Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 178, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE AND UNIQUENESS OF M-SOLUTIONS FOR BACKWARD STOCHASTIC VOLTERRA INTEGRAL EQUATIONS

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Abstract. In this article, we study general backward stochastic Volterra integral equations (BSVIEs). Combining the contractive-mapping principle, step-by-step iteration method and mathematical induction, we establish the existence and uniqueness theorem of M-solution for the BSVIEs. This theorem could be applied directly to many models, for example, using the result to a kind of financial models provides a new and easy method to discuss the existence of dynamic risk measure.

1. Introduction

Backward stochastic differential equations (BSDEs) and backward stochastic Volterra integral equation (BSVIE) are applied widely in finance and stochastic control etc.[3, 6, 7]. The theoretical foundation of BSDEs have been established by Pardoux and Peng[5]. The development of BSDEs greatly promoted the evolution of economics and finance. For example, economists Diffie and Epstein introduced BSDEs into economics in 1992, and stochastic analyst El Karoai et al [2] discovered the important role of BSDEs in finance. In 2006, Yong [8] introduce the BSVIEs, in which play a major role in considering the properties of forward stochastic Volterra integral equations, which describe the stochastic optimal control problem with memories, and in the proof of stochastic Pontryagin maximum principle [7] etc.. Besides, in [9], a class of dynamic convex and coherent risk measures are identified as a component of the adapted M-solutions to certain BSVIEs.

In this article, we consider general backward stochastic Volterra integral equation (BSVIE)

$$\begin{aligned} Y(t) = F(t, Y(t)) &+ \int_t^T g(t, s, Y(s), Y(t), Z(t, s), Z(s, t))\,\mathrm{d}s \\ &- \int_t^T h(t, s, Y(s), Y(t), Z(t, s))\,\mathrm{d}W(s), \quad t \in [0, T]. \end{aligned} \tag{1.1}$$

2000 *Mathematics Subject Classification.* 45D05, 60H17, 34A12, 60H20.
Key words and phrases. Backward stochastic Volterra integral equations; existence; uniqueness; dynamic risk measure.

Submitted August 2, 2013. Published August 21, 2014.

A class of important BSVIEs (1.1) is the the following

$$Y(t) = \xi + \int_t^T g(s, Y(s), Z(s))\, \mathrm{d}s + \int_t^T Z(s)\, \mathrm{d}W(s), \quad t \in [0,T]. \tag{1.2}$$

Yong [8] introduced another form of BSVIE (1.1):

$$Y(t) = \phi(t) - \int_t^T g\Big(t, s, Y(s), Z(t,s), Z(s,t)\Big)\, \mathrm{d}s - \int_t^T Z(t,s)\, \mathrm{d}W(s), \quad t \in [0,T], \tag{1.3}$$

and gave the following definition of adapted solution.

Definition 1.1. Any pair of stochastic processes $(Y(\cdot), Z(\cdot,\cdot)) \in \mathcal{H}^1[S,T]$ (defined as (3.1)) satisfying (1.3) is called an adapted solution of (1.3).

Also, the conditions of the existence and uniqueness of the adapted solution to (1.3) are given. However, it is difficult to consider the uniqueness of the adapted solution to (1.3) under Definition 1.1. For example, for the BSVIE

$$Y(t) = \int_t^T g\Big(t, s, Y(s), Z(t,s)\Big)\, \mathrm{d}s - \int_t^T Z(t,s)\, \mathrm{d}W(s), \quad t \in [0,T], \tag{1.4}$$

in which g satisfies the conditions of existence and uniqueness theorem, suppose $(Y(\cdot), Z(\cdot,\cdot))$ is the uniqueness adapted solution of (1.4). But it is easy to check that $(\hat{Y}(\cdot), \hat{Z}(\cdot,\cdot))$ satisfies

$$\begin{gathered}
\hat{Y}(t) = Y(t), \quad t \in [0,T], \\
\hat{Z}(t,s) = Z(t,s), \quad (t,s) \in [0,T] \times [t,T], \\
\hat{Z}(t,s) = \varsigma(t,s), \quad (t,s) \in [0,T] \times [0,t],
\end{gathered}$$

is also an adapted solution of (1.4) for any $\varsigma(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R})$. It is contradictive.

The purpose of this article is to discuss the existence and uniqueness of adapted M-solutions (defined later), rather than the adapted solution to (1.1) in Definition 1.1. We give some sufficient conditions for the existence and uniqueness of M-solution to (1.1), by combining contractive-mapping principle, step-by-step iteration method and mathematical induction. These results could be applied directly to many models, such as those described as BSVIEs, in which a component of the M-solution of BSVIEs has a close relation with the dynamic risk measure. By applying the main result to the financial models, it provides a new and easy method to discuss the existence of dynamic risk measure.

2. Motivation and main results

2.1. **Experimental motivation.** In this article, we use (1.1) to describe a class of economic problems as certain portfolio, such as European option, some current cash flows, mutual funds etc. Here, $Y(t)$ denotes the price of merchandize, $F(t, Y(t))$ stands for the total wealth of certain portfolio and g is referred to as the generator of (1.1). Since the component $Y(t)$ of the M-solution to (1.1) has a close relationship with the dynamic risk measure, it is significant to consider the existence and uniqueness of the M-solution. Here the dynamic risk measure is defined as follows:

Definition 2.1 ([9])**.** A map $\rho : L^2_{\mathcal{F}_T}(0,T) \to L^2_{\mathbb{F}}(0,T)$ is called a dynamic risk measure if the following conditions hold:

1. For any $\psi(\cdot),\bar\psi(\cdot)\in L^2_{\mathcal{F}_T}(0,T)$, if $\psi(\cdot)=\bar\psi(\cdot)$, a.s. $s\in[t,T]$, for some $t\in[0,T)$, then
$$\rho(t,\psi(\cdot))=\rho(t,\bar\psi(\cdot)),\quad \text{a.s.}.$$
2. For any $\psi(\cdot),\bar\psi(\cdot)\in L^2_{\mathcal{F}_T}(0,T)$, if $\psi(\cdot)\le\bar\psi(\cdot)$, a.s. $s\in[t,T]$, for some $t\in[0,T)$, then
$$\rho(t,\psi(\cdot))\ge\rho(t,\bar\psi(\cdot)),\quad \text{a.s. } s\in[t,T].$$

In fact, the model is on the basis of some classical financial models. El Karoui et al [2] described the problem of European option pricing by applying BSDE (1.2). In this case, ξ represents square-integrable contingent claim and $Y(t)$ represents the price of European option. For their model, if there exists unique solution, then the map $\rho:\xi\to Y(t)$ defined by (1.2) is a dynamic risk measure. Yong [9], extended (1.2) into (1.3). There $\phi(t)$ represents the total wealth of certain portfolio. The author gave dynamic risk measure $\rho(t,\phi(t))=Y(t)$ for (1.3).

2.2. **Model assumptions and novelty.** Now we give some assumptions for (1.1) such that it has unique solution.

(H1) Let $g:\Delta^c\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^{m\times d}\times\Omega\to\mathbb{R}^m$ be $\mathfrak{B}(\Delta^c\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^{m\times d})\otimes\mathcal{F}_T$-measurable, and for all $(t,\zeta,\eta,\xi,\varsigma)\in[0,T]\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^{m\times d}$, $g(t,\cdot,\zeta,\eta,\xi,\varsigma)$ is $\mathbb{F}$-adapted and satisfies
$$E\int_0^T\Big(\int_t^T|g(t,s,0,0,0,0)|\,\mathrm{d}s\Big)^2\,\mathrm{d}t<\infty,\tag{2.1}$$
and for all $(t,s)\in\Delta^c,\zeta,\bar\zeta,\eta,\bar\eta\in\mathbb{R}^m,\xi,\bar\xi,\varsigma,\bar\varsigma\in\mathbb{R}^{m\times d}$
$$\begin{aligned}&|g(t,s,\zeta,\eta,\xi,\varsigma)-g(t,s,\bar\zeta,\bar\eta,\bar\xi,\bar\varsigma))|\\&\le L_\zeta(t,s)|\zeta-\bar\zeta|+L_\eta(t,s)|\eta-\bar\eta|+L_\xi(t,s)|\xi-\bar\xi|+L_\varsigma(t,s)|\varsigma-\bar\varsigma| \text{ a.s.},\end{aligned}\tag{2.2}$$
where for some $\varepsilon>0$, $L_\zeta(t,s),L_\eta(t,s),L_\xi(t,s),L_\varsigma(t,s):\Delta^c\to\mathbb{R}$ satisfy
$$\sup_{t\in[0,T]}\int_t^T\Big[L_\zeta(t,s)^{2+\varepsilon}+L_\eta(t,s)^{2+\varepsilon}+L_\xi(t,s)^{2+\varepsilon}+L_\varsigma(t,s)^{2+\varepsilon}\Big]\,\mathrm{d}s=A<\infty,$$
and
$$\sup_{t\in[0,T]}\Big(\int_0^T L_\eta(t,s)\,\mathrm{d}s\Big)^2=K<\frac{1}{8C^2},\tag{2.3}$$
in which C is the same as the one in (3.5).

(H2) Let $F:\mathbb{R}^1\times\mathbb{R}^m\times\Omega\to\mathbb{R}^m$ be $\mathfrak{B}(\mathbb{R}^1\times\mathbb{R}^m)\otimes\mathcal{F}_T$-measurable, and
$$E\int_0^T|F(t,0)|\,\mathrm{d}t<\infty.\tag{2.4}$$
Moreover,
$$\begin{gathered}|F(t,\varsigma)-F(t,\bar\varsigma))|\le L_\varsigma(t)|\varsigma-\bar\varsigma|,\quad t\in\mathbb{R},\varsigma,\bar\varsigma\in\mathbb{R}^m \text{ a.s.},\\ \sup_{t\in[0,T]}|L_\varsigma(t)|^2\le D,\\ 2C_LD<1.\end{gathered}\tag{2.5}$$
hold. Here C_L is determined by (3.11).

(H3) Let $h : \Delta^c \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \Omega \to \mathbb{R}^{m\times d}$ be $\mathfrak{B}(\Delta^c \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{m\times d}) \otimes \mathcal{F}_T$-measurable, and for all $(t, \zeta, \eta, \xi) \in [0, T] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$, $h(t, \cdot, \zeta, \eta, \xi)$ is $\mathbb{F}$-adapted and has the following relations:

$$
\begin{gathered}
|h(t, s, \zeta, \eta, \xi) - h(t, s, \bar{\zeta}, \bar{\eta}, \bar{\xi}) - (\xi - \bar{\xi})|^2 \le L_\zeta |\zeta - \bar{\zeta}|^2 + L_\eta |\eta - \bar{\eta}|^2 + L_\xi |\xi - \bar{\xi}|^2, \\
\forall (t, s) \in \Delta^c, \zeta, \bar{\zeta}, \eta, \bar{\eta} \in \mathbb{R}^m,\ \xi, \bar{\xi} \in \mathbb{R}^{m\times d},\ \text{a.s.}, \\
E \int_0^T \int_t^T |h(t, s, 0, 0, 0)|^2 \,\mathrm{d}s \,\mathrm{d}t < \infty, \\
\max \Big\{ 4C_F L_\xi, C_F (L_\zeta + L_\eta) T \Big\} < 1,
\end{gathered}
\tag{2.6}
$$

where C_F is determined by (3.21).

It is easy to verify that conditions (H1)–(H3) will degenerate into (H) in [10] as (1.1) equal to (1.3). In the rest of the subsection, we will show the novelty in this paper from the following points: theory and application.

(1) It is noted that (1.2) can not show the rule of the total wealth changing with the time; and both (1.2) and (1.3) cannot build up the relation between the total wealth with the price of merchandise. To get rid of the two defects, we reconstruct the model as BSVIE (1.1).

In [2] (or [9]), by building the relation between ξ and $Y(t)$, (or $\phi(t)$ and $Y(t)$), the dynamic risk measure for (1.2) (or (1.3)) is found. However, model (1.1) gives directly the explicit relation of $\varphi(t)$ and $F(t, Y(t))$, i. e. $\varphi(t) = F(t, Y(t))$. Hence, if $F(t, \cdot)$ has well properties, we could arrive a dynamic risk measure $F^{-1} : \varphi(t) \to Y(t)$. Sequently, in order to study the properties of the dynamic risk measure, such as dynamic convex or coherent risk measures, it only needs to restrict the $F(t, \cdot)$ further. Then applying the main results to this model, it is convenient to find the dynamic risk measure. (2) It is well-known that step-by-step iteration method and fixed point theorem are important method to prove the existence and uniqueness of solution to equations [1, 4, 5, 8, 10]. But sometimes, for the equations having complicated forms, it is hard to derive the existence and uniqueness theorem, by using only method of them. However, in this paper we combine step-by-step iteration method, fixed point theorem, mathematic induction and Martingale representation theorem to provide a proof of the existence and uniqueness of M-solution to (1.1). For different domain of definition of $Z(\cdot, \cdot)$, we use different method as in Figure 1.

2.3. **Main results.** Firstly, we introduce the definition of M-solution, and then give the main results of the paper: existence and uniqueness theorem of M-solution to BSVIE (1.1). The proof of it is left in the next section.

Definition 2.2. A pair $(Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{H}^1[S, T]$ is called an adapted M-solution of BSVIE (1.1) on $[S, T]$, if $(Y(\cdot), Z(\cdot, \cdot))$ satisfies (1.1) in the usual *Itôs* sense for almost all $t \in [S, T]$ and, in addition, the following holds:

$$
Y(t) = E(Y(t)|\mathcal{F}_S) + \int_S^t Z(t, s) \,\mathrm{d}W(s), \quad \text{a.e. } t \in [S, T].
$$

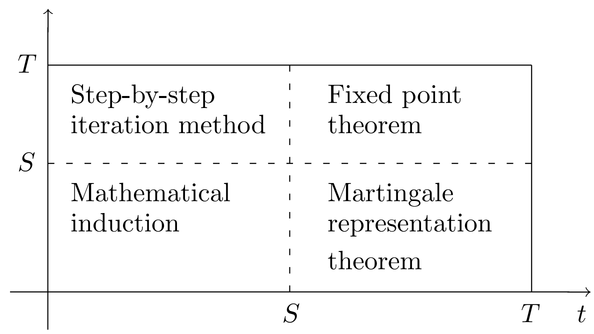

FIGURE 1. Diagram of methods used in different domains of definition of $Z(\cdot,\cdot)$ to prove the main result

Theorem 2.3. *Let* (H1)–(H3) *hold, then* (1.1) *admits a unique adapted M-solution* $(Y(\cdot), Z(\cdot,\cdot))$*, and the following estimate holds:*

$$\begin{aligned} &\|(Y(\cdot), Z(\cdot,\cdot))\|^2_{\mathcal{H}^2[R,T]} \\ &\le C_h E\Big\{ \int_R^T |F(t,0)|^2 \,\mathrm{d}t + \int_R^T \Big(\int_t^T |g(t,s,0,0,0,0)| \,\mathrm{d}s \Big)^2 \mathrm{d}t \\ &\quad + \int_R^T \int_t^T |h(t,s,0,0,0)|^2 \,\mathrm{d}s \,\mathrm{d}t \Big\}, \quad R \in [0,T]. \end{aligned} \tag{2.7}$$

Furthermore, if $\bar g, \bar F, \bar h$ *satisfy* (H1)–(H3)*, respectively, and* $(\bar Y(\cdot), \bar Z(\cdot,\cdot))$ *is the M-solution, in which* (g,F,h) *is replaced by* $(\bar g, \bar F, \bar h)$*. Then*

$$\begin{aligned} &E\Big[\int_R^T |Y(t) - \bar Y(t)|^2 \,\mathrm{d}t + \int_R^T \int_R^T |Z(t,s) - \bar Z(t,s)|^2 \,\mathrm{d}s \,\mathrm{d}t \Big] \\ &\le C_h E\Big\{ \int_R^T |F(t,Y(t)) - \bar F(t,Y(t))|^2 \,\mathrm{d}t \\ &\quad + \int_R^T \Big(\int_t^T |g(t,s,Y(s),Y(t),Z(t,s),Z(t,s)) \\ &\quad - \bar g(t,s,Y(s),Y(t),Z(t,s),Z(t,s))| \,\mathrm{d}s \Big)^2 \mathrm{d}t \\ &\quad + \int_R^T \int_t^T |h(t,s,Y(s),Y(t),Z(t,s)) - \bar h(t,s,Y(s),Y(t),Z(t,s))|^2 \,\mathrm{d}s \,\mathrm{d}t \Big\}, \end{aligned} \tag{2.8}$$

for $R \in [0,T]$.

3. Proof of the main theorem

Before proving Theorem 2.3, we show some useful notion from [10], and some lemmas. Let

$$\begin{aligned} L^p_{\mathcal{F}_S}(\Omega; L^q(0,T)) = \Big\{ &\phi : (0,T) \times \Omega \to \mathbb{R}^m : \phi(\cdot) \text{ is } \mathcal{B}([0,T]) \otimes \mathcal{F}_S\text{-measurable}, \\ &E\Big(\int_0^T |\phi(t)|^q \,\mathrm{d}t \Big)^{\frac{p}{q}} < \infty \Big\}. \end{aligned}$$

$$L^p_{\mathbb{F}}(\Omega;L^q(0,T))=\Big\{\phi(\cdot)\in L^p(\Omega;L^q(0,T)):\phi(\cdot)\text{ is }\mathbb{F}\text{-adapted}\Big\}.$$

For any $p,q\ge 1$, let $L^q(0,T;L^p_{\mathbb{F}}(\Omega;L^2(0,T)))$ be the set of all processes $Z:[0,T]^2\times\Omega\to\mathbb{R}^{m\times d}$, such that for almost all $t\in[0,T]$, $Z(t,\cdot)\in L^p_{\mathbb{F}}(\Omega;L^2(0,T))$, there is

$$\int_0^T\Big\{E\Big(\int_0^T|Z(t,s)|^2\,\mathrm{d}s\Big)^{p/2}\Big\}^{q/p}\mathrm{d}t<\infty.$$

For convenience, denote

$$\Delta[R,S]=\{(t,s)\in[R,S]^2:R\le s\le t\le S\},$$
$$\Delta^c[R,S]=\{(t,s)\in[R,S]^2:R\le t<s\le S\},$$

and for any $0\le R\le S\le T$,

$$\mathcal{H}^p[R,S]=L^p_{\mathbb{F}}(\Omega;L^p(0,T))\times L^p(0,T;L^p_{\mathbb{F}}(\Omega;L^2(0,T))). \tag{3.1}$$

If we define

$$\|y(\cdot),z(\cdot,\cdot)\|_{\mathcal{H}^2[R,S]}\equiv\Big\{E\Big[\int_R^S|y(t)|^2\,\mathrm{d}t+\int_R^S\int_R^S|z(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t\Big]\Big\}^{1/2}.$$

Then $\|\cdot\|_{\mathcal{H}^2[R,S]}$ could define a metric on $\mathcal{H}^2[R,S]$ and the space is complete under this metric, clearly.

For any $R,S\in[0,T)$, consider the stochastic integral equation

$$\lambda(t,r)=\psi(t)+\int_r^T k(t,s,\mu(t,s))\,\mathrm{d}s-\int_r^T\mu(t,s)\,\mathrm{d}W(s), \tag{3.2}$$

for $r\in[R,T]$ and $t\in[S,T]$, where $k:[S,T]\times[R,T]\times\mathbb{R}^{m\times d}\times\Omega\to\mathbb{R}^m$ is given. $(\lambda(t,\cdot),\mu(t,\cdot))$ is $\mathbb{F}$-adapted for any $t\in[R,T]$. Introduce the following assumption for k:

(H0) Let $R,S\in[0,T)$, and $k:[S,T]\times[R,T]\times\mathbb{R}^{m\times d}\times\Omega\to\mathbb{R}^m$ be $\mathfrak{B}([S,T]\times[R,T]\times\mathbb{R}^{m\times d})\otimes\mathcal{F}_T$-measurable such that $k(t,\cdot,z)$ is $\mathbb{F}$-progressively measurable for $(t,z)\in[S,T]\times R^{m\times d}$, and

$$E\Big(\int_R^T|k(t,s,0)|\,\mathrm{d}s\Big)^p<\infty,\ a.e.\ t\in[S,T], \tag{3.3}$$

for some $p>1$. Moreover, the following holds:

$$|k(t,s,z)-k(t,s,\bar z)|\le L_z(t,s)|z-\bar z|,\ (t,s)\in[S,T]\times[R,T],\ z,\bar z\in\mathbb{R}^{m\times d},a.s,$$

where $L_z:[S,T]\times[R,T]\to[0,\infty)$ is a deterministic function, such that for some $\varepsilon>0$,

$$\sup_{t\in[S,T]}\int_R^T L_z(t,s)^{2+\varepsilon}\,\mathrm{d}s<\infty.$$

Let $r=S\in[R,T)$ be fixed. Define

$$\psi^S(t)=\lambda(t,S),\ Z(t,s)=\mu(t,s),\quad t\in[R,S],\ s\in[S,T].$$

Then (3.2) is rewritten as stochastic Fredholm integral equations (SFIEs):

$$\psi^S(t)=\psi(t)+\int_r^T k(t,s,Z(t,s))\,\mathrm{d}s-\int_r^T Z(t,s)\,\mathrm{d}W(s),\quad t\in[S,T]. \tag{3.4}$$

We call $(\psi^S(\cdot),Z(\cdot,\cdot))\in L^p_{\mathcal{F}_S}(R,S)\times L^p(R,S;L^2_{\mathbb{F}}(S,T))$ as an adapted solution of (3.4), if it satisfies (3.4) in the sense of Itô.

Lemma 3.1 ([10]). *Let* (H0) *hold. Then for any* $\psi(\cdot) \in L^p_{\mathcal{F}_T}(R,S)$, (3.4) *admits a unique adapted solution* $(\psi^S(\cdot), Z(\cdot,\cdot)) \in L^p_{\mathcal{F}_S}(R,S) \times L^p(R,S; L^2_{\mathbb{F}}(S,T))$, *and the following estimate holds:*

$$E\Big\{|\psi^S(t)|^p+\Big(\int_S^T |Z(t,s)|^2\,\mathrm{d}s\Big)^{p/2}\Big\} \le CE\Big\{|\psi(t)|^p+\Big(\int_S^T |k(t,s,0)|\,\mathrm{d}s\Big)^p\Big\}, \tag{3.5}$$

for $t \in [R,S]$. *If* $\bar{k} : [R,S] \times [S,T] \times \mathbb{R}^{m\times d} \times \Omega \to \mathbb{R}^m$ *satisfies (H0),* $\bar{\psi}(\cdot) \in L^p_{\mathcal{F}_T}(R,S)$, *and* $(\bar{\psi}^S(\cdot), \bar{Z}(\cdot,\cdot)) \in L^p_{\mathcal{F}_S}(R,S) \times L^p(R,S; L^2_{\mathbb{F}}(S,T))$ *is the unique adapted solution of* (3.4) *in which* (k,ψ) *is replaced by* $(\bar{k},\bar{\psi})$, *then*

$$\begin{aligned} &E\Big\{|\psi^S(t)-\bar{\psi}^S(t)|^p+\Big(\int_S^T |Z(t,s)-\bar{Z}(t,s)|^2\,\mathrm{d}s\Big)^{p/2}\Big\} \\ &\le CE\Big\{|\psi(t)-\bar{\psi}(t)|^p+\Big(\int_S^T |k(t,s,Z(t,s))-\bar{k}(t,s,Z(t,s))|\,\mathrm{d}s\Big)^p\Big\}, \end{aligned} \tag{3.6}$$

for $t \in [R,S]$.

Let $S = R$, and define

$$\begin{gathered} Y(t) = \lambda(t,t), \quad t \in [S,T], \\ Z(t,s) = \mu(t,s), \quad (t,s) \in \Delta^c[S,T]. \end{gathered}$$

Then (3.2) can be rewritten as

$$Y(t) = \psi(t) + \int_t^T k(t,s,Z(t,s))\,\mathrm{d}s - \int_t^T Z(t,s)\,\mathrm{d}W(s), \quad t \in [S,T]. \tag{3.7}$$

Lemma 3.2 ([10]). *Let* (H0) *hold. Then for any* $\psi(\cdot) \in L^p_{\mathcal{F}_T}(R,S)$, (3.7) *admits a unique adapted M-solution* $(Y(\cdot), Z(\cdot,\cdot)) \in \mathcal{H}^p[S,T]$, *and the following estimate holds:*

$$E\Big\{|Y(t)|^p+\Big(\int_S^T |Z(t,s)|^2\,\mathrm{d}s\Big)^{p/2}\Big\} \le CE\Big\{|\psi(t)|^p+\Big(\int_t^T |k(t,s,0)|\,\mathrm{d}s\Big)^p\Big\}, \tag{3.8}$$

for $t \in [S,T]$. *If* $\bar{k} : [R,S] \times [S,T] \times \mathbb{R}^{m\times d} \times \Omega \to \mathbb{R}^m$ *also satisfies* (H0), $\bar{\psi}(\cdot) \in L^p_{\mathcal{F}_T}(S,T)$, *and* $(\bar{Y}(\cdot), \bar{Z}(\cdot,\cdot)) \in \mathcal{H}^p[S,T]$ *is the unique adapted M-solution of BSVIE* (3.7) *in which* (k,ψ) *is replaced by* $(\bar{k},\bar{\psi})$, *then*

$$\begin{aligned} &E\Big\{|Y(t)-\bar{Y}(t)|^p+\Big(\int_S^T |Z(t,s)-\bar{Z}(t,s)|^2\,\mathrm{d}s\Big)^{p/2}\Big\} \\ &\le CE\Big\{|\psi(t)-\bar{\psi}(t)|^p+\Big(\int_t^T |k(t,s,Z(t,s))-\bar{k}(t,s,Z(t,s))|\,\mathrm{d}s\Big)^p\Big\}, \end{aligned} \tag{3.9}$$

for $t \in [S,T]$.

The proof of Theorem 2.3 is split into three steps, in which we find solutions for the three BSVIE's: (3.10), (3.20), and (1.1).

3.1. Existence and uniqueness of M-solution for the BSVIE.

$$Y(t) = \psi(t) + \int_t^T g(t,s,Y(s),Y(t),Z(t,s),Z(s,t))\,\mathrm{d}s - \int_t^T Z(t,s)\,\mathrm{d}W(s), \tag{3.10}$$

for $t \in [0,T]$.

Theorem 3.3. *Let* (H1) *hold. Then for any* $\psi(\cdot) \in L^2_{\mathcal{F}_T}(0,T)$, (3.10) *admits a unique adapted M-solution* $(Y(\cdot), Z(\cdot,\cdot)) \in \mathcal{H}^2[0,T]$. *Moreover, the following estimate holds:*

$$\begin{aligned}
&\|(Y(\cdot), Z(\cdot,\cdot))\|^2_{\mathcal{H}^2[R,T]}\\
&\equiv E\Big\{\int_R^T |Y(t)|^2\,\mathrm{d}t + \int_R^T\int_R^T |Z(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t\Big\}\\
&\le C_L E\Big\{\int_R^T |\psi(t)|^2\,\mathrm{d}t + \int_R^T\Big(\int_t^T |g(t,s,0,0,0,0)|\,\mathrm{d}s\Big)^2\mathrm{d}t\Big\}, \quad R\in[0,T].
\end{aligned} \tag{3.11}$$

If $\bar g$ *also satisfies (H1),* $\bar\psi(\cdot) \in L^2_{\mathcal{F}_T}(0,T)$, *and* $(\bar Y(\cdot), \bar Z(\cdot,\cdot)) \in \mathcal{H}^2[0,T]$ *is the adapted M-solution of* (3.10) *in which* (g,ψ) *is replaced by* $(\bar g, \bar\psi)$, *then*

$$\begin{aligned}
&E\Big\{\int_R^T |Y(t)-\bar Y(t)|^2\,\mathrm{d}t + \int_R^T\int_R^T |Z(t,s)-\bar Z(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t\Big\}\\
&\le C_L E\Big\{\int_R^T |\psi(t)-\bar\psi(t)|^2\,\mathrm{d}t + \int_R^T\Big(\int_t^T |g(t,s,Y(s),Y(t),Z(t,s),Z(s,t))\\
&\quad - \bar g(t,s,Y(s),Y(t),Z(t,s),Z(s,t))|\,\mathrm{d}s\Big)^2\mathrm{d}t\Big\}, \quad R\in[0,T].
\end{aligned} \tag{3.12}$$

Proof. We split the proof into three steps.

Step 1: Let $\mathcal{M}^2[S,T]$ be a subspace of $\mathcal{H}^2[S,T]$, and for any $(y(\cdot), z(\cdot,\cdot)) \in \mathcal{M}^2[S,T]$ satisfies

$$y(t) = E(y(t)|\mathcal{F}_S) + \int_S^t z(t,s)\,\mathrm{d}W(s), \ \ a.e.\ t\in[S,T], \text{a.s.}, \tag{3.13}$$

and define

$$\|y(\cdot), z(\cdot,\cdot)\|_{\mathcal{M}^2[S,T]} \equiv \Big\{E\Big[\int_S^T |y(t)|^2\,\mathrm{d}t + \int_S^T\int_t^T |z(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t\Big]\Big\}^{1/2}.$$

Clearly, $\mathcal{M}^2[S,T]$ is a nontrivial closed subspace of $\mathcal{H}^2[S,T]$.

For any $\psi(\cdot) \in L^2_{\mathcal{F}_T}(S,T)$, $(y(\cdot), z(\cdot,\cdot)) \in \mathcal{M}^2[S,T]$, consider the BSVIE:

$$Y(t) = \psi(t) + \int_t^T g(t,s,y(s),y(t),Z(t,s),z(s,t))\,\mathrm{d}s - \int_t^T Z(t,s)\,\mathrm{d}W(s), \quad t\in[S,T].$$

Applying Lemma 3.2, this BSVIE admits a unique adapted M-solution $(Y(\cdot), Z(\cdot,\cdot))$ in $\mathcal{H}^2[S,T]$. On the other hand, $(Y(\cdot), Z(\cdot,\cdot))$ salsifies (3.13), hence $(Y(\cdot), Z(\cdot,\cdot)) \in \mathcal{M}^2[S,T]$.

Define map $\Lambda : \mathcal{M}^2[S,T] \to \mathcal{M}^2[S,T]$ by

$$\Lambda(y(\cdot), z(\cdot,\cdot)) = (Y(\cdot), Z(\cdot,\cdot)). \tag{3.14}$$

If $(\bar y(\cdot), \bar z(\cdot,\cdot)) \in \mathcal{M}^2[S,T]$, and $\Lambda(\bar y(\cdot), \bar z(\cdot,\cdot)) = (\bar Y(\cdot), \bar Z(\cdot,\cdot))$, then by (3.9) we obtain

$$\begin{aligned}
&E\Big[|Y(t)-\bar Y(t)|^2 + \int_S^T |Z(t,s)-\bar Z(t,s)|^2\,\mathrm{d}s\Big]\\
&\le CE\Big[\int_t^T |g(t,s,y(s),y(t),Z(t,s),z(s,t)) - \bar g(t,s,\bar y(s),\bar y(t),Z(t,s),\bar z(s,t))|\,\mathrm{d}s\Big]^2
\end{aligned}$$

$$
\begin{aligned}
&\le CE\Big[\int_t^T \Big(L_\zeta(t,s)|y(s)-\bar y(s)|+L_\eta(t,s)|y(t)-\bar y(t)|+L_\varsigma(t,s)|z(s,t) \\
&\quad -\bar z(s,t)|\Big)\,\mathrm{d}s\Big]^2 \\
&\le 9CA^{\frac{2}{2+\varepsilon}}(T-t)^{\frac{\varepsilon}{2+\varepsilon}}E\Big[\int_t^T \Big(|y(s)-\bar y(s)|^2+|y(t)-\bar y(t)|^2+|z(s,t) \\
&\quad -\bar z(s,t)|^2\Big)\,\mathrm{d}s\Big].
\end{aligned}
$$

Consequently, we obtain that

$$
\begin{aligned}
&\|\Lambda(y(\cdot),z(\cdot,\cdot))-\Lambda(\bar y(\cdot),\bar z(\cdot,\cdot))\|^2_{\mathcal{M}^2[S,T]} \\
&\equiv E\Big[\int_S^T |Y(t)-\bar Y(t)|^2\,\mathrm{d}t+\int_S^T\int_t^T |Z(t,s)-\bar Z(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t\Big] \\
&\le 9CA^{\frac{2}{2+\varepsilon}}(T-t)^{\frac{\varepsilon}{2+\varepsilon}}\max\{1,2(T-S)\}E\Big\{\int_S^T |y(t)-\bar y(t)|^2\,\mathrm{d}t \\
&\quad +\int_S^T\int_t^T |z(s,t)-\bar z(s,t)|^2\,\mathrm{d}s\,\mathrm{d}t\Big\} \\
&\le C_3(T-S)^{\frac{\varepsilon}{2+\varepsilon}}\|(y(\cdot),z(\cdot,\cdot))-(\bar y(\cdot),\bar z(\cdot,\cdot))\|^2_{\mathcal{M}^2[S,T]},
\end{aligned}
$$

where $C_3=9CA^{\frac{2}{2+\varepsilon}}\max\{1,2(T-S)\}$. So $\Lambda:\mathcal{M}^2[S,T]\to\mathcal{M}^2[S,T]$ is contracting, if $T-S$ is sufficiently small. Then there exists a unique fixed point in $\mathcal{M}^2[S,T]$. Hence (3.10) admits a unique adapted M-solution $(Y(\cdot),Z(\cdot,\cdot))\in\mathcal{M}^2[S,T]$.

Now (3.8) yields

$$
\begin{aligned}
&E\Big[|Y(t)|^2+\int_t^T |Z(t,s)|^2\,\mathrm{d}s\Big] \\
&\le CE\Big\{|\psi(t)|^2+\Big(\int_t^T |g(t,s,Y(s),Y(t),0,Z(s,t))|\,\mathrm{d}s\Big)^2\Big\} \\
&\le 4CE\Big\{|\psi(t)|^2+\Big(\int_t^T |g(t,s,0,0,0,0)|\,\mathrm{d}s\Big)^2 \\
&\quad +\Big(\int_t^T L_\zeta(t,s)^2\,\mathrm{d}s\Big)\Big(\int_t^T |Y(s)|^2\,\mathrm{d}s\Big)+\Big(\int_t^T L_\eta(t,s)^2\,\mathrm{d}s\Big)\Big(\int_t^T |Y(t)|^2\,\mathrm{d}s\Big) \\
&\quad +\Big(\int_t^T L_\varsigma(t,s)^2\,\mathrm{d}s\Big)\Big(\int_t^T |Z(s,t)|^2\,\mathrm{d}s\Big)\Big\}.
\end{aligned} \tag{3.15}
$$

Consequently,

$$
\begin{aligned}
&\|(Y(\cdot),Z(\cdot,\cdot))\|^2_{\mathcal{H}^2[S,T]} \\
&\equiv E\Big[\int_S^T |Y(t)|^2\,\mathrm{d}t+\int_S^T\int_S^T |Z(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t\Big] \\
&\le 4CE\Big\{\int_S^T |\psi(t)|^2\,\mathrm{d}t+\int_S^T\Big(\int_t^T |g(t,s,0,0,0,0)|\,\mathrm{d}s\Big)^2\,\mathrm{d}t \\
&\quad +2(T-S)^{\frac{\varepsilon}{2+\varepsilon}+1}A^{\frac{2}{2+\varepsilon}}\int_S^T |Y(s)|^2\,\mathrm{d}s
\end{aligned}
$$

$$+ (T-S)^{\frac{\varepsilon}{2+\varepsilon}} A^{\frac{2}{2+\varepsilon}} \int_S^T \int_S^T |Z(s,t)|^2 \,\mathrm{d}s \,\mathrm{d}t \Big\}.$$

If $T-S$ is so small that $4C(T-S)^{\frac{\varepsilon}{2+\varepsilon}} A \max\{2(T-S), 1\} < 1/2$, then (3.11) holds with $C_L = 8C$.

Next, let us talk about the stability estimate. Let $(Y(\cdot), Z(\cdot,\cdot))$ and $(\bar{Y}(\cdot), \bar{Z}(\cdot,\cdot))$ be adapted M-solutions of (3.10) corresponding to (g, ψ) and $(\bar{g}, \bar{\psi})$, respectively. Denote

$$\begin{gathered} \hat{Y}(t) = Y(t) - \bar{Y}(t), \quad \hat{Z}(t,s) = Z(t,s) - \bar{Z}(t,s), \quad \hat{\psi}(t) = \psi(t) - \bar{\psi}(t) \\ \hat{g}(t,s) = g(t,s,Y(s),Y(t),Z(t,s),Z(s,t)) - \bar{g}(t,s,Y(s),Y(t),Z(t,s),Z(s,t)). \end{gathered}$$

Then by Hadamard formula, we obtain

$$\begin{aligned} \hat{Y}(t) &= \hat{\psi}(t) + \int_t^T \Big[\alpha_1(t,s)\hat{Y}(s) + \alpha_2(t,s)\hat{Y}(t) \\ &\quad + \sum_{i=1}^d (\beta_i(t,s)\hat{Z}_i(t,s) + \bar{\beta}_i(t,s)\hat{Z}_i(s,t)) + \hat{g}(t,s)\Big] \,\mathrm{d}s - \int_t^T \hat{Z}(t,s) \,\mathrm{d}W(s). \end{aligned}$$

Applying (3.11), we have the stability estimate (3.12) holds with $C_L = 8C$.

In this step we determine the unique solution $(Y(t), Z(t,s))$ to (3.10) for $t, s \in [S,T]$.

Step 2: Since $E[Y(t)|\mathcal{F}_S] \in L^2(S,T; L^2_{\mathcal{F}_S}(\Omega))$, by the Martingale Representation Theorem we could find a unique $Z(\cdot,\cdot) \in L^2(S,T; L^2_{\mathcal{F}_S}(2S-T,S))$ such that

$$E[Y(t)|\mathcal{F}_S] = E[Y(t)|\mathcal{F}_{2S-T}] + \int_{2S-T}^S Z(t,s) \,\mathrm{d}W(s), \quad t \in [S,T]. \tag{3.16}$$

By (3.16) we conclude that

$$E\int_{2S-T}^S |Z(t,s)|^2 \,\mathrm{d}s = E|Y(t)|^2 - |EY(t)|^2, \quad t \in [S,T].$$

Furthermore,

$$\begin{aligned} &E\int_S^T \int_{2S-T}^S |Z(t,s)|^2 \,\mathrm{d}s \,\mathrm{d}t \\ &\le E\int_S^T |Y(t)|^2 \,\mathrm{d}t \\ &\le 8CE\Big\{\int_S^T |\psi(t)|^2 \,\mathrm{d}t + \int_S^T \Big(\int_t^T |g(t,s,0,0,0,0)| \,\mathrm{d}s\Big)^2 \,\mathrm{d}t\Big\}. \end{aligned}$$

In this step we determine the unique solution $(Y(t), Z(t,s))$ of (3.10) for $t \in [S,T], s \in [2S-T,S]$.

Step 3: Denote

$$\begin{gathered} Y_1(t) = Y(t), \quad t \in [2S-T,S], \quad Y_2(t) = Y(t), \quad t \in [S,T], \\ Z_{11}(t,s) = Z(t,s), t \in [2S-T,S] \times [S,T], \quad Z_{12}(t,s) = Z(t,s), t \in [S,T] \times [S,T], \\ Z_{21}(t,s) = Z(t,s), t \in [2S-T,S] \times [2S-T,S], \\ Z_{22}(t,s) = Z(t,s), t \in [S,T] \times [2S-T,S]. \end{gathered}$$

Set $Y_1^{(0)}(t)=0$, and for $n=1,2,\cdots$, define the Picard iterations:

$$\begin{aligned}\psi_n^S(t)&=\psi(t)+\int_S^T g(t,s,Y_2(s),Y_1^{(n-1)}(t),Z_{11}(t,s),Z_{22}(s,t))\,\mathrm{d}s\\&\quad-\int_S^T Z_{11}(t,s)\,\mathrm{d}W(s),\\Y_1^{(n)}(t)&=\psi_n^S(t)+\int_t^S g(t,s,Y_1^{(n)}(s),Y_1^{(n)}(t),Z_{21}^{(n)}(t,s),Z_{21}^{(n)}(s,t))\,\mathrm{d}s\\&\quad-\int_t^S Z_{21}^{(n)}(t,s)\,\mathrm{d}W(s),\end{aligned}\tag{3.17}$$

for $t\in[2S-T,S]$. Obviously, $(Y_1^{(n)}(t),Z_{21}^{(n)}(t,s))\in\mathcal{M}^2[2S-T,S]$. Moreover, by stability estimate (3.12) we can obtain

$$\begin{aligned}&E\Big[\int_{2S-T}^S|Y_1^n(t)-Y_1^{n-1}(t)|^2\,\mathrm{d}t+\int_{2S-T}^S\int_{2S-T}^S|Z_{21}^{(n)}(t,s)-Z_{21}^{(n-1)}(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t\Big]\\&\le 8CE\Big\{\int_{2S-T}^S|\psi_n^S(t)-\psi_{n-1}^S(t)|^2\,\mathrm{d}t\Big\}.\end{aligned}$$

By (2.3) and (3.6), it is not difficult to see that

$$\begin{aligned}&E\Big[|\psi_n^S(t)-\psi_{n-1}^S(t)|^2+\int_S^T|Z_{11}^n(t,s)-Z_{11}^{n-1}(t,s)|^2\,\mathrm{d}s\Big]\\&\le CKE|Y_1^{n-1}(t)-Y_1^{n-2}(t)|^2,\end{aligned}$$

and

$$8CE\int_{2S-T}^S|\psi_n^S(t)-\bar{\psi}_{n-1}^S(t)|^2\,\mathrm{d}t\le 8C^2KE\int_{2S-T}^S|Y_1^{n-1}(t)-Y_1^{n-2}(t)|^2\,\mathrm{d}t.$$

Consequently,

$$\begin{aligned}&E\Big[\int_{2S-T}^S|Y_1^n(t)-Y_1^{n-1}(t)|^2\,\mathrm{d}t+\int_{2S-T}^S\int_{2S-T}^S|Z_{21}^{(n)}(t,s)-Z_{21}^{(n-1)}(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t\Big]\\&\le 8C^2KE\int_{2S-T}^S|Y_1^{n-1}(t)-Y_1^{n-2}(t)|^2\,\mathrm{d}t\\&\le\cdots\le(8C^2K)^{n-1}E\int_{2S-T}^S|Y_1^1(t)-Y_1^0(t)|^2\,\mathrm{d}t.\end{aligned}$$

By (2.3) it is easy to see that $8C^2K<1$, then we obtain $(Y_1^{(n)}(\cdot),Z_{21}^{(n)}(\cdot,\cdot))$ and $(\psi_n^S(\cdot),Z_{11}^{(n)}(\cdot,\cdot))$ are Cauchy sequences on $\mathcal{M}^2[2S-T,S]$ and $L^p_{\mathcal{F}_T}(S,T)\times L^p(2S-T,S;L^2_{\mathbb{F}}(S,T))$, respectively. If $n\to\infty$ in (3.17), we could obtain the unique adapted M-solution $(Y(t),Z(t,s))$ of (3.10) for $t\in[2S-T,S]$, $s\in[2S-T,T]$.

Now, we give the estimate of solution for $(t,s)\in[2S-T,S]\times[2S-T,T]$. Since

$$\begin{aligned}&E\Big[\int_{2S-T}^S|Y_1(t)|^2\,\mathrm{d}t+\int_{2S-T}^S\int_{2S-T}^S|Z_{21}(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t\Big]\\&\le 8CE\Big\{\int_{2S-T}^S|\psi^S(t)|^2\,\mathrm{d}t+\int_{2S-T}^S\Big(\int_t^S|g(t,s,0,0,0,0)|\,\mathrm{d}s\Big)^2\,\mathrm{d}t\Big\},\end{aligned}$$

and

$$
\begin{aligned}
&E\Big[|\psi^S(t)|^2 + \int_S^T |Z_{11}(t,s)|^2\,\mathrm{d}s\Big] \\
&\le CE\Big\{|\psi(t)|^2 + \Big(\int_S^T |g(t,s,Y_2(s),Y_1(t),0,Z_{22}(t,s))|\,\mathrm{d}s\Big)^2\Big\} \\
&\le CE\Big\{|\psi(t)|^2 + 4\Big(\int_S^T |g(t,s,0,0,0,0)|\,\mathrm{d}s\Big)^2\Big\} \\
&\quad + 4C(T-S)^{\frac{\varepsilon}{2+\varepsilon}}A^{\frac{2}{2+\varepsilon}}E\Big[\int_S^T |Y_2(s)|^2\,\mathrm{d}s \\
&\quad + \int_S^T |Z_{22}(s,t)|^2\,\mathrm{d}s\Big] + 4C(T-S)^{\frac{\varepsilon}{2+\varepsilon}+1}A^{\frac{2}{2+\varepsilon}}E|Y_1(t)|^2.
\end{aligned}
$$

So, if $32(T-S)^{\frac{2\varepsilon}{2+\varepsilon}}A^{\frac{2}{2+\varepsilon}}C^2\max\{1,T-S\} < \frac{1}{2}$, we have

$$
\begin{aligned}
&E\Big[\int_{2S-T}^S |\psi^S(t)|^2\,\mathrm{d}t + \int_{2S-T}^S\int_S^T |Z_{11}(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t\Big] \\
&\le (8C+3)E\Big\{\int_{2S-T}^T |\psi(t)|^2\,\mathrm{d}t + \int_{2S-T}^T\Big(\int_t^T |g(t,s,0,0,0,0)|\,\mathrm{d}s\Big)^2\,\mathrm{d}t\Big\},
\end{aligned} \tag{3.18}
$$

and

$$
\begin{aligned}
&E\Big[\int_{2S-T}^S |Y_1(t)|^2\,\mathrm{d}t + \int_{2S-T}^S\int_{2S-T}^S |Z_{21}(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t\Big] \\
&\le (8C+4)E\Big\{\int_{2S-T}^T |\psi(t)|^2\,\mathrm{d}t + \int_{2S-T}^T\Big(\int_t^T |g(t,s,0,0,0,0)|\,\mathrm{d}s\Big)^2\,\mathrm{d}t\Big\}.
\end{aligned} \tag{3.19}
$$

Combining (3.18) and (3.19), we show that

$$
\begin{aligned}
&E\Big\{\int_{2S-T}^T |Y(t)|^2\,\mathrm{d}t + \int_{2S-T}^T\int_{2S-T}^T |Z(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t\Big\} \\
&\le (8C+4)E\Big\{\int_{2S-T}^T |\psi(t)|^2\,\mathrm{d}t + \int_{2S-T}^T\Big(\int_t^T |g(t,s,0,0,0,0)|\,\mathrm{d}s\Big)^2\,\mathrm{d}t\Big\}
\end{aligned}
$$

Similar to Step 1, stability estimate (3.12) holds for $t \in [2S-T,T]$. Then we can use induction method to finish the theorem. □

3.2. **Existence and uniqueness of M-solution for the BSVIE.**

$$
\begin{aligned}
Y(t) = F(t,Y(t)) &+ \int_t^T g(t,s,Y(s),Y(t),Z(t,s),Z(s,t))\,\mathrm{d}s \\
&- \int_t^T Z(t,s)\,\mathrm{d}W(s),\ t\in[0,T].
\end{aligned} \tag{3.20}
$$

Theorem 3.4. *If* (H1) *and* (H2) *hold for* (3.20). *Then there admits unique adapted M-solution* $(Y(\cdot),Z(\cdot,\cdot)) \in \mathcal{H}^2[0,T]$, *and the following estimate holds:*

$$
\begin{aligned}
&\|(Y(\cdot),Z(\cdot,\cdot))\|^2_{\mathcal{H}^2[R,T]} \\
&\le C_F E\Big\{\int_R^T |F(t,0)|^2\,\mathrm{d}t + \int_R^T\Big(\int_t^T |g(t,s,0,0,0,0)|\,\mathrm{d}s\Big)^2\,\mathrm{d}t\Big\},\ R\in[0,T],
\end{aligned} \tag{3.21}
$$

where $C_F = \frac{2C_L}{1-2C_LD}$. Furthermore, if $\bar g$ also satisfies (H1), $\bar F$ satisfies (H2), and $(\bar Y(\cdot), \bar Z(\cdot,\cdot)) \in \mathcal{H}^2[0,T]$ is the adapted M-solution of (3.20), in which (g,F) is replaced by $(\bar g, \bar F)$. Then

$$
\begin{aligned}
&E\Big[\int_R^T |Y(t)-\bar Y(t)|^2\,\mathrm{d}t + \int_R^T\int_R^T |Z(t,s)-\bar Z(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t\Big]\\
&\le C_F E\Big\{\int_R^T |F(t,Y(t))-\bar F(t,Y(t))|^2\,\mathrm{d}t\\
&\quad + \int_R^T \Big(\int_t^T |g(t,s,Y(s),Y(t),Z(t,s),Z(t,s))\\
&\quad - \bar g(t,s,Y(s),Y(t),Z(t,s),Z(t,s))|\,\mathrm{d}s\Big)^2\,\mathrm{d}t\Big\},\ R\in[0,T].
\end{aligned}
\tag{3.22}
$$

Proof. Let $Y^{(0)}(t)=0$. Since $F(t,Y^{(0)}(t)) \in L^2_{\mathcal{F}_T}(0,T)$, then for $n=1,2,\cdots$ we can define the Picard iterations:

$$
\begin{aligned}
Y^{(n)}(t) &= F(t,Y^{(n-1)}(t)) + \int_t^T g(t,s,Y^{(n)}(s),Y^{(n)}(t),Z^{(n)}(t,s),Z^{(n)}(s,t))\,\mathrm{d}s\\
&\quad - \int_t^T Z^{(n)}(t,s)\,\mathrm{d}W(s), \quad t\in[0,T].
\end{aligned}
\tag{3.23}
$$

In view of (3.12), we have that

$$
\begin{aligned}
&E\Big[\int_0^T |Y^{(n)}(t)-Y^{n-1}(t)|^2\,\mathrm{d}t + \int_0^T\int_0^T |Z^n(t,s)-Z^{n-1}(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t\Big]\\
&\le C_LDE\int_0^T |Y_1^{n-1}(t)-Y_1^{n-2}(t)|^2\,\mathrm{d}t \le \cdots\\
&\le (C_LD)^{n-1}E\int_0^T |Y_1^1(t)-Y_1^0(t)|^2\,\mathrm{d}t.
\end{aligned}
$$

By (2.5) we are sure that $(Y^{(n)}(\cdot), Z^{(n)}(\cdot,\cdot))$ is a Cauchy sequence on $\mathcal{M}^2[0,T]$. Let $n\to\infty$ in (3.23) we could obtain the unique solution of (3.20). From (3.11), we have

$$
\begin{aligned}
&E\Big[\int_0^T |Y(t)|^2\,\mathrm{d}t + \int_0^T\int_0^T |Z(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t\Big]\\
&\le C_LE\Big\{2\int_0^T |F(t,0)|^2\,\mathrm{d}t + 2D\int_0^T |Y(t)|^2\,\mathrm{d}t\\
&\quad + \int_0^T\Big(\int_t^T |g(t,s,0,0,0,0)|\,\mathrm{d}s\Big)^2\,\mathrm{d}t\Big\}.
\end{aligned}
\tag{3.24}
$$

So from (2.5) and (3.24) we conclude that

$$
\begin{aligned}
&E\Big[\int_0^T |Y(t)|^2\,\mathrm{d}t + \int_0^T\int_0^T |Z(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t\Big]\\
&\le C_FE\Big\{\int_0^T |F(t,0)|^2\,\mathrm{d}t + \int_0^T\Big(\int_t^T |g(t,s,0,0,0,0)|\,\mathrm{d}s\Big)^2\,\mathrm{d}t\Big\},
\end{aligned}
$$

where $C_F = \frac{2C_L}{1-2C_LD}$. Similar to Step 1 of Theorem 3.3, the stability estimate holds. The proof is complete. $\square$

3.3. **Existence and uniqueness of M-solution for** (1.1)**.**

Proof. Let $(Y^{(0)}(t), Z^{(0)}(t,s)) = (0,0)$, then for $n = 1,2,\cdots$ we can have the Picard iterations:

$$\begin{aligned} Y^n(t) = F(t, Y^{n-1}(t)) - \int_t^T &\Big[h(t,s,Y^{(n-1)}(s), Y^{(n-1)}(t), Z^{(n-1)}(t,s)) \\ &- Z^{(n-1)}(t,s)\Big]\,\mathrm{d}W(s) \\ &+ \int_t^T g(t,s,Y^n(s), Y^n(t), Z^n(t,s), Z^n(s,t))\,\mathrm{d}s \\ &- \int_t^T Z^n(t,s)\,\mathrm{d}W(s), \quad t \in [0,T]. \end{aligned} \tag{3.25}$$

By (3.22) and Itô's isometry, we have

$$\begin{aligned} &E\Big[\int_0^T |Y^{(n)}(t) - Y^{(n-1)}(t)|^2\,\mathrm{d}t + \int_0^T\int_0^T |Z^{(n)}(t,s) - Z^{(n-1)}(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t\Big] \\ &\le C_F E\Big\{\int_0^T\int_t^T |h(t,s,Y^{(n-1)}(s), Y^{(n-1)}(t), Z^{(n-1)}(t,s)) - Z^{(n-1)}(t,s) \\ &\quad - [h(t,s,Y^{(n-2)}(s), Y^{(n-2)}(t), Z^{(n-2)}(t,s)) - Z^{(n-2)}(t,s)]|^2\,\mathrm{d}s\,\mathrm{d}t\Big\} \\ &\le C_F L_\xi E\int_0^T\int_0^T |Z^{(n-1)}(t,s) - Z^{(n-2)}(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t \\ &\quad + C_F(L_\zeta + L_\eta)TE\int_0^T (Y^{(n-1)}(s) - Y^{(n-2)}(s))\,\mathrm{d}s. \end{aligned} \tag{3.26}$$

By (2.6), we see that $\max\{C_F L_\zeta, C_F(L_\zeta + L_\eta)T\} = M < 1$, and then

$$\begin{aligned} &E\Big[\int_0^T |Y^{(n)}(t) - Y^{(n-1)}(t)|^2\,\mathrm{d}t + \int_0^T\int_0^T |Z^{(n)}(t,s) - Z^{(n-1)}(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t\Big] \\ &\le ME\Big[\int_0^T |Y^{(n-1)}(t) - Y^{(n-2)}(t)|^2\,\mathrm{d}t \\ &\quad + \int_0^T\int_0^T |Z^{(n-1)}(t,s) - Z^{(n-2)}(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t\Big] \le \cdots \\ &\le M^{n-1}E\Big[\int_0^T |Y^{(1)}(t) - Y^{(0)}(t)|^2\,\mathrm{d}t + \int_0^T\int_0^T |Z^{(1)}(t,s) - Z^{(0)}(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t\Big] \end{aligned}$$

holds. So we obtain $(Y^{(n)}(\cdot), Z^{(n)}(\cdot,\cdot))$ is a Cauchy sequence on $\mathcal{M}^2[0,T]$. Let $n \to \infty$ in (3.25) we obtain the unique adapted M-solution of (1.1). From (3.21), we have

$$\begin{aligned} &E\Big[\int_0^T |Y(t)|^2\,\mathrm{d}t + \int_0^T\int_0^T |Z(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t\Big] \\ &\le C_F E\Big\{\int_0^T |F(t,0) - \int_t^T \Big[h(t,s,0,0,Z(t,s)) - Z(t,s)\Big]\,\mathrm{d}W(s)|^2\,\mathrm{d}t \\ &\quad + \int_0^T \Big(\int_t^T |g(t,s,0,0,0,0)|\,\mathrm{d}s\Big)^2\,\mathrm{d}t\Big\} \end{aligned}$$

$$
\begin{aligned}
&\le C_F E\Big\{\int_0^T |F(t,0)|^2\,\mathrm{d}t + 4L_\xi \int_0^T\int_0^T |Z(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + 4\int_0^T\int_t^T |h(t,s,0,0,0)|^2\,\mathrm{d}s\,\mathrm{d}t + \int_0^T \Big(\int_t^T |g(t,s,0,0,0,0)|\,\mathrm{d}s\Big)^2\,\mathrm{d}t\Big\}.
\end{aligned}
$$

So

$$
\begin{aligned}
&E\Big[\int_0^T |Y(t)|^2\,\mathrm{d}t + \int_0^T\int_0^T |Z(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t\Big] \\
&\le C_h E\Big\{\int_0^T |F(t,0)|^2\,\mathrm{d}t + \int_0^T \Big(\int_t^T |g(t,s,0,0,0,0)|\,\mathrm{d}s\Big)^2\,\mathrm{d}t \\
&\quad + \int_0^T\int_t^T |h(t,s,0,0,0)|^2\,\mathrm{d}s\,\mathrm{d}t\Big\}
\end{aligned}
\tag{3.27}
$$

where $C_h = \frac{4C_F}{1-4L_\xi C_F}$. Similar to Step 1 of Theorem 3.3, applying the Hadamard formula it is not difficult to see that (2.8) holds. □

Acknowledgements. This research was supported by the NNSF of China (Nos. 11301112, 11171081 and 11171056), by the NNSF of Shandong Province (No. ZR2013AQ003), by China Postdoctoral Science Foundation funded project (Nos. 2013M541352, 2014T70313), by HIT.IBRSEM.A.2014014 and by the Key Project of Science and Technology of Weihai (No.2013DXGJ04).

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