \quad \text{for all } x\in P\cap\partial\Omega_1. \tag{3.6}$$

Further, let $x\in P$ with $\|x\|=p_1$. Then from (H5), we obtain

$$\begin{aligned}
\|Ax\| \le{}& (1+\sup_{t\in\mathbb{R}}\frac{c}{p(t)})\Big[\lambda\int_{-\infty}^{\infty} G(s,s)q(s)f(s,x(s),x'(s))ds\\
&+\frac{\max\{\theta(+\infty),\varphi(-\infty)\}}{D}\int_{-\infty}^{\infty}[g_1(s,x(s),x'(s))+g_2(s,x(s),x'(s))]\psi(s)ds\Big]\\
\le{}& (1+\sup_{t\in\mathbb{R}}\frac{c}{p(t)})\Big[\frac{p_1}{\lambda B}\lambda\int_{-\infty}^{\infty} G(s,s)q(s)u_1(s)ds\\
&+\frac{p_1\max\{\theta(+\infty),\varphi(-\infty)\}}{CD}\int_{-\infty}^{\infty}(v_1(s)+v_2(s))\psi(s)ds\Big]\\
={}& p_1=\|x\|.
\end{aligned}$$

Thus,

$$\|Ax\| \le \|x\| \quad \text{for all } x \in P \cap \partial\Omega_2, \tag{3.7}$$

where $\Omega_2 = \{x \in B : \|x\| < p_1\}$. Lemma 3.1, (3.6) and (3.7) imply that there exists a fixed point x_1 in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$ satisfying $r \le \|x_1\| \le p_1$.

On the other hand, let $R_1 = R/w$ and $\Omega_3 = \{x \in B : \|x\| < R_1\}$. Then $x \in P$ with $\|x\| = R_1$, $k \in (0, \infty)$, $t \in [-k, k]$ implies

$$|x(t)| + |x'(t)| \ge w\|x\| = R \quad \text{for } t \in [-k, k].$$

Therefore, from (H4) again, we have

$$\begin{aligned}
\|Ax\| &\ge \lambda w \int_{-k}^{k} G(s,s)q(s)f(s,x(s),x'(s))ds \\
&\ge \frac{1}{\lambda A}\lambda w \int_{-k}^{k} G(s,s)q(s)u_2(s)[x(s) + |x'(s)|]ds \\
&\ge \frac{w^2\gamma_0}{A}\|x\| \int_{-k}^{k} G(s,s)q(s)u_1(s)ds = \|x\|.
\end{aligned}$$

Therefore,

$$\|Ax\| \ge \|x\| \quad \text{for all } x \in P \cap \partial\Omega_3. \tag{3.8}$$

Lemma 3.1, (3.7) and (3.8) imply that there exists a fixed point x_2 in $P \cap (\overline{\Omega_3} \setminus \Omega_2)$ satisfying $p_1 \le \|x_2\| \le R_1$.

Both x_1 and x_2 are nonnegative solutions of (1.3)-(1.4) and $0 < \|x_1\| \le p_1 \le \|x_2\|$ holds. $\square$

In Theorem 3.4, we will assume the following conditions on $h_i(x, y)$ $(i = 1, 2, 3)$.

(H6) There exist numbers $0 < r < R < +\infty$ such that for all $t \in \mathbb{R}$,

$$h_3(x,y) \le \frac{|x| + |y|}{\lambda B}, \quad h_i(x,y) \le \frac{|x| + |y|}{C} \quad (i = 1, 2)$$

for $0 \le |x| + |y| \le r$ and $0 \le |x| + |y| \le R$.

(H7) There exist numbers $0 < r < p_2 < R < +\infty$ $(r < \frac{A}{B}p_2)$ such that for all $t \in \mathbb{R}$,

$$h_3(x,y) \ge \frac{|x| + |y|}{\lambda A}, \quad 0 \le |x| + |y| \le p_2.$$

Theorem 3.4. *Assume that* (H1)-(H3), (H6), (H7) *are satisfied. Then* (1.3)-(1.4) *has at least two nonnegative solutions* x_1, x_2, $t \in \mathbb{R}$ *such that* $0 < \|x_1\| \le p_2 \le \|x_2\|$.

Proof. For $x \in P$ with $\|x\| = r$, we have

$$\begin{aligned}
\|Ax\| &\le (1 + \sup_{t\in\mathbb{R}} \frac{c}{p(t)})\Big[\lambda \int_{-\infty}^{\infty} G(s,s)q(s)f(s,x(s),x'(s))ds \\
&\quad + \frac{\max\{\theta(+\infty), \varphi(-\infty)\}}{D} \int_{-\infty}^{\infty} [g_1(s,x(s),x'(s)) + g_2(s,x(s),x'(s))]\psi(s)ds\Big] \\
&\le (1 + \sup_{t\in\mathbb{R}} \frac{c}{p(t)})\Big[\frac{r}{\lambda B}\lambda \int_{-\infty}^{\infty} G(s,s)q(s)u_1(s)ds \\
&\quad + \frac{\max\{\theta(+\infty), \varphi(-\infty)\}r}{CD} \int_{-\infty}^{\infty} (v_1(s) + v_2(s))\psi(s)ds\Big] \\
&= r = \|x\|.
\end{aligned}$$

Hence,

$$\|Ax\| \le \|x\| \quad \forall x \in P \cap \partial\Omega_1, \tag{3.9}$$

where $\Omega_1 = \{x \in B : \|x\| < r\}$.

On the other hand, let $x \in P$ with $\|x\| = p_2$, then for any $t \in [-k, k]$, we have

$$\begin{aligned}
\|Ax\| &\ge \lambda w \int_{-k}^{k} G(s,s)q(s)f(s,x(s),x'(s))ds \\
&\ge \frac{1}{\lambda A}\lambda w \int_{-k}^{k} G(s,s)q(s)u_2(s)[x(s)+|x'(s)|]ds \\
&\ge \frac{w^2\gamma_0}{A}\|x\| \int_{-k}^{k} G(s,s)q(s)u_1(s)ds = \|x\|.
\end{aligned}$$

Therefore, if we choose $\Omega_2 = \{x \in B : \|x\| < p_2\}$, then

$$\|Ax\| \ge \|x\| \quad \forall x \in P \cap \partial\Omega_2. \tag{3.10}$$

Lemma 3.1, (3.9) and (3.10) imply that there exists a fixed point x_1 in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$ satisfying $r \le \|x_1\| \le p_2$.

Next set $\Omega_3 = \{x \in B : \|x\| < R\}$. Then $x \in P$ with $\|x\| = R$, so by (H6), we have

$$\begin{aligned}
\|Ax\| \le{} & (1 + \sup_{t\in\mathbb{R}} \frac{c}{p(t)})\Big[\lambda \int_{-\infty}^{\infty} G(s,s)q(s)f(s,x(s),x'(s))ds \\
&+ \frac{\max\{\theta(+\infty),\varphi(-\infty)\}}{D} \int_{-\infty}^{\infty} [g_1(s,x(s),x'(s)) + g_2(s,x(s),x'(s))]\psi(s)ds\Big] \\
\le{} & (1 + \sup_{t\in\mathbb{R}} \frac{c}{p(t)})\Big[\frac{R}{\lambda B}\lambda \int_{-\infty}^{\infty} G(s,s)q(s)u_1(s)ds \\
&+ \frac{R}{C}\frac{\max\{\theta(+\infty),\varphi(-\infty)\}}{D} \int_{-\infty}^{\infty} (v_1(s)+v_2(s))\psi(s)ds\Big] \\
={} & R = \|x\|.
\end{aligned}$$

Therefore,

$$\|Ax\| \le \|x\| \quad \text{for all } x \in P \cap \partial\Omega_3. \tag{3.11}$$

Lemma 3.1, (3.10) and (3.11) imply that there exists a fixed point x_2 in $P \cap (\overline{\Omega_3} \setminus \Omega_2)$ satisfying $p_2 \le \|x_2\| \le R$. Both x_1 and x_2 are nonnegative solutions of (1.3)-(1.4) and $0 < \|x_1\| \le p_2 \le \|x_2\|$ holds. $\square$

In the next theorem, we assume the following condition on $h_i(x,y)$ $(i = 1,2,3)$.

(H8) There exist numbers $0 < r < R < +\infty$ $(r < AR/B)$ such that for all $t \in \mathbb{R}$,

$$\begin{gathered}
h_3(x,y) \ge \frac{1}{\lambda A}[|x| + |y|], \quad 0 \le |x| + |y| \le r, \\
h_3(x,y) \le \frac{R}{\lambda B}, \quad h_i(x,y) \le \frac{R}{C}, \quad (i = 1,2)\ 0 \le |x| + |y| \le R.
\end{gathered}$$

Theorem 3.5. *Assume that* (H1)-(H3), (H8) *are satisfied. Then* (1.3)-(1.4) *has at least one nonnegative solution $x(t)$, $t \in \mathbb{R}$ such that*

$$\begin{gathered}
wr \le x(t) \le R, \quad t \in [-k,k]; \\
0 \le x(t) \le R, \quad t \in (-\infty,-k) \cup (k,\infty), \\
-R \le x'(t) \le R, \quad t \in \mathbb{R}.
\end{gathered}$$

In the next theorem, we assume the following condition on $h_i(x,y)$ $(i=1,2,3)$.

(H9) There exist numbers $0<r<R<+\infty$ such that for all $t\in\mathbb{R}$,
$$h_3(x,y)\le \frac{r}{\lambda B},\quad h_i(x,y)\le \frac{r}{C},\quad (i=1,2)\quad 0\le |x|+|y|\le r,$$
$$h_3(x,y)\ge \frac{R}{\lambda A},\quad R\le |x|+|y|<\infty.$$

Theorem 3.6. *Assume that* (H1)-(H3), (H9) *are satisfied. Then* (1.3)-(1.4) *has at least one nonnegative solution $x(t)$, $t\in\mathbb{R}$ such that*
$$wr\le x(t)\le \frac{R}{w},\quad t\in[-k,k];$$
$$0\le x(t)\le \frac{R}{w},\quad t\in(-\infty,-k)\cup(k,\infty),\ -\frac{R}{w}\le x'(t)\le \frac{R}{w},\quad t\in\mathbb{R}.$$

The proofs of Theorem 3.5 and 3.6 are similar to those of Theorem 3.3 and 3.4. So, they are omitted.

Remark 3.7. If
$$\lim_{|x|+|y|\to 0^+}\frac{h_i(x,y)}{|x|+|y|}=0\quad (i=1,2,3)$$
and
$$\lim_{|x|+|y|\to \infty}\frac{h_3(x,y)}{|x|+|y|}=\infty$$
for all $t\in\mathbb{R}$, then (H9) will be satisfied for $r>0$ sufficiently small and $R>0$ sufficiently large.

Example. Consider the second-order integral boundary-value problem
$$((1+t^2)x'(t))'+\lambda\frac{[x(t)+|x'(t)|]^2}{t^2+1}=0,\quad t\in\mathbb{R}, \tag{3.12}$$
$$\begin{gathered}\lim_{t\to-\infty}x(t)-\lim_{t\to-\infty}(1+t^2)x'(t)=\int_{-\infty}^{\infty}\frac{[x(t)+|x'(t)|]^2}{(t^2+1)^2}dt,\\ \lim_{t\to+\infty}(1+t^2)x'(t)=0.\end{gathered} \tag{3.13}$$

Here, $p(t)=1+t^2$, $q(t)=1$, $a_1=1$, $a_2=0$, $b_1=b_2=1$, $\psi(t)=\frac{1}{1+t^2}$,
$$f(t,x(t),y(t))=g_1(t,x(t),y(t))=\frac{[x(t)+|y(t)|]^2}{t^2+1},\quad g_2(t,x,y)=0.$$
It is obvious that $f,g_1,g_2\in\mathcal{C}(\mathbb{R}\times[0,\infty)\times\mathbb{R},[0,\infty))$. Set $h_1(x,y)=h_3(x,y)=(x+|y|)^2$, $h_2(x,y)=0$, $u_1(t)=v_1(t)=\frac{1}{t^2+1}$, and $v_2(t)=0$. It is clear that $h_i\in\mathcal{C}([0,\infty)\times\mathbb{R},[0,\infty))$ $(i=1,2,3)$, $u_1,v_1\in L(\mathbb{R},(0,\infty))$ and
$$\int_{-\infty}^{\infty}G(s,s)q(s)u_1(s)ds=\frac{\pi(\pi+2)}{2}<+\infty\quad (i=1,2).$$
In addition to this, it is easy to see that
$$\lim_{|x|+|y|\to 0^+}\frac{h_i(x,y)}{|x|+|y|}=0,\quad (i=1,2,3),$$
$$\lim_{|x|+|y|\to \infty}\frac{h_3(x,y)}{|x|+|y|}=\infty$$

for all $t \in \mathbb{R}$; that is, condition (H9) is satisfied for $r > 0$ sufficiently small and $R > 0$ sufficiently large. Hence, by Remark 3.7 and Theorem 3.6, the boundary-value problem (3.12)-(3.13) has at least one nonnegative solution.

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Fulya Yoruk Deren

Department of Mathematics, Ege University, 35100 Bornova, Izmir, Turkey

E-mail address: `fulya.yoruk@ege.edu.tr`

Nuket Aykut Hamal

Department of Mathematics, Ege University, 35100 Bornova, Izmir, Turkey

E-mail address: `nuket.aykut@ege.edu.tr`