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EXISTENCE AND UNIQUENESS OF STRONG SOLUTIONS FOR NONLOCAL EVOLUTION EQUATIONS

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Abstract. The aim of this article is to study the existence and uniqueness of strong solutions for a class of semilinear evolution equations with nonlocal initial conditions. The discussions are based on analytic semigroup theory and fixed point theorems. An example illustrates the main results.

1. Introduction

The nonlocal Cauchy problem for abstract evolution equation was first investigated by Byszewski and Lakshmikantham [5], where, by using the Banach fixed point theorem, the authors obtained the existence and uniqueness of mild solutions for nonlocal differential equations. The nonlocal problem was motivated by physical problems. Indeed, it is demonstrated that the nonlocal problems have better effects in applications than the classical Cauchy problems. For example, it is used to represent mathematical models for evolution of various phenomena, such as nonlocal neural networks, nonlocal pharmacokinetics, nonlocal pollution and nonlocal combustion (see McKibben [18]). For this reason, differential or integro-differential equations with nonlocal initial conditions were studied by many authors and some basic results on nonlocal problems have been obtained, see the references in this article and their references. Particularly, in 1999, Byszewski [8] obtained the existence and uniqueness of classical solution to a class of abstract functional differential equations with nonlocal conditions of the form

$$u'(t) = f(t, u(t), u(a(t))), \quad t \in I, \tag{1.1}$$

$$u(t_0) + \sum_{k=1}^{p} c_k u(t_k) = x_0, \tag{1.2}$$

where $I := [t_0, t_0 + T]$, $t_0 < t_1 < \cdots < t_p \leq t_0 + T$, $T > 0$; $f : I \times E^2 \to E$ and $a : I \to I$ are given functions satisfying some assumptions; E is a Banach space, $x_0 \in E$, $c_k \neq 0$ ($k = 1, 2, \ldots, p$) and $p \in \mathbb{N}$. The author pointed out that if $c_k \neq 0$, $k = 1, 2, \ldots, p$, then the results of the paper can be applied to kinematics to determine the location evolution $t \to u(t)$ of a physical object for which we do not know the positions $u(0), u(t_1), \ldots, u(t_p)$, but we know that the nonlocal

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condition (1.2) holds. The nonlocal condition of type (1.2) has also been used by Deng [10] to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In this case, condition (1.2) allows the additional measurements at t_k, $k = 1, 2, \dots, p$, which is more precise than the measurement just at $t = t_0$. Consequently, to describe some physical phenomena, the nonlocal condition can be more useful than the standard initial condition.

Recently, Vrabie [21] studied the existence of global C^0-solutions for a class of nonlinear functional differential evolution inclusions of the form

$$\begin{gathered} u'(t) \in Au(t) + f(t), \quad t \geq 0, \\ f(t) \in F(t, u(t), u_t), \quad t \geq 0, \\ u(t) = g(u)(t), \quad t \in [-\tau, 0], \end{gathered} \tag{1.3}$$

where X is a real Banach space, A is the infinitesimal generator of a nonlinear compact semigroup, $\tau \geq 0$, $F : [0, +\infty) \times X \times C([-\tau, +\infty); \overline{D(A)}) \to X$ is a nonempty convex and weakly compact value multi-function and $g : C_b([-\tau, +\infty); \overline{D(A)}) \to C([-\tau, 0); \overline{D(A)})$.

In [26], by using the approach of geometry of Banach space, Hausdroff metric, the measure of noncompactness and fixed point theorem, Zhu, Huang and Li studied the existence of integral solutions for the following nonlinear set-valued differential inclusion with nonlocal initial conditions

$$\begin{gathered} u'(t) \in Au(t) + F(t, u(t)), \quad 0 < t \leq T, \\ u(0) = g(u), \end{gathered} \tag{1.4}$$

where $A : D(A) \subseteq X \to X$ is a nonlinear m-dissipative operator which generates a contraction semigroup $T(t)$ and F is weakly upper semi-continuous multifunction with respect to its second variable in a real Banach space X.

In most of the existing articles, such as [6, 2, 3, 4, 7, 11, 12, 13, 15, 16, 23, 24, 25], the existence of mild solutions for nonlocal evolution equations have been studied extensively, but there are very few paper studied the regularity of nonlocal evolution equations. Motivated by the above-mentioned aspects, in this work we discuss the existence and uniqueness of strong solutions for a class of semilinear evolution equations with nonlocal initial conditions

$$u'(t) + Au(t) = f(t, u(t)), \quad t \geq 0, \tag{1.5}$$

$$u(0) = \sum_{k=1}^{p} c_k u(t_k), \tag{1.6}$$

where H is a Hilbert space, $A : D(A) \subset H \to H$ is a positive definite self-adjoint operator, $J = [0, K]$, $K > 0$ is a constant, $f : J \times H \to H$ is a given function satisfying some assumptions, $0 < t_1 < t_2 < \dots < t_p \leq K$, $p \in \mathbb{N}$, c_k are real numbers, $c_k \neq 0$, $k = 1, 2, \dots, p$.

In the following section we first introduce some notation and preliminaries which are used throughout this paper, at the same time the existence of strong solution for linear evolution equation nonlocal problem has been obtained. In section 3 we state and prove the existence and uniqueness of strong solutions for nonlinear evolution equation nonlocal problem. In the last paragraph we give an example to illustrate our main results.

2. Preliminaries

Let H be a Hilbert space with inner product $(\cdot,\cdot)$, then $\|\cdot\| = \sqrt{(\cdot,\cdot)}$ is the norm on H induced by inner product. We denote by $C(J,H)$ the Banach space of all continuous functions from J to H endowed with the maximum norm $\|u\|_C = \max_{t\in J}\|u(t)\|$ and by $\mathcal{L}(H)$ the Banach space of all linear and bounded operators on H.

Let $A : D(A) \subset H \to H$ be a positive definite self-adjoint operator in Hilbert space H and it have compact resolvent. By the spectral resolution theorem of self-adjoint operator, the spectrum $\sigma(A)$ only consists of real eigenvalues and it can be arrayed in sequences as

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \dots, \quad \lambda_n \to \infty \text{ as } n \to \infty. \tag{2.1}$$

By the positive definite property of A, the first eigenvalue $\lambda_1 > 0$. From [9, 14, 19], we know that $-A$ generates an analytic operator semigroup $T(t)(t \geq 0)$ on H, which is exponentially stable and satisfies

$$\|T(t)\| \leq e^{-\lambda_1 t}, \quad \forall t \geq 0. \tag{2.2}$$

Since the positive definite self-adjoint operator A has compact resolvent, the embedding $D(A) \hookrightarrow H$ is compact, and therefore $T(t)(t \geq 0)$ is also a compact semigroup.

We recall some concepts and conclusions on the fractional powers of A. For $\alpha > 0$, $A^{-\alpha}$ is defined by

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty s^{\alpha-1}T(s)ds, \tag{2.3}$$

where $\Gamma(\cdot)$ is the Euler gamma function. $A^{-\alpha} \in \mathcal{L}(H)$ is injective, and A^α can be defined by $A^\alpha = (A^{-\alpha})^{-1}$ with the domain $D(A^\alpha) = A^{-\alpha}(H)$. For $\alpha = 0$, let $A^\alpha = I$. We endow an inner product $(\cdot,\cdot)_\alpha = (A^\alpha\cdot, A^\alpha\cdot)$ to $D(A^\alpha)$. Since A^α is a closed linear operator, it follows that $(D(A^\alpha),(\cdot,\cdot)_\alpha)$ is a Hilbert space. We denote by H_α the Hilbert space $(D(A^\alpha),(\cdot,\cdot)_\alpha)$. Especially, $H_0 = H$ and $H_1 = D(A)$. For $0 \leq \alpha < \beta$, H_β is densely embedded into H_α and the embedding $H_\beta \hookrightarrow H_\alpha$ is compact. For the details of the properties of the fractional powers, we refer to [14, 22].

It is well known [19, Chapter 4, Theorem 2.9] that for any $u_0 \in D(A)$ and $h \in C^1(J,H)$, the initial value problem of linear evolution equation (LIVP)

$$\begin{gathered} u'(t) + Au(t) = h(t), \quad t \in J, \\ u(0) = u_0, \end{gathered} \tag{2.4}$$

has a unique classical solution $u \in C^1(J,H) \cap C(J,D(A))$ expressed by

$$u(t) = T(t)u_0 + \int_0^t T(t-s)h(s)ds. \tag{2.5}$$

If $u_0 \in H$ and $h \in L^1(J,H)$, the function u given by (2.5) belongs to $C(J,H)$, which is known as a mild solution of (2.4). If a mild solution u of (2.4) belongs to $W^{1,1}(J,H) \cap L^1(J,D(A))$ and satisfies the equation for a.e. $t \in J$, we call it a strong solution.

Throughout this paper, we assume that

(P0) $\sum_{k=1}^p |c_k| < e^{\lambda_1 t_1}$.

From this assumption, $\|\sum_{k=1}^p c_k T(t_k)\| \le \sum_{k=1}^p |c_k| e^{-\lambda_1 t_1} < 1$. By operator spectrum theorem, we know that the operator

$$B := \Big(I - \sum_{k=1}^p c_k T(t_k)\Big)^{-1} \tag{2.6}$$

exists and it is bounded. Furthermore, by Neumann expression, B can be written as

$$B = \sum_{n=0}^{\infty} \Big(\sum_{k=1}^p c_k T(t_k)\Big)^n. \tag{2.7}$$

Therefore,

$$\|B\| \le \sum_{n=0}^{\infty} \|\sum_{k=1}^p c_k T(t_k)\|^n = \frac{1}{1 - \|\sum_{k=1}^p c_k T(t_k)\|} \le \frac{1}{1 - e^{-\lambda_1 t_1} \sum_{k=1}^p |c_k|}. \tag{2.8}$$

To prove our main results, for any $h \in C(J,H)$, we consider the linear evolution equation nonlocal problem (LNP)

$$u'(t) + Au(t) = h(t), \quad t \in J, \tag{2.9}$$

$$u(0) = \sum_{k=1}^p c_k u(t_k). \tag{2.10}$$

Lemma 2.1. *If condition* (P0) *holds, then* (2.9)–(2.10) *has a unique mild solution* $u \in C(J,H)$ *given by*

$$u(t) = \sum_{k=1}^p c_k T(t) B \int_0^{t_k} T(t_k - s) h(s) ds + \int_0^t T(t-s) h(s) ds, \quad t \in J. \tag{2.11}$$

Moreover, $u \in W^{1,2}(J,H) \cap L^2(J, D(A))$ *is a strong solution of* (2.9)–(2.10).

Proof. By (2.4) and (2.5), we know that (2.9) has a unique mild solution $u \in C(J,H)$ which can be expressed as

$$u(t) = T(t)u(0) + \int_0^t T(t-s)h(s)ds. \tag{2.12}$$

From (2.12),

$$u(t_k) = T(t_k)u(0) + \int_0^{t_k} T(t_k - s)h(s)ds, \quad k = 1, 2, \dots, p. \tag{2.13}$$

By (2.10) and (2.13),

$$u(0) = \sum_{k=1}^p c_k T(t_k) u(0) + \sum_{k=1}^p c_k \int_0^{t_k} T(t_k - s)h(s)ds. \tag{2.14}$$

Since $I - \sum_{k=1}^p c_k T(t_k)$ has a bounded inverse operator B,

$$u(0) = \sum_{k=1}^p c_k B \int_0^{t_k} T(t_k - s)h(s)ds. \tag{2.15}$$

From (2.12) and (2.15), we know that u satisfies (2.11).

Inversely, we can verify directly that the function $u \in C(J,H)$ given by (2.11) is a mild solution of (2.9)–(2.10).

By the maximal regularity of linear evolution equations with positive definite operator in Hilbert spaces (see [20, Chapter II, Theorem 3.3]), when $u(0) = u_0 \in H_{1/2}$, the mild solution of the (2.4) has the regularity

$$u \in W^{1,2}(J, H) \cap L^2(J, D(A)) \cap C(J, H_{1/2}) \tag{2.16}$$

and it is a strong solution.

We note that $u(t)$ defined by (2.11) is the mild solution of (2.4) for $u(0) = \sum_{k=1}^{p} c_k B \int_0^{t_k} T(t_k - s)h(s)ds$. By the representation (2.5) of mild solution, $u(t) = T(t)u(0) + v(t)$, where $v(t) = \int_0^t T(t-s)h(s)ds$. Since the function $v(t)$ is a mild solution of (2.4) with the null initial value $u(0) = \theta$, v has the regularity (2.16). By the analytic property of the semigroup $T(t)$, $T(t_k)u(0) \in D(A) \subset H_{1/2}$. Hence, $u(0) = \sum_{k=1}^{p} c_k T(t_k)u(0) + \sum_{k=1}^{p} c_k v(t_k) \in H_{1/2}$. Using the regularity (2.16) again, we obtain that $u \in W^{1,2}(J, H) \cap L^2(J, D(A))$ and it is a strong solution of (2.9)–(2.10). This completes the proof. □

For any $r > 0$, let

$$\Omega_r = \{u \in C(J, H) : \|u\|_C \le r\},$$

then Ω_r is a closed ball in $C(J, H)$ with center θ and radius r.

3. Main results

Theorem 3.1. *Let A be a positive definite self-adjoint operator in Hilbert space H, and having compact resolvent. Let $f : J \times H \to H$ be continuous. If conditions* (P0) *and*

(P1) *There exist positive constants η and M with*

$$\eta < \frac{\lambda_1(1 - e^{-\lambda_1 t_1} \sum_{k=1}^{p} |c_k|)}{\sum_{k=1}^{p} |c_k| + 1}$$

such that

$$\|f(t, u)\| \le \eta \|u\| + M, \quad t \in J,\ u \in H\,,$$

are satisfied then (1.5)–(1.6) *has at least one strong solution $u \in W^{1,2}(J, H) \cap L^2(J, D(A))$.*

Proof. We consider the operator $\mathcal{F}$ on $C(J, H)$ defined by

$$\mathcal{F}u(t) = \sum_{k=1}^{p} c_k T(t) B \int_0^{t_k} T(t_k - s) f(s, u(s))ds + \int_0^t T(t-s) f(s, u(s))ds, \tag{3.1}$$

$t \in J$. By condition (P0) and Lemma 2.1, it is easy to see that the mild solution of problem (1.5)-(1.6) is equivalent to the fixed point of the operator $\mathcal{F}$. In the following, we will prove that $\mathcal{F}$ has a fixed point by using the Schauder fixed point theorem. At first, we can prove that $\mathcal{F} : C(J, H) \to C(J, H)$ is continuous by condition (P1) and the usual techniques (see, e.g. [12, 24]).

Subsequently, we prove that $\mathcal{F} : C(J, H) \to C(J, H)$ is a compact operator. Let $0 \le \alpha < \frac{1}{2}$, $0 < \nu < \frac{1}{2} - \alpha$. By [1], we can prove that the operator $\mathcal{F}$ defined by (3.1) maps $C(J, H)$ into $C^\nu(J, H_\alpha)$. By Arzela-Ascoli's theorem, the embedding $C^\nu(J, H_\alpha) \hookrightarrow C(J, H)$ is compact. This implies that $\mathcal{F} : C(J, H) \to C(J, H)$ is a compact operator. Combining this with the continuity of $\mathcal{F}$ on $C(J, H)$, we know that $\mathcal{F} : C(J, H) \to C(J, H)$ is a completely continuous operator.

Next, we prove that there exists a positive constant R big enough, such that $Q(\Omega_R) \subset \Omega_R$. For any $u \in C(J,H)$, by the condition (P1), we have

$$\|f(t,u(t))\| \le \eta\|u(t)\| + M \le \eta\|u\|_C + M, \quad t \in J. \tag{3.2}$$

Choose

$$R \ge \frac{M(1+\sum_{k=1}^{p}|c_k|)}{\lambda_1(1-e^{-\lambda_1 t_1}\sum_{k=1}^{p}|c_k|) - \eta(1+\sum_{k=1}^{p}|c_k|)}. \tag{3.3}$$

For any $u \in \Omega_R$ and $t \in J$, we have

$$\begin{aligned}
\|\mathcal{F}u(t)\| &\le \sum_{k=1}^{p}|c_k|e^{-\lambda_1 t}\|B\|\int_0^{t_k} e^{-\lambda_1(t_k-s)}\|f(s,u(s))\|ds \\
&\quad + \int_0^t e^{-\lambda_1(t-s)}\|f(s,u(s))\|ds \\
&\le \frac{\sum_{k=1}^{p}|c_k|e^{-\lambda_1 t}}{1-e^{-\lambda_1 t_1}\sum_{k=1}^{p}|c_k|}\int_0^{t_k} e^{-\lambda_1(t_k-s)}\big(\eta\|u\|_C + M\big)ds \\
&\quad + \int_0^t e^{-\lambda_1(t-s)}\big(\eta\|u\|_C + M\big)ds \\
&\le \frac{\sum_{k=1}^{p}|c_k|+1}{\lambda_1(1-e^{-\lambda_1 t_1}\sum_{k=1}^{p}|c_k|)}\big(\eta R + M\big) \le R.
\end{aligned}$$

Thus, $\|\mathcal{F}u\|_C \le R$. Therefore, $\mathcal{F}(\Omega_R) \subset \Omega_R$. By Schauder fixed point theorem, we know that $\mathcal{F}$ has at least one fixed point $u \in \Omega_R$. Since u is mild solution of (2.9)–(2.10) for $h(\cdot) = f(\cdot, u(\cdot))$, by Lemma 2.1, $u \in W^{1,2}(J,H) \cap L^2(J,D(A))$ is a strong solution of the problem (1.5)–(1.6). This completes the proof. □

Theorem 3.2. *Let A be a positive definite self-adjoint operator in Hilbert space H and it have compact resolvent, $f\colon J \times H \to H$ be continuous. If the condition (P0) and the condition*

(P2) *There exists a positive constant*

$$\eta < \frac{\lambda_1(1-e^{-\lambda_1 t_1}\sum_{k=1}^{p}|c_k|)}{\sum_{k=1}^{p}|c_k|+1}$$

such that

$$\|f(t,u) - f(t,v)\| \le \eta\|u - v\|, \quad \forall u, v \in H,$$

holds then (1.5)–(1.6) *has a unique strong solution* $\widehat{u} \in W^{1,2}(J,H) \cap L^2(J,D(A))$.

Proof. By the proof of Theorem 3.1, we know that the operator $\mathcal{F} : C(J,H) \to C(J,H)$ is completely continuous and the mild solution of problem (1.5)–(1.6) is equivalent to the fixed point of $\mathcal{F}$. For any $u, v \in C(J,H)$, from the assumption

(P2) and (3.1), we have

$$\begin{aligned}
\|\mathcal{F}u(t)-\mathcal{F}v(t)\| &\le \sum_{k=1}^{p}|c_k|e^{-\lambda_1 t}\|B\|\int_0^{t_k} e^{-\lambda_1(t_k-s)}\|f(s,u(s))-f(s,v(s))\|ds\\
&\quad+\int_0^t e^{-\lambda_1(t-s)}\|f(s,u(s))-f(s,v(s))\|ds\\
&\le \frac{\sum_{k=1}^{p}|c_k|e^{-\lambda_1 t}}{1-e^{-\lambda_1 t_1}\sum_{k=1}^{p}|c_k|}\int_0^{t_k} e^{-\lambda_1(t_k-s)}\eta\|u-v\|_C ds\\
&\quad+\int_0^t e^{-\lambda_1(t-s)}\eta\|u-v\|_C ds\\
&\le \frac{\eta(\sum_{k=1}^{p}|c_k|+1)}{\lambda_1(1-e^{-\lambda_1 t_1}\sum_{k=1}^{p}|c_k|)}\|u-v\|_C.
\end{aligned} \tag{3.4}$$

Therefore, we have

$$\|\mathcal{F}u-\mathcal{F}v\|_C \le \frac{\eta(\sum_{k=1}^{p}|c_k|+1)}{\lambda_1(1-e^{-\lambda_1 t_1}\sum_{k=1}^{p}|c_k|)}\|u-v\|_C. \tag{3.5}$$

Thus, by the assumption (P2) and (3.5), we know that $\mathcal{F}$ is a contraction operator on $C(J,H)$, and therefore $\mathcal{F}$ has a unique fixed point $\widehat{u}$ on $C(J,H)$. Since $\widehat{u}$ is mild solution of (2.9)–(2.10) for $h(\cdot)=f(\cdot,\widehat{u}(\cdot))$, by Lemma 2.1, $\widehat{u}\in W^{1,2}(J,H)\cap L^2(J,D(A))$ is a unique strong solution of (1.5)–(1.6). This completes the proof of Theorem 3.2. □

4. Application

To illustrate our results, we consider the following semilinear heat equation with nonlocal condition

$$\begin{gathered}
\frac{\partial}{\partial t}w(x,t)-\kappa\frac{\partial^2}{\partial x^2}w(x,t)=g(x,t,w(x,t)),\quad (x,t)\in[a,b]\times J,\\
w(a,t)=w(b,t)=0,\quad t\in J,\\
w(x,0)=\sum_{k=1}^{p}\arctan\frac{1}{2k^2}w(x,k),\quad x\in[a,b],
\end{gathered} \tag{4.1}$$

where $\kappa>0$ is the coefficient of heat conductivity, $J=[0,K]$, $g:[a,b]\times J\times\mathbb{R}\to\mathbb{R}$ is continuous.

Let $H=L^2(a,b)$ with the norm $\|\cdot\|_2$. Define an operator A in Hilbert space H by

$$D(A)=H^2(a,b)\cap H_0^1(a,b),\quad Au=-\kappa\frac{\partial^2}{\partial x^2}u, \tag{4.2}$$

where $H^2(a,b)=W^{2,2}(a,b)$, $H_0^1(a,b)=W_0^{1,2}(a,b)$. From [14, 19], we know that A is a positive definite self-adjoint operator on H and $-A$ is the infinitesimal generator of an analytic, compact semigroup $T(t)(t\ge 0)$. Moreover, A has discrete spectrum with eigenvalues $\lambda_n=\kappa n^2\pi^2/(b-a)^2$, $n\in\mathbb{N}$, associated normalized eigenvectors $v_n(x)=\sqrt{2/z}\sin n\pi x/(b-a)$, $z=\sqrt{b-a+(\sin 2n\pi a-\sin 2n\pi b)/(2n\pi)}$, the set $\{v_n: n\in\mathbb{N}\}$ is an orthonormal basis of H and

$$T(t)u=\sum_{n=1}^{\infty}e^{-\frac{\kappa n^2\pi^2 t}{(b-a)^2}}(u,v_n)v_n,\quad \|T(t)\|\le e^{-\frac{\kappa\pi^2 t}{(b-a)^2}},\quad \forall t\ge 0. \tag{4.3}$$

Let $u(t)=w(\cdot,t)$, $f(t,u(t))=g(\cdot,t,w(\cdot,t))$, $c_k=\arctan\frac{1}{2k^2}$, $t_k=k$, $k=1,2,\dots,p$, then (4.1) can be rewritten into the abstract form of problem (1.5)–(1.6).

Theorem 4.1. *Assume that the nonlinear term g satisfies the following conditions:*

(G1) *there exist positive constants η and M with $\eta<\frac{\kappa\pi^2}{(b-a)^2(\pi+4)}\big(4-\pi e^{-\frac{\kappa\pi^2}{(b-a)^2}}\big)$ such that*
$$|g(x,t,w)|\le \eta|w|+M,\quad x\in[a,b],\ t\in J,\ w\in\mathbb{R};$$

(G2) *there exists a function $c:\mathbb{R}^+\to\mathbb{R}^+$ such that*
$$|g(x,t,\xi)-g(y,s,\eta)|\le c(\rho)\big(|x-y|^{\mu}+|t-s|^{\mu/2}+|\xi-\eta|\big),$$
for any $\rho>0$, $\mu\in(0,1)$ and (x,t,ξ), $(y,s,\eta)\in[a,b]\times J\times[-\rho,\rho]$.

Then (4.1) *has at least one classical solution $u\in C^{2+\mu,1+\mu/2}([a,b]\times J)$.*

Proof. Since
$$\sum_{k=1}^{p}|c_k|\le\sum_{k=1}^{\infty}\arctan\frac{1}{2k^2}=\pi/4<e^{\frac{\kappa\pi^2}{(b-a)^2}},$$
condition (P0) holds. From (G1), we see that the condition (P1) is satisfied. Hence by Theorem 3.1, problem (4.1) has a strong solution $u\in C(J,H_0^1(a,b))\cap L^2(J,H^2(a,b))\cap W^{1,2}(J,L^2(a,b))$ in the $L^2(a,b)$ sense. Since the nonlinear term g satisfies (G2), by using a similar regularization method in [1, Lemma 4.2], we can prove that $u\in C^{2+\mu,1+\mu/2}([a,b]\times J)$ is a classical solution of (4.1). $\square$

Similarly, from Theorem 3.2 we obtain the following result.

Theorem 4.2. *Assume that the nonlinear term g satisfies* (G2) *and*

(G3) *there exists a positive constant*
$$\eta<\frac{\kappa\pi^2}{(b-a)^2(\pi+4)}\big(4-\pi e^{-\frac{\kappa\pi^2}{(b-a)^2}}\big)$$
such that
$$|g(x,t,w)-g(x,t,v)|\le\eta|w-v|,\quad x\in[a,b],\ t\in J,\ w,v\in\mathbb{R}.$$

Then (4.1) *has a unique classical solution $\widehat{u}\in C^{2+\mu,1+\mu/2}([a,b]\times J)$.*

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