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WEAK INVERSE PROBLEMS FOR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS CONTAINING TWO KERNELS

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Abstract. An inverse problem to determine a coefficient and two kernels in a parabolic integro-differential equation is considered. A corresponding direct problem is supposed to be in the weak form. Existence of the quasi-solution is proved and issues related to Fréchet differentiation of the cost functional are treated.

1. Introduction

Inverse problems to determine coefficients and kernels in integro-differential heat equations are well-studied in the smooth case when the medium is continuous and corresponding direct problems hold in the classical sense (selection of references: [2, 4, 5, 9, 10, 12, 13, 15, 16, 17, 19]). For instance, in [10] problems to determine space-dependent coefficients by means of final over-determination of the solution of the direct problem are dealt with. This paper exploits and generalizes methods developed earlier in the usual parabolic case [3, 7].

Results are known for particular non-smooth cases, as well. For instance, identification problems for parabolic transmission problems are considered in [11] under additional smoothness assumptions in neighbourhoods of observation areas. Several papers deal with degenerate cases (see [8] and references therein). In [14] problems to reconstruct free terms and coefficients in a weak parabolic problem containing a single kernel (heat flux relaxation kernel) are considered. In particular, a new method that enables to deduce formulas for Fréchet derivatives for cost functionals of inverse problems is proposed.

In the present article we consider the inverse problem of determining two kernels and a coefficient in a parabolic integro-differential equation. The corresponding direct problem is posed in the weak form. We prove the Fréchet differentiability of the cost functional related to the inverse problem and deduce a suitable form for the Fréchet derivative in terms of an adjoint problem. In this connection we use an integrated convolutional form of the weak direct problem that enables to use test functions without classical time derivatives. Finally, we prove the existence of the quasi-solution of the inverse problem under certain restrictions.

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Inverse problems for smooth models with two kernels were formerly considered in [5, 12, 19].

2. Formal statement of problems

Let Ω be a n-dimensional domain, where $n \geq 1$ and $\Gamma = \partial\Omega$. Further, let $\Gamma = \Gamma_1 \cup \Gamma_2$ with $\operatorname{meas}\Gamma_1 \cap \Gamma_2 = 0$, $\operatorname{meas}\Gamma_2 > 0$ and either $\Gamma_1 = \emptyset$ or $\operatorname{meas}\Gamma_1 > 0$. In case $n \geq 2$ we assume Γ to be sufficiently smooth. Define

$$\Omega_t = \Omega \times (0,t), \quad \Gamma_{1,t} = \Gamma_1 \times (0,t), \quad \Gamma_{2,t} = \Gamma_2 \times (0,t)$$

for $t \geq 0$.

Let $T > 0$. We pose the formal direct problem: find $u(x,t) : \Omega_T \to \mathbb{R}$ such that

$$u_t + (\mu * u)_t = Au - m * Au + f + \nabla \cdot \phi + \varphi_t \quad \text{in } \Omega_T, \tag{2.1}$$

$$u = u_0 \quad \text{in } \Omega \times \{0\}, \tag{2.2}$$

$$u = g \quad \text{in } \Gamma_{1,T}, \tag{2.3}$$

$$-\nu_A \cdot \nabla u + m * \nu_A \cdot \nabla u = \vartheta u + h + \nu \cdot \phi \quad \text{in } \Gamma_{2,T}, \tag{2.4}$$

where

$$Av = \sum_{i,j=1}^{n} \big(a_{ij} v_{x_j}\big)_{x_i} + av, \quad \nu_A = \Big(\sum_{j=1}^{n} a_{ij}\nu_j\big|_{i=1,\dots,n}\Big),$$

$\nu = (\nu_1, \dots, \nu_n)$ is the outer normal of Γ_2, $a_{ij}, a, u_0 : \Omega \to \mathbb{R}$, $f, \varphi : \Omega_T \to \mathbb{R}$, $\phi : \Omega_T \to \mathbb{R}^n$, $g : \Omega_T \to \mathbb{R}$, $\vartheta : \Gamma_2 \to \mathbb{R}$, $h : \Gamma_{2,T} \to \mathbb{R}$, $\mu, m : (0,T) \to \mathbb{R}$ are given functions and

$$z * w(t) = \int_0^t z(t-\tau) w(\tau) d\tau$$

is the convolution with respect to the variable t. In the case $\Gamma_1 = \emptyset$, the boundary condition (2.3) is omitted. The second and third addend of the free term of the equation (2.1); i.e., $\nabla \cdot \phi$ and φ_t may be singular distributions.

The problem (2.1)–(2.4) governs the heat conduction in the body Ω filled with material with memory, where μ and m are the relaxation kernels of the internal energy and the heat flux, respectively and u is the temperature [1, 4, 5, 18]. Then the condition (2.4) corresponds to the third kind boundary condition, namely it contains the heat flux to the co-normal direction $-\nu_A \cdot \nabla u + m * \nu_A \cdot \nabla u$.

Let us formulate the inverse problem:

IP. Find a, m and μ such that the solution of (2.1)–(2.4) satisfies the following final and integral additional conditions:

$$u = u_T \quad \text{in } \Omega \times \{T\}, \tag{2.5}$$

$$\int_{\Gamma_2} \kappa_j(x,\cdot) u(x,\cdot) d\Gamma = v_j \quad \text{in } (0,T),\ j = 1,2, \tag{2.6}$$

where $u_T : \Omega \to \mathbb{R}$, $\kappa_j : \Gamma_{2,T} \to \mathbb{R}$ and $v_j : (0,T) \to \mathbb{R}$ are prescribed functions.

Remark 2.1. In the case $n = 1$ and $\Omega = (c,d)$, the integral $\int_{\Gamma_2} z(x) d\Gamma$ is merely the sum $\sum_{l=1}^{L} z(x_l)$, where $x_l \in \Gamma_2 \subseteq \{c; d\}$ and L is the number of points in Γ_2 (i.e $L \in \{1; 2\}$). Then the conditions (2.6) read

$$\sum_{l=1}^{L} \kappa_j(x_l,\cdot) u(x_l,\cdot) = v_j \quad \text{in } (0,T), \quad j = 1,2. \tag{2.7}$$

3. Results concerning direct problem

Let us start by a rigorous mathematical formulation of the direct problem. Define the following functional spaces:

$$\begin{gathered}
\mathcal{U}(\Omega_t) = C([0,t];L^2(\Omega)) \cap L^2((0,t);W_2^1(\Omega)),\\
\mathcal{U}_0(\Omega_t) = \Big\{\eta \in \mathcal{U}(\Omega_t) : \eta|_{\Gamma_{1,t}} = 0 \ \text{ in case } \Gamma_1 \neq \emptyset\Big\},\\
\mathcal{T}(\Omega_t) = \big\{\eta \in L^2((0,t);W_2^1(\Omega)) : \eta_t \in L^2((0,t);L^2(\Omega))\big\},\\
\mathcal{T}_0(\Omega_t) = \Big\{\eta \in \mathcal{T}(\Omega_t) : \eta|_{\Gamma_{1,t}} = 0 \text{ in case } \Gamma_1 \neq \emptyset\Big\}
\end{gathered}$$

and introduce the following basic assumptions on the data of the direct problem:

$$a_{ij} \in L^\infty(\Omega), \quad a_{ij} = a_{ji}, \quad \vartheta \in C(\overline{\Gamma}_2), \quad \vartheta \geq 0, \tag{3.1}$$

$$\sum_{i,j=1}^n a_{ij}(x)\lambda_i\lambda_i \geq \epsilon|\lambda|^2, \quad x \in \Omega,\ \lambda \in \mathbb{R}^n \text{ with some } \epsilon > 0, \tag{3.2}$$

$$a \in L^{q_1}(\Omega), \quad \text{where } q_1 = 1 \text{ if } n = 1,\ q_1 > \frac{n}{2} \text{ if } n \geq 2, \tag{3.3}$$

$$\mu \in L^2(0,T), \quad m \in L^1(0,T)\,, \tag{3.4}$$

$$u_0 \in L^2(\Omega), \quad g \in \mathcal{T}(\Omega_T), \quad h \in L^2(\Gamma_{2,T}), \tag{3.5}$$

$$\begin{gathered}
f \in L^2((0,T);L^{q_2}(\Omega)),\ \text{where } q_2 = 1 \text{ if } n = 1,\\
q_2 \in (1,q_1) \text{ if } n = 2,\ \ q_2 = \frac{2n}{n+2} \text{ if } n \geq 3,
\end{gathered} \tag{3.6}$$

$$\phi = (\phi_1,\dots,\phi_n) \in (L^2(\Omega_T))^n\,, \tag{3.7}$$

$$\varphi \in \mathcal{U}(\Omega_T),\ \text{in case } \Gamma_1 \neq \emptyset \tag{3.8}$$

$$;\exists g_\varphi \in \mathcal{T}(\Omega_T) : \varphi = g_\varphi \text{ in } \Gamma_{1,T}. \tag{3.9}$$

If we assume additional conditions $a_{ij} \in W_2^1(\Omega)$, $\frac{\partial}{\partial x_i}\phi_i \in L^2(\Omega_T)$, $i = 1,\dots,n$, $\varphi_t \in L^2(\Omega_T)$ and suppose that (2.1)–(2.4) has a classical solution $u \in L^2(\Omega_T)$ such that $u_t, u_{x_i}, u_{x_ix_j} \in L^2(\Omega_T)$, $i,j = 1,\dots,n$, then multiplying (2.1) with a test function $\eta \in \mathcal{T}_0(\Omega_T)$ and integrating by parts we come to the relation

$$\begin{aligned}
0 &= \int_\Omega [(u+\mu*u)(x,T)\eta(x,T) - u_0(x)\eta(x,0)]\,dx - \iint_{\Omega_T}(u+\mu*u)\eta_t\,dx\,dt\\
&\quad + \iint_{\Omega_T}\Big[\sum_{i,j=1}^n a_{ij}(u_{x_j} - m*u_{x_j})\eta_{x_i} - a(u - m*u)\eta\Big]\,dx\,dt\\
&\quad + \iint_{\Gamma_{2,T}}(\vartheta u + h)\eta\,d\Gamma dt - \iint_{\Omega_T}(f\eta - \phi\cdot\nabla\eta)\,dx\,dt\\
&\quad - \int_\Omega[\varphi(x,T)\eta(x,T) - \varphi(x,0)\eta(x,0)]dx + \iint_{\Omega_T}\varphi\eta_t\,dx\,dt.
\end{aligned} \tag{3.10}$$

This relation makes sense also in a more general case when a_{ij}, ϕ, φ satisfy (3.1), (3.7), (3.8) and $u \in \mathcal{U}(\Omega_T)$.

We call a *weak solution* of the problem (2.1)–(2.4) a function belonging to $\mathcal{U}(\Omega_T)$ that satisfies the relation (3.10) for any $\eta \in \mathcal{T}_0(\Omega_T)$ and, in case $\Gamma_1 \neq \emptyset$, that fulfills the boundary condition (2.3).

Theorem 3.1. *Problem* (2.1)–(2.4) *has a unique weak solution. This solution satisfies the estimate*

$$\begin{aligned} \|u\|_{\mathcal{U}(\Omega_T)} \le C_0 \Big[&\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T);L^{q_2}(\Omega))} + \|\phi\|_{(L^2(\Omega_T))^n} \\ &+ \|\varphi\|_{\mathcal{U}(\Omega_T)} + \theta\{\|g\|_{\mathcal{T}(\Omega_T)} + \|g_\varphi\|_{\mathcal{T}(\Omega_T)}\} + \|h\|_{L^2(\Gamma_{2,T})} \Big], \end{aligned} \tag{3.11}$$

where $\theta = 0$ in case $\Gamma_1 = \emptyset$ and C_0 is a constant independent of $u_0, f, \phi, \varphi, g, h$.

Proof. Since $\mu \in L^2(0,T)$, the Volterra equation of the second kind

$$\widehat{\mu} + \mu * \widehat{\mu} = -\mu \quad \text{in } (0,T). \tag{3.12}$$

has a unique solution $\widehat{\mu} \in L^2(0,T)$ [6]. We call $\widehat{\mu}$ the resolvent kernel of μ. Further, let us consider the following problem:

$$\widehat{u}_t = A\widehat{u} - \widehat{m} * A\widehat{u} + \widehat{f} + \nabla \cdot \widehat{\phi} \quad \text{in } \Omega_T, \tag{3.13}$$

$$\widehat{u} = \widehat{u}_0 \quad \text{in } \Omega \times \{0\}, \tag{3.14}$$

$$\widehat{u} = \widehat{g} \quad \text{in } \Gamma_{1,T}, \tag{3.15}$$

$$-\nu_A \cdot \nabla \widehat{u} + \widehat{m} * \nu_A \cdot \nabla \widehat{u} = \vartheta \widehat{u} + \vartheta \widehat{\mu} * \widehat{u} + \widehat{h} + \nu \cdot \widehat{\phi} \quad \text{in } \Gamma_{2,T}, \tag{3.16}$$

where

$$\widehat{m} = m - \widehat{\mu} + m * \widehat{\mu}, \quad \widehat{f} = f + a\varphi - \widehat{m} * a\varphi,$$
$$\widehat{\phi}_i = \phi_i + \sum_{j=1}^n a_{ij}\varphi_{x_j} - \widehat{m} * \sum_{j=1}^n a_{ij}\varphi_{x_j},$$
$$\widehat{h} = h + \vartheta\varphi + \vartheta\widehat{\mu} * \varphi, \quad \widehat{g} = g + \mu * g - g_\varphi, \quad \widehat{u}_0 = u_0 - \varphi(\cdot, 0).$$

By the properties of m and $\widehat{\mu}$ we have $\widehat{m} \in L^1(0,T)$. Further, [14, Lemma 1] yields

$$\begin{gathered} \mathcal{U}(\Omega_T) \hookrightarrow L^2((0,T); L^{q_3}(\Omega)), \quad \text{where } q_3 = \infty \text{ if } n = 1, \\ q_3 > \frac{q_1 q_2}{q_1 - q_2} \text{ if } n = 2,\ q_3 = \frac{2n}{n-2} \text{ if } n \ge 3 \end{gathered} \tag{3.17}$$

and

$$\begin{gathered} av \in L^2((0,T); L^{q_2}(\Omega)) \text{ if } a \in L^{q_1}(\Omega),\ v \in L^2((0,T); L^{q_3}(\Omega)), \\ \|av\|_{L^2((0,T);L^{q_2}(\Omega))} \le C\|a\|_{L^{q_1}(\Omega)} \|v\|_{L^2((0,T);L^{q_3}(\Omega))}, \end{gathered} \tag{3.18}$$

where C is a constant. Using the relations (3.17), (3.18), the properties of $\widehat{m}$, $\widehat{\mu}$, the assumptions (3.1)–(3.8), trace theorems and the Young theorem for convolutions we obtain

$$\begin{gathered} \widehat{d} := (\widehat{f}, \widehat{\phi}, \widehat{u}_0, \widehat{g}, \widehat{h}) \in \mathcal{X} \\ := L^2((0,T); L^{q_2}(\Omega)) \times (L^2(\Omega_T))^n \times L^2(\Omega) \times \mathcal{T}(\Omega_T) \times L^2(\Gamma_{2,T}), \\ \|\widehat{d}\|_{\mathcal{X}} \le \bar{C}\|d\|_{\bar{\mathcal{X}}} \end{gathered} \tag{3.19}$$

where $d = (f, \phi, u_0, g, h, \varphi, g_\varphi)$ and

$$\bar{\mathcal{X}} = L^2((0,T); L^{q_2}(\Omega)) \times (L^2(\Omega_T))^n \times L^2(\Omega) \times \mathcal{T}(\Omega_T) \times L^2(\Gamma_{2,T}) \times \mathcal{U}(\Omega_T) \times \mathcal{T}(\Omega_T)$$

and $\bar{C}$ is a constant. It was proved in [14, Theorem 1] that problem (2.1)–(2.4) in case $\mu = 0$ and $\varphi = 0$ has for any $(f, \phi, u_0, g, h) \in \mathcal{X}$ a unique weak solution and the corresponding solution operator $\mathcal{B}$ belongs to $\mathcal{L}(\mathcal{X}; \mathcal{U}(\Omega_T))$. (Here $\mathcal{L}(X, Y)$ stands for the space of linear bounded operators from a Banach space X to a Banach space

Y.) This implies that problem (3.13)–(3.16) is equivalent in $\mathcal{U}(\Omega_T)$ to the following operator equation:

$$\widehat{u} = \mathcal{Q}\widehat{u} \quad \text{with} \quad \mathcal{Q}\widehat{u} = \mathcal{B}(0,0,0,0,\vartheta\widehat{\mu} * \widehat{u}) + \mathcal{B}\widehat{d}. \tag{3.20}$$

To study this equation, we will use the inequality

$$\|\widehat{\mu} * y\|_{L^2(\Omega_t)} \le \int_0^t |\widehat{\mu}(t-\tau)|\, \|y\|_{L^2(\Omega_\tau)} d\tau\,, \quad t \in [0,T] \tag{3.21}$$

that holds for any $y \in L^2(\Omega_T)$. This was proved in [14, inequality (3.12)].

Let $\widehat{u}^1, \widehat{u}^2 \in \mathcal{U}(\Omega_T)$, denote $v = \widehat{u}^1 - \widehat{u}^2$ and estimate $\mathcal{Q}\widehat{u}^1 - \mathcal{Q}\widehat{u}^2 = \mathcal{B}(0,0,0,0,\vartheta\widehat{\mu} * v)$. To this end, fix $t \in [0,T]$ and define

$$P_t w = \begin{cases} w & \text{in } \Gamma_{2,t} \\ 0 & \text{in } \Gamma_{2,T} \setminus \Gamma_{2,t} \end{cases} \tag{3.22}$$

for $w : \Gamma_{2,T} \to \mathbb{R}$. Due to the causality, we have $\mathcal{B}(0,0,0,P_t\vartheta\widehat{\mu} * v)(x,\tau) = \mathcal{B}(0,0,0,\vartheta\widehat{\mu} * v)(x,\tau)$ for any $(x,\tau) \in \Omega_t$. Since $\mathcal{B} \in \mathcal{L}(\mathcal{X};\mathcal{U}(\Omega_T))$, the continuity of ϑ, the trace theorem and the inequality (3.21) with $y = v, v_{x_i}$, $i = 1,\dots,n$, it follows that

$$\begin{aligned} \|\mathcal{Q}\widehat{u}^1 - \mathcal{Q}\widehat{u}^2\|_{\mathcal{U}(\Omega_t)} &= \|\mathcal{B}(0,0,0,0,\vartheta\widehat{\mu} * v)\|_{\mathcal{U}(\Omega_t)} \\ &= \|\mathcal{B}(0,0,0,0,P_t\vartheta\widehat{\mu} * v)\|_{\mathcal{U}(\Omega_t)} \\ &\le \|\mathcal{B}(0,0,0,0,P_t\vartheta\widehat{\mu} * v)\|_{\mathcal{U}(\Omega_T)} \\ &\le \|\mathcal{B}\|\|P_t\vartheta\widehat{\mu} * v\|_{L^2(\Gamma_{2,T})} = \|\mathcal{B}\|\|\vartheta\widehat{\mu} * v\|_{L^2(\Gamma_{2,t})} \\ &\le C_1\|\widehat{\mu} * v\|_{L^2((0,t);W_2^1(\Omega))} \\ &\le C_2\int_0^t |\widehat{\mu}(t-\tau)|\, \|v\|_{L^2((0,\tau);W_2^1(\Omega))} d\tau \end{aligned} \tag{3.23}$$

with some constants C_1 and C_2. Let us define the weighted norm in $\mathcal{U}(\Omega_T)$: $\|v\|_\sigma = \sup_{0<t<T} e^{-\sigma t}\|v\|_{\mathcal{U}(\Omega_t)}$ where $\sigma \ge 0$. In view of (3.23) and $\mathcal{U}(\Omega_t) \hookrightarrow L^2((0,t);W_2^1(\Omega))$ we get

$$\begin{aligned} \|\mathcal{Q}\widehat{u}^1 - \mathcal{Q}\widehat{u}^2\|_\sigma &\le C_3 \sup_{0<t<T} e^{-\sigma t}\int_0^t |\widehat{\mu}(t-\tau)|\, \|v\|_{\mathcal{U}(\Omega_\tau)} d\tau \\ &= C_3 \sup_{0<t<T} \int_0^t e^{-\sigma(t-\tau)}|\widehat{\mu}(t-\tau)|\, e^{-\sigma\tau}\, \|v\|_{\mathcal{U}(\Omega_\tau)} d\tau \\ &\le C_3 \int_0^T e^{-\sigma s}|\widehat{\mu}(s)|ds \sup_{0<\tau<T} e^{-\sigma\tau}\, \|v\|_{\mathcal{U}(\Omega_\tau)} \\ &= C_2\int_0^T e^{-\sigma s}|\widehat{\mu}(s)|ds\, \|v\|_\sigma \end{aligned}$$

with some constant C_3. By the dominated convergence theorem, $\int_0^T e^{-\sigma s}|\widehat{\mu}(s)|ds \to 0$ as $\sigma \to \infty$. Thus, there exists σ_0 such that

$$C_3\int_0^T e^{-\sigma_0 s}|\widehat{\mu}(s)|ds \le \frac{1}{2}.$$

Therefore, $\|\mathcal{Q}\widehat{u}^1 - \mathcal{Q}\widehat{u}^2\|_{\sigma_0} \le \frac{1}{2}\|\widehat{u}^1 - \widehat{u}^2\|_{\sigma_0}$. The operator $\mathcal{Q}$ is a contraction in $\mathcal{U}(\Omega_T)$. This implies that (3.13)–(3.16) has a unique weak solution in $\mathcal{U}(\Omega_T)$.

Moreover, observing (3.20) and the relation $\mathcal{Q}0 = \mathcal{B}\widehat{d}$, for the solution of (3.13)–(3.16) we obtain the estimate

$$\|\widehat{u}\|_{\sigma_0} = \|\mathcal{Q}\widehat{u} - \mathcal{Q}0 + \mathcal{Q}0\|_{\sigma_0} \le \|\mathcal{Q}\widehat{u} - \mathcal{Q}0\|_{\sigma_0} + \|\mathcal{B}\widehat{d}\|_{\sigma_0} \le \frac{1}{2}\|\widehat{u}\|_{\sigma_0} + \|\mathcal{B}\widehat{d}\|_{\sigma_0}$$

which implies

$$\|\widehat{u}\|_{\sigma_0} \le 2\|\mathcal{B}\widehat{d}\|_{\sigma_0} \le 2\|\mathcal{B}\widehat{d}\|_{\mathcal{U}(\Omega_T)} \le 2\|\mathcal{B}\|\|\widehat{d}\|_{\mathcal{X}}.$$

Observing the relation $e^{-\sigma_0 T}\|\widehat{u}\|_{\mathcal{U}(\Omega_T)} \le \|\widehat{u}\|_{\sigma_0}$ and (3.19) we arrive at the estimate

$$\|\widehat{u}\|_{\mathcal{U}(\Omega_T)} \le C_4\|d\|_{\bar{\mathcal{X}}} \tag{3.24}$$

with a constant C_4.

Further, let us define

$$u = \widehat{u} + \varphi + \widehat{\mu} * (\widehat{u} + \varphi). \tag{3.25}$$

Then $\widehat{u}$ is expressed in terms of u as

$$\widehat{u} = u + \mu * u - \varphi. \tag{3.26}$$

One can immediately check that the implications $u \in \mathcal{U}(\Omega_T) \Leftrightarrow \widehat{u} \in \mathcal{U}(\Omega_T)$ are valid. Moreover, it is easy to see that $\widehat{u}$ is a weak solution of (3.13)–(3.16) if and only if u is weak solution of (2.1)–(2.4). In view of the above-presented arguments we can conclude that (2.1)–(2.4) has a unique weak solution. From (3.25) we obtain

$$\|u\|_{\mathcal{U}(\Omega_T)} \le C_4(\|\widehat{u}\|_{\mathcal{U}(\Omega_T)} + \|\varphi\|_{\mathcal{U}(\Omega_T)})$$

with a constant C_5. This with (3.24) implies (3.11). The proof is complete. □

It is possible to give an equivalent form to the relation (3.10) that does not contain the derivative of the test function with respect to t. Namely, the following theorem holds.

Theorem 3.2. *The function $u \in \mathcal{U}(\Omega_T)$ satisfies the relation* (3.10) *for any $\eta \in \mathcal{T}_0(\Omega_T)$ if and only if it satisfies the relation*

$$\begin{aligned} 0 &= \int_\Omega (u + \mu * u - \varphi) * \eta \, dx - \int_\Omega \int_0^t (u_0(x) - \varphi(x,0))\eta(x,\tau)\, d\tau\, dx \\ &\quad + \int_\Omega 1 * \Big[\sum_{i,j=1}^n a_{ij}(u_{x_j} - m * u_{x_j}) * \eta_{x_i} - a(u - m * u) * \eta\Big] dx \\ &\quad + \int_{\Gamma_2} 1 * (\vartheta u + h) * \eta \, d\Gamma - \int_\Omega 1 * \Big(f * \eta - \sum_{i=1}^n \phi_i * \eta_{x_i}\Big)\, dx, \end{aligned} \tag{3.27}$$

for any $t \in [0,T]$ and $\eta \in \mathcal{U}_0(\Omega_T)$.

Proof. It is analogous to the proof of [14, Theorem 2] that considers the case $\varphi = 0$, $\mu = 0$. We have only to replace u by $\widehat{u} = u + \mu * u - \varphi$ in the term $K_1(t)$ appearing in formulas [14, (3.19), (3.20)] to get the desired result. □

Remark 3.3. Theorems 3.1 and 3.2 remain valid also in the case $\Gamma_2 = \emptyset$. In this case the terms $\|h\|_{\Gamma_{2,T}}$ and $\int_{\Gamma_2} 1 * (\vartheta u + h) * \eta \, d\Gamma$ are missing in (3.11) and (3.27), respectively.

4. Quasi-solution of IP. Fréchet derivative of cost functional

Assume that $n \in \{1;2;3\}$. Moreover, let us set $q_1 = 2$ if $n = 2$. Then any coefficient a that belongs to $L^2(\Omega)$ satisfies (3.3). For the weight functions κ_j we assume that

$$\kappa_j \in L^\infty((0,T);L^2(\Gamma_2)), \quad j = 1,2. \tag{4.1}$$

In the case $n = 1$ this assumption is simply $\kappa(x_l,\cdot) \in L^\infty(0,T)$, $x_l \in \Gamma_2 \subseteq \{c;d\}$. According to Theorem 3.1, $u \in \mathcal{U}(\Omega_T)$, thus $u(\cdot,T) \in L^2(\Omega)$, and the condition (2.5) is well-defined for $u_T \in L^2(\Omega)$. Moreover, by a trace theorem we have $u \in L^2(\Gamma_{2,T})$. This implies that $\int_{\Gamma_2} \kappa_j(x,\cdot)u(x,\cdot)d\Gamma \in L^2(0,T)$, $j = 1,2$, hence the condition (2.6) is well-defined for $v_j \in L^2(0,T)$, $j = 1,2$.

Let $M \subseteq \mathcal{Z} := L^2(\Omega) \times (L^2(0,T))^2$. We call the *quasi-solution* of IP in the set M an element $z^* \in \arg\min_{z\in M} J(z)$, where J is the cost functional

$$J(z) = \|u(\cdot,T;z) - u_T\|^2_{L^2(\Omega)} + \sum_{j=1}^{2} \| \int_{\Gamma_2} \kappa_j(x,\cdot)u(x,\cdot;z)d\Gamma - v_j\|^2_{L^2(0,T)}$$

and $u(x,t;z)$ is the weak solution of the direct problem (2.1)–(2.4) corresponding to given $z = (a,m,\mu)$. In case $n = 1$ the integral $\int_{\Gamma_2} \kappa_j(x,t)u(x,t;z)d\Gamma$ in the definition of J is replaced by $\sum_{l=1}^{L} \kappa_j(x_l,t)u(x_l,t;z)$.

Theorem 4.1. *The functional J is Fréchet differentiable in $\mathcal{Z}$ and*

$$\begin{aligned} &J'(z)\Delta z \\ &= 2\int_\Omega [u(x,T;z) - u_T(x)]\,\Delta u(x,T)dx \\ &\quad + 2\sum_{j=1}^{2} \int_0^T \Big[\int_{\Gamma_2} \kappa_j(y,t)u(y,t;z)d\Gamma - v_j(t)\Big] \int_{\Gamma_2} \kappa_j(x,t)\Delta u(x,t)d\Gamma dt, \end{aligned} \tag{4.2}$$

where $\Delta z = (\Delta a, \Delta m, \Delta\mu) \in \mathcal{Z}$ and $\Delta u \in \mathcal{U}(\Omega_T)$ is the z- and Δz-dependent weak solution of the following problem:

$$\begin{aligned} \Delta u_t + (\mu * \Delta u)_t &= A\Delta u - m * A\Delta u + \Delta a[u - m * u] - \Delta m * au \\ &\quad - \nabla\cdot\big[\Delta m * \sum_{j=1}^{n} a_{ij}u_{x_j}\big] - (\Delta\mu * u)_t \quad \text{in } \Omega_T, \end{aligned} \tag{4.3}$$

$$\Delta u = 0 \quad \text{in } \Omega \times \{0\}, \tag{4.4}$$

$$\Delta u = 0 \quad \text{in } \Gamma_{1,T}, \tag{4.5}$$

$$\begin{aligned} &-\nu_A\cdot\nabla\Delta u + m * \nu_A\cdot\nabla\Delta u \\ &= \vartheta\Delta u - \nu\cdot\Big[\Delta m * \sum_{j=1}^{n} a_{ij}u_{x_j}\Big] \quad \text{in } \Gamma_{2,T}. \end{aligned} \tag{4.6}$$

Proof. Denote $\widetilde{\Delta}u = u(x,t;z+\Delta z) - u(x,t;z)$ and define $\widehat{\Delta}u = \widetilde{\Delta}u - \Delta u$. Then we can represent the difference of J as follows:

$$J(z+\Delta z) - J(z) = \mathrm{RHS} + \Theta, \tag{4.7}$$

where RHS is the right-hand side of the equality (4.2) and

$$\Theta = 2\int_\Omega [u(x,T) - u_T(x)]\widehat{\Delta}u(x,T)dx$$

$$
\begin{aligned}
&+ 2\sum_{j=1}^{2}\int_0^T \Big[\int_{\Gamma_2}\kappa_j(y,t)u(y,t)d\Gamma - v_j(t)\Big]\int_{\Gamma_2}\kappa_j(x,t)\widehat{\Delta}u(x,t)d\Gamma dt\\
&+ \int_\Omega \Big\{(\Delta u+\widehat{\Delta}u)(x,T)\Big\}^2 dx\\
&+ \sum_{j=1}^{2}\int_0^T \Big\{\int_{\Gamma_2}\kappa_j(x,t)(\Delta u+\widehat{\Delta}u)(x,t)d\Gamma\Big\}^2 dt.
\end{aligned}
$$

Let us study problem (4.3)–(4.6). To this end we estimate the terms in the right-hand side of (4.3). Observing the relations $u \in \mathcal{U}(\Omega_T)$, (3.17), (3.18), $L^2(\Omega) \hookrightarrow L^{q_1}(\Omega)$ and using the Young and Cauchy inequalities we deduce

$$
\begin{aligned}
&\|\Delta a[u - m * u] - \Delta m * au\|_{L^2((0,T);L^{q_2}(\Omega))}\\
&\le c_1\|u\|_{\mathcal{U}(\Omega_T)}\big[(1+\|m\|_{L^2(0,T)})\|\Delta a\|_{L^2(\Omega)} + \|a\|_{L^2(\Omega)}\|\Delta m\|_{L^2(0,T)}\big]\\
&\le c_2(z,u)\|\Delta z\|,
\end{aligned}
\tag{4.8}
$$

where c_1 is a constant, c_2 is a coefficient depending on $z = (a, m, \mu)$, u and $\|\cdot\|$ denotes the norm in $\mathcal{Z}$. Taking the boundendness of a_{ij} into account we similarly get

$$
\|\Delta m * \sum_{j=1}^{n} a_{ij}u_{x_j}\|_{(L^2(\Omega_T))^n} \le c_3\|u\|_{\mathcal{U}(\Omega_T)}\|\Delta m\|_{L^2(0,T)} \tag{4.9}
$$

with a constant c_3. Next let us estimate the term $\Delta\mu * u$ at the right-hand side of (4.3). Since $u \in C([0,T];L^2(\Omega))$ and $\Delta\mu \in L^2(0,T)$, it is easy to check that $\Delta\mu * u \in C([0,T];L^2(\Omega))$ and $\|\Delta\mu * u\|_{C([0,T];L^2(\Omega))} \le T^{1/2}\|u\|_{C([0,T];L^2(\Omega))}\|\Delta\mu\|_{L^2(0,T)}$. Similarly, $\|\Delta\mu * u\|_{L^2((0,T);W_2^1(\Omega))} \le T^{1/2}\|u\|_{L^2((0,T);W_2^1(\Omega))}\|\Delta\mu\|_{L^2(0,T)}$. Taking these estimates together, we have

$$
\|\Delta\mu * u\|_{\mathcal{U}(\Omega_T)} \le T^{1/2}\|u\|_{\mathcal{U}(\Omega_T)}\|\Delta\mu\|_{L^2(0,T)}. \tag{4.10}
$$

Since $u = g$ in $\Gamma_{1,T}$, we find that

$$
\Delta\mu * u = \Delta\mu * g \quad \text{in } \Gamma_{1,T}. \tag{4.11}
$$

Using the assumption $g \in \mathcal{T}(\Omega_T)$ and the Young and Cauchy inequalities again, we obtain

$$
\begin{aligned}
\|\Delta\mu * g\|_{\mathcal{T}(\Omega_T)} &= \|\Delta\mu * g\|_{L^2((0,T);W_2^1(\Omega))} + \|(\Delta\mu * g)_t\|_{L^2((0,T);L^2(\Omega))}\\
&= \|\Delta\mu * g\|_{L^2((0,T);W_2^1(\Omega))} + \|\Delta\mu * g_t\|_{L^2((0,T);L^2(\Omega))}\\
&\quad + \|\Delta\mu\, g(\cdot,0)\|_{L^2((0,T);L^2(\Omega))}\\
&\le c_4\|\Delta\mu\|_{L^2(0,T)}
\end{aligned}
\tag{4.12}
$$

with a constant c_4. Relations (4.8)–(4.12) show that Theorem 3.1 holds for problem (4.3)–(4.6), hence it has a unique weak solution $\Delta u \in \mathcal{U}(\Omega_T)$. Using the estimate (3.11) for the solution of this problem we obtain

$$
\begin{aligned}
&\|\Delta u\|_{\mathcal{U}(\Omega_T)}\\
&\le C_0\Big[\|\Delta a[u - m * u] + \Delta m * au\|_{L^2((0,T);L^{q_2}(\Omega))}\\
&\quad + \|\Delta m * \sum_{j=1}^{n} a_{ij}u_{x_j}\|_{(L^2(\Omega_T))^n} + \|\Delta\mu * u\|_{\mathcal{U}(\Omega_T)} + \theta\|\Delta\mu * g\|_{\mathcal{T}(\Omega_T)}\Big]\\
&\quad \le c_5(z,u)\|\Delta z\|
\end{aligned}
\tag{4.13}
$$

with a coefficient c_5 depending on z, u.

The function $\widetilde{\Delta}u$ satisfies the problem

$$\widehat{\Delta}u_t + (\mu * \widehat{\Delta}u)_t = A\widehat{\Delta}u - m * A\widehat{\Delta}u + f + \widehat{f} + \nabla\cdot\phi + \nabla\cdot\widehat{\phi} + \varphi_t + \widehat{\varphi}_t \quad \text{in } \Omega_T, \tag{4.14}$$

$$\widehat{\Delta}u = 0 \quad \text{in } \Omega\times\{0\}, \tag{4.15}$$

$$\widehat{\Delta}u = 0 \quad \text{in } \Gamma_{1,T}, \tag{4.16}$$

$$-\nu_A\cdot\nabla\widehat{\Delta}u + m * \nu_A\cdot\nabla\widehat{\Delta}u = \vartheta\widehat{\Delta}u + \nu\cdot\phi + \nu\cdot\widehat{\phi} \quad \text{in } \Gamma_{2,T}, \tag{4.17}$$

where

$$\begin{gathered}
f = \Delta a\Delta u - (m+\Delta m) * \Delta a\Delta u - \Delta m * \Delta a u - \Delta m * a\Delta u,\\
\widehat{f} = \Delta a\widehat{\Delta}u - (m+\Delta m) * \Delta a\widehat{\Delta}u - \Delta m * a\widehat{\Delta}u,\\
\phi = -\Delta m * \sum_{j=1}^n a_{ij}\Delta u_{x_j}, \quad \widehat{\phi} = -\Delta m * \sum_{j=1}^n a_{ij}\widehat{\Delta}u_{x_j},\\
\varphi = -\Delta\mu * \Delta u, \quad \widehat{\varphi} = -\Delta\mu * \widehat{\Delta}u.
\end{gathered}$$

Similarly to (4.8)–(4.10) we deduce the following estimates:

$$\begin{aligned}
&\|f\|_{L^2((0,T);L^{q_2}(\Omega))}\\
&\le c_6\Big\{(1+\|m\|_{L^2(0,T)} + \|\Delta m\|_{L^2(0,T)})\|\Delta a\|_{L^2(\Omega)}\|\Delta u\|_{\mathcal{U}(\Omega_T)}\\
&\quad + \|u\|_{\mathcal{U}(\Omega_T)}\|\Delta m\|_{L^2(0,T)}\|\Delta a\|_{L^2(\Omega)} + \|a\|_{L^2(\Omega)}\|\Delta m\|_{L^2(0,T)}\|\Delta u\|_{\mathcal{U}(\Omega_T)}\Big\}\\
&\le c_7(z,u)\Big\{\big[\|\Delta z\| + \|\Delta z\|^2\big]\|\Delta u\|_{\mathcal{U}(\Omega_T)} + \|\Delta z\|^2\Big\},
\end{aligned}$$

$$\begin{gathered}
\|\widehat{f}\|_{L^2(0,T;L^{q_2}(\Omega))} \le c_8(z)\,\big[\|\Delta z\| + \|\Delta z\|^2\big]\,\|\widehat{\Delta}u\|_{\mathcal{U}(\Omega_T)},\\
\|\phi\|_{(L^2(\Omega_T))^n} \le c_9\|\Delta z\|\|\Delta u\|_{\mathcal{U}(\Omega_T)},\\
\|\widehat{\phi}\|_{(L^2(\Omega_T))^n} \le c_9\|\Delta z\|\|\widehat{\Delta}u\|_{\mathcal{U}(\Omega_T)},\\
\|\varphi\|_{\mathcal{U}(\Omega_T)} \le T^{1/2}\|\Delta z\|\|\Delta u\|_{\mathcal{U}(\Omega_T)},\\
\|\widehat{\varphi}\|_{\mathcal{U}(\Omega_T)} \le T^{1/2}\|\Delta z\|\|\widehat{\Delta}u\|_{\mathcal{U}(\Omega_T)}
\end{gathered}$$

with some coefficients $c_6,\dots,c_9$. Moreover, since $\Delta u = \widetilde{\Delta}u = 0$ in $\Gamma_{1,T}$, we have $\varphi = \widehat{\varphi} = 0$ in $\Gamma_{1,T}$. Applying the estimate (3.11) to the solution of the problem (4.14)–(4.17) we get

$$\|\widehat{\Delta}u\|_{\mathcal{U}(\Omega_T)} \le c_{10}(z,u)\Big\{\,\big[\|\Delta z\| + \|\Delta z\|^2\big]\,\Big\{\|\Delta u\|_{\mathcal{U}(\Omega_T)} + \|\widehat{\Delta}u\|_{\mathcal{U}(\Omega_T)}\Big\} + \|\Delta z\|^2\Big\}$$

with a coefficient c_{10}. Provided $\|\Delta z\|$ is sufficiently small; i.e., $\|\Delta z\| + \|\Delta z\|^2 \le \frac{1}{2c_{10}(z,u)}$, we have

$$\|\widehat{\Delta}u\|_{\mathcal{U}(\Omega_T)} \le 2c_{10}(z,u)\Big\{\,\big[\|\Delta z\| + \|\Delta z\|^2\big]\,\|\Delta u\|_{\mathcal{U}(\Omega_T)} + \|\Delta z\|^2\Big\}.$$

Due to (4.13), this yields

$$\|\widehat{\Delta}u\|_{\mathcal{U}(\Omega_T)} \le c_{11}(z,u)\,\big[\|\Delta z\|^2 + \|\Delta z\|^3\big] \tag{4.18}$$

with a coefficient c_{11}.

In view of (4.13), (4.18) and the assumption $\kappa_j \in L^\infty((0,T);L^2(\Gamma_2))$ the right-hand side of (4.2) RHS and the quantity Θ satisfy the estimates

$$st|\mathrm{RHS}| \le c_{12}(z,u)\|\Delta z\|, \quad |\Theta| \le c_{13}(z,u)\sum_{l=2}^{6}\|\Delta z\|^l, \tag{4.19}$$

st where c_{12} and c_{13} are some coefficients. Moreover, RHS is linear with respect to Δz. This with (4.7) shows that J is Fréchet differentiable in $\mathcal{Z}$ and $J'(z)\Delta z$ equals RHS. $\square$

Theorem 4.2. *Assume $g = 0$. Then the Fréchet derivative of J admits the form*

$$J'(z)\Delta z = \int_\Omega \gamma_1(x)\Delta a(x)dx + \int_0^T \gamma_2(t)\Delta m(t)dt + \int_0^T \gamma_3(t)\Delta\mu(t)dt, \tag{4.20}$$

where

$$\gamma_1(x) = [(u - m*u)*\psi](x,T), \tag{4.21}$$

$$\gamma_2(t) = -\int_\Omega \Big[au*\psi + \sum_{i,j=1}^n a_{ij}\psi_{x_i} * u_{x_j}\Big](x,T-t)dx, \tag{4.22}$$

$$\begin{aligned}
&\gamma_3(t)\\
&= -\int_\Omega \Big[au*\psi + au*\psi*[\widehat{\mu} - m - m*\widehat{\mu}]\\
&\quad + \sum_{i,j=1}^n a_{ij}\psi_{x_i}*u_{x_j} + \sum_{i,j=1}^n a_{ij}\psi_{x_i}*u_{x_j}*[\widehat{\mu} - m - m*\widehat{\mu}]\Big](x,T-t)dx\\
&\quad - \int_{\Gamma_2}[\vartheta(u + \widehat{\mu}*u)*\psi](x,T-t)d\Gamma\\
&\quad - 2\int_\Omega \{u(x,T) - u_T(x)\}[u + \widehat{\mu}*u](x,T-t)dx\\
&\quad - 2\sum_{j=1}^2\int_t^T\Big[\int_{\Gamma_2}\kappa_j(y,\tau)u(y,\tau)d\Gamma - v_j(\tau)\Big]\int_{\Gamma_2}\kappa_j(x,\tau)\big[u\\
&\quad + \widehat{\mu}*u\big](x,\tau-t)d\Gamma d\tau,
\end{aligned} \tag{4.23}$$

where $\widehat{\mu}$ is the solution of (3.12), $u(x,t) = u(x,t;z)$ and $\psi \in \mathcal{U}(\Omega_T)$ is the z-dependent weak solution of the following "adjoint" problem:

$$\Delta\psi_t + (\mu*\Delta\psi)_t = A\Delta\psi - m*A\Delta\psi \quad \text{in } \Omega_T, \tag{4.24}$$

$$\Delta\psi = 2[u(\cdot,T) - u_T] \quad \text{in } \Omega\times\{0\}, \tag{4.25}$$

$$\Delta\psi = 0 \quad \text{in } \Gamma_{1,T}, \tag{4.26}$$

$$-\nu_A\cdot\nabla\Delta\psi + m*\nu_A\cdot\nabla\Delta\psi = \vartheta\Delta\psi + h^\circ \quad \text{in } \Gamma_{2,T}, \tag{4.27}$$

where

$$\begin{aligned}
&h^\circ(x,t)\\
&= -2\sum_{j=1}^2\kappa_j(x,T-t)\Big[\int_{\Gamma_2}\kappa_j(y,T-t)u(y,T-t)d\Gamma - v_j(T-t)\Big].
\end{aligned} \tag{4.28}$$

Proof. Define $\Delta w = \Delta u + \Delta\mu * u + \widehat{\mu} * \Delta\mu * u$. Since $u, \Delta u \in \mathcal{U}(\Omega_T)$, we have $\Delta w \in \mathcal{U}(\Omega_T)$. Moreover, using (3.12) it is easy to see that $\Delta u + \mu * \Delta u + \Delta\mu * u = \Delta w + \mu * \Delta w$. Using this relation for the time derivatives in (4.3) and the equality $\Delta u = \Delta w - \Delta\mu * u - \widehat{\mu} * \Delta\mu * u$ for other terms containing Δu in (4.3)–(4.6) we see that Δw is the weak solution of the problem

$$\Delta w_t + (\mu * \Delta w)_t = A\Delta w - m * A\Delta w + f^\dagger + \nabla \cdot \phi^\dagger \quad \text{in } \Omega_T, \tag{4.29}$$

$$\Delta w = 0 \quad \text{in } \Omega \times \{0\}, \tag{4.30}$$

$$\Delta w = 0 \quad \text{in } \Gamma_{1,T}, \tag{4.31}$$

$$-\nu_A \cdot \nabla \Delta u + m * \nu_A \cdot \nabla \Delta u = \vartheta \Delta u + h^\dagger + \nu \cdot \phi^\dagger \quad \text{in } \Gamma_{2,T}, \tag{4.32}$$

where

$$f^\dagger = \Delta a[u - m * u] - a\Delta m * u - a\Delta\mu * u - a\Delta\mu * u * [\widehat{\mu} - m - m * \widehat{\mu}], \tag{4.33}$$

$$\phi^\dagger = (\phi_1^\dagger, \dots, \phi_n^\dagger),$$

$$\phi_i^\dagger = -\Delta m * \sum_{j=1}^n a_{ij} u_{x_j} - \Delta\mu * \sum_{j=1}^n a_{ij} u_{x_j} - \Delta\mu * \sum_{j=1}^n a_{ij} u_{x_j} * [\widehat{\mu} - m - m * \widehat{\mu}], \tag{4.34}$$

$$h^\dagger = -\vartheta \Delta\mu * [u + \widehat{\mu} * u]. \tag{4.35}$$

Let us write the weak form (3.27) for the problem for Δw and use the test function $\eta = \psi$. Then we obtain

$$\begin{aligned} 0 = &\int_\Omega (\Delta w + \mu * \Delta w) * \psi \, dx + \int_\Omega 1 * \Big[\sum_{i,j=1}^n a_{ij} (\Delta w_{x_j} - m * \Delta w_{x_j}) * \psi_{x_i} \\ &- a(\Delta w - m * \Delta w) * \psi\Big] dx + \int_{\Gamma_2} 1 * (\vartheta \Delta w + h^\dagger) * \psi \, d\Gamma \\ &- \int_\Omega 1 * \Big(f^\dagger * \psi - \sum_{i=1}^n \phi_i^\dagger * \psi_{x_i}\Big) dx. \end{aligned} \tag{4.36}$$

Next we write the weak form (3.27) for the problem for ψ and use the test function $\eta = \Delta w$ to get

$$\begin{aligned} 0 = &\int_\Omega (\psi + \mu * \psi) * \Delta w \, dx - 2\int_\Omega \int_0^t u(x,T) - u_T(x)] \Delta w(x,\tau) d\tau dx \\ &+ \int_\Omega 1 * \Big[\sum_{i,j=1}^n a_{ij} (\psi_{x_j} - m * \psi_{x_j}) * \Delta w_{x_i} - a(\psi - m * \psi) * \Delta w\Big] dx \\ &+ \int_{\Gamma_2} 1 * (\vartheta \psi + h^\circ) * \Delta w \, d\Gamma. \end{aligned} \tag{4.37}$$

Subtracting (4.36) from (4.37), differentiating with respect to t and setting $t = T$ we have

$$\begin{aligned} &2\int_\Omega [u(x,T) - u_T(x)] \Delta w(x,T) d\tau dx - \int_{\Gamma_2} (h^\circ * \Delta w)(x,T) \, d\Gamma \\ &= \int_\Omega \Big(f^\dagger * \psi - \sum_{i=1}^n \phi_i^\dagger * \psi_{x_i}\Big)(x,T) \, dx - \int_{\Gamma_2} (h^\dagger * \psi)(x,T) \, d\Gamma. \end{aligned}$$

Observing the relations $\Delta w = \Delta u + \Delta\mu * u + \widehat{\mu} * \Delta\mu * u$, (4.28) and (4.2) we obtain the formula

$$\begin{aligned} J'(z)\Delta z = &\int_\Omega \Big(f^\dagger * \psi - \sum_{i=1}^n \phi_i^\dagger * \psi_{x_i}\Big)(x,T)dx - \int_{\Gamma_2} (h^\dagger * \psi)(x,T)\,d\Gamma \\ &- 2\int_\Omega [u(x,T) - u_T(x)]\big\{(\Delta\mu + \widehat{\mu} * \Delta\mu) * u\big\}(x,T)dx \\ &- 2\sum_{j=1}^2 \int_0^T \Big[\int_{\Gamma_2} \kappa_j(y,t)u(y,t;z)d\Gamma - v_j(t)\Big] \\ &\times \int_{\Gamma_2} \kappa_j(x,t)\big\{(\Delta\mu + \widehat{\mu} * \Delta\mu) * u\big\}(x,t)d\Gamma dt. \end{aligned}$$

Rearranging the terms yields (4.20) with (4.21)-(4.23). □

The formula (4.20) shows that the vector $(\gamma_1, \gamma_2, \gamma_3)$ is a representation of $J'(z)$ in the space $\mathcal{Z}$. It can be used in gradient-type minimization algorithms (cf. [13, 14]).

5. Existence of quasi-solutions

Theorem 5.1. *Let M be compact. Then IP has a quasi-solution in M.*

Proof. It coincides with the proof of [14, Theorem 7 (ii)]. We use the continuity of J that is a consequence of the Fréchet differentiability of J proved in the previous section. □

Theorem 5.2. *Let $n = 1$, $\Omega = (c,d)$, $\varphi = g_\varphi = 0$, $g(x,0) = 0$ and M be bounded, closed and convex. Then IP has a quasi-solution in M.*

Proof. This theorem follows from Weierstrass existence theorem [20] provided we are able to show that J is weakly sequentially lower semi-continuous in M. We will prove that J is in fact weakly sequentially continuous in M.

Let us choose some sequence $z_k = (a_k, m_k, \mu_k) \in M$ such that $z_k \rightharpoonup z = (a, m, \mu) \in M$. Then it is easy to see that $a_k \rightharpoonup a$ in $L^2(c,d)$ and $m_k \rightharpoonup m$, $\mu_k \rightharpoonup \mu$ in $L^2(0,T)$. As in the proof of Theorem 3.1, let $\widehat{\mu} \in L^2(0,T)$ be the solution of (3.12). Similarly, let $\widehat{\mu}_k \in L^2(0,T)$ be the solution of the equation $\widehat{\mu}_k + \mu_k * \widehat{\mu}_k = -\mu_k$ in $(0,T)$. Let us show that $\widehat{\mu}_k \rightharpoonup \widehat{\mu}$ in $L^2(0,T)$. To this end we firstly verify the boundedness of the sequence $\widehat{\mu}_k$. Multiplying the equation of $\widehat{\mu}_k$ by $e^{-\sigma t}$, $\sigma > 0$, and estimating by means of the Young and Cauchy inequalities we obtain

$$\begin{aligned} \|e^{-\sigma t}\widehat{\mu}_k\|_{L^2(0,T)} &\le \|e^{-\sigma t}\mu_k * e^{-\sigma t}\widehat{\mu}_k\|_{L^2(0,T)} + \|e^{-\sigma t}\mu_k\|_{L^2(0,T)} \\ &\le \|e^{-\sigma t}\mu_k\|_{L^1(0,T)}\|e^{-\sigma t}\widehat{\mu}_k\|_{L^2(0,T)} + \|e^{-\sigma t}\mu_k\|_{L^2(0,T)} \\ &\le \|e^{-\sigma t}\|_{L^2(0,T)}\|\mu_k\|_{L^2(0,T)}\|e^{-\sigma t}\widehat{\mu}_k\|_{L^2(0,T)} + \|e^{-\sigma t}\mu_k\|_{L^2(0,T)}. \end{aligned}$$

Observing that $\|e^{-\sigma t}\|_{L^2(0,T)} \le 1/\sqrt{2\sigma}$ and choosing $\sigma = \sigma_1 = 2[\sup\|\mu_k\|_{L^2(0,T)}]^2$ we get

$$\|e^{-\sigma_1 t}\widehat{\mu}_k\|_{L^2(0,T)} \le 2\|e^{-\sigma_1 t}\mu_k\|_{L^2(0,T)} \Rightarrow \|\widehat{\mu}_k\|_{L^2(0,T)} \le 2e^{\sigma_1 T}\sup\|\mu_k\|_{L^2(0,T)}.$$

This shows that the sequence $\widehat{\mu}_k$ is bounded. The difference $\widehat{\mu}_k - \widehat{\mu}$ can be expressed as

$$\widehat{\mu}_k - \widehat{\mu} = -(\mu_k - \mu) - v_k * (\mu_k - \mu),$$

where $v_k = \widehat{\mu} + \widehat{\mu}_k + \widehat{\mu} * \widehat{\mu}_k$ is a bounded sequence in $L^2(0,T)$. Denote by $\langle\cdot,\cdot\rangle$ the inner product in $L^2(0,T)$. With an arbitrary $\zeta \in L^2(0,T)$ we have

$$\langle\widehat{\mu}_k-\widehat{\mu},\zeta\rangle = -\langle\mu_k-\mu,\zeta\rangle - N_k,\ N_k = \int_0^T v_k(\tau)\int_0^{T-\tau}(\mu_k-\mu)(s)\zeta(\tau+s)ds d\tau. \tag{5.1}$$

Since $\zeta(\tau+\cdot) \in L^2(0,T-\tau)$ for $\tau \in (0,T)$, it holds $\int_0^{T-\tau}(\mu_k-\mu)(s)\zeta(\tau+s)ds \to 0$ for $\tau \in (0,T)$. Moreover, since μ_k is bounded in $L^2(0,T)$, the sequence of τ-dependent functions $|\int_0^{T-\tau}(\mu_k-\mu)(s)\zeta(\tau+s)ds|$ is bounded by a constant. By the Cauchy inequality and the dominated convergence theorem, we find

$$|N_k| \le \|v_k\|_{L^2(0,T)}\|\int_0^{T-\cdot}(\mu_k-\mu)(s)\zeta(\cdot+s)ds\|_{L^2(0,T)} \to 0.$$

Thus, from (5.1), in view of $\mu_k \rightharpoonup \mu$, we obtain $\widehat{\mu}_k \rightharpoonup \widehat{\mu}$.

Let us define

$$\widehat{u} = u + \mu * u\,, \quad \widehat{u}_k = u_k + \mu_k * u_k,$$

where $u = u(x,t;z)$ and $u_k = u(x,t;z_k)$ are the weak solutions of (2.1)–(2.4) corresponding to the vectors z and z_k, respectively. The relations $u, u_k \in \mathcal{U}(\Omega_T)$ and $\mu, \mu_k \in L^2(0,T)$ imply $\widehat{u}, \widehat{u}_k \in \mathcal{U}(\Omega_T)$. Observing the definitions of the resolvent kernels $\widehat{\mu}$ and $\widehat{\mu}_k$ we deduce

$$\begin{gathered} u = \widehat{u} + \widehat{\mu} * \widehat{u}\,, \quad u_k = \widehat{u}_k + \widehat{\mu}_k * \widehat{u}_k, \\ u_k - u = \widehat{u}_k - \widehat{u} + \widehat{\mu}_k * (\widehat{u}_k - \widehat{u}) + (\widehat{\mu}_k - \widehat{\mu}) * \widehat{u}. \end{gathered}$$

In view of the latter relation we express the difference of values of the functional J as follows:

$$\begin{aligned} &J(z_k) - J(z) \\ &= \int_c^d (u_k-u)^2(x,T)dx + 2\int_c^d [u(x,T)-u_T(x)](u_k-u)(x,T)dx \\ &\quad + \sum_{j=1}^2\int_0^T \Big[\sum_{l=1}^L \kappa_j(x_l,t)(u_k-u)(x_l,t)\Big]^2 dt \\ &\quad + 2\sum_{j=1}^2\int_0^T \Big[\sum_{l=1}^L \kappa_j(x_l,t)u(x_l,t) - v_j(t)\Big]\Big[\sum_{l=1}^L \kappa_j(x_l,t)(u_k-u)(x_l,t)\Big]dt \\ &= I_k^1 + I_k^2 + I_k^3 + I_k^4, \end{aligned} \tag{5.2}$$

where

$$I_k^1 = \int_c^d \Big(\widehat{u}_k - \widehat{u} + \widehat{\mu}_k * (\widehat{u}_k - \widehat{u}) + (\widehat{\mu}_k - \widehat{\mu}) * \widehat{u}\Big)^2(x,T)dx,$$

$$I_k^2 = 2\int_c^d [u(x,T)-u_T(x)]\Big(\widehat{u}_k - \widehat{u} + \widehat{\mu}_k * (\widehat{u}_k - \widehat{u}) + (\widehat{\mu}_k - \widehat{\mu}) * \widehat{u}\Big)(x,T)dx,$$

$$I_k^3 = \sum_{j=1}^2\int_0^T \Big[\sum_{l=1}^L \kappa_j(x_l,t)\Big(\widehat{u}_k - \widehat{u} + \widehat{\mu}_k * (\widehat{u}_k - \widehat{u}) + (\widehat{\mu}_k - \widehat{\mu}) * \widehat{u}\Big)(x_l,t)\Big]^2 dt,$$

$$
\begin{aligned}
I_k^4 = 2\sum_{j=1}^{2}\int_0^T & \Big[\sum_{l=1}^{L}\kappa_j(x_l,t)u(x_l,t)-v_j(t)\Big] \\
& \times \Big[\sum_{l=1}^{L}\kappa_j(x_l,t)\Big(\widehat{u}_k-\widehat{u}+\widehat{\mu}_k*(\widehat{u}_k-\widehat{u})+(\widehat{\mu}_k-\widehat{\mu})*\widehat{u}\Big)(x_l,t)\Big]dt.
\end{aligned}
$$

Using the Cauchy inequality, $\widehat{\mu}\in L^2(0,T)$, $\widehat{u}_k,\widehat{u}\in\mathcal{U}(\Omega_T)$ and the boundedness of the sequence $\widehat{\mu}_k$ in $L^2(0,T)$ we obtain

$$
\begin{aligned}
|I_k^1| &\le \|\big(\widehat{u}_k-\widehat{u}+\widehat{\mu}_k*(\widehat{u}_k-\widehat{u})\big)(\cdot,T)\|_{L^2(c,d)}^2 \\
&\quad + 2\|\big((\widehat{\mu}_k-\widehat{\mu})*\widehat{u}\big)(\cdot,T)\|_{L^2(c,d)}\|\big(\widehat{u}_k-\widehat{u}+\widehat{\mu}_k*(\widehat{u}_k-\widehat{u})\big)(\cdot,T)\|_{L^2(c,d)}+R_k^1 \\
&\le \tilde{C}_1\big(\|\widehat{u}_k-\widehat{u}\|_{\mathcal{U}(\Omega_T)}^2+\|\widehat{u}_k-\widehat{u}\|_{\mathcal{U}(\Omega_T)}\big)+R_k^1
\end{aligned}
$$

with a constant $\tilde{C}_1$ and

$$
R_k^1=\int_c^d\Big[\int_0^T(\widehat{\mu}_k-\widehat{\mu})(\tau)\widehat{u}(x,T-\tau)d\tau\Big]^2dx.
$$

Since $\widehat{u}\in\mathcal{U}(\Omega_T)\subset L^2(\Omega_T)$, by Tonelli's theorem it holds $\widehat{u}(x,\cdot)\in L^2(0,T)$ a.e. $x\in(c,d)\Rightarrow\widehat{u}(x,T-\cdot)\in L^2(0,T)$ a.e. $x\in(c,d)$. Thus, in view of $\mu_k\rightharpoonup\widehat{\mu}$ in $L^2(0,T)$ we have $\int_0^T(\widehat{\mu}_k-\widehat{\mu})(\tau)\widehat{u}(x,T-\tau)d\tau\to 0$ a.e. $x\in(c,d)$. Moreover, $\Big[\int_0^T(\widehat{\mu}_k-\widehat{\mu})(\tau)\widehat{u}(x,T-\tau)d\tau\Big]^2\le\tilde{C}_{11}\int_0^T[\widehat{u}(x,\tau)]^2d\tau\in L^1(c,d)$ with a constant $\tilde{C}_{11}$, because the sequence $\widehat{\mu}_k$ is bounded in $L^2(0,T)$. Therefore, by the dominated convergence theorem we obtain $R_k^1\to 0$. Similarly for I_k^2 we get

$$
\begin{aligned}
|I_k^2| &\le 2\|u(\cdot,T)-u_T\|_{L^2(c,d)}\|\big(\widehat{u}_k-\widehat{u}+\widehat{\mu}_k*(\widehat{u}_k-\widehat{u})\big)(\cdot,T)\|_{L^2(c,d)}+R_k^2 \\
&\le \tilde{C}_2\|\widehat{u}_k-\widehat{u}\|_{\mathcal{U}(\Omega_T)}+R_k^2,
\end{aligned}
$$

where $\tilde{C}_2$ is a constant and

$$
stR_k^2=\int_c^d[u(x,T)-u_T(x)]\int_0^T(\widehat{\mu}_k-\widehat{\mu})(\tau)\widehat{u}(x,T-\tau)d\tau dx. \tag{5.3}
$$

st By the same reasons as above, it holds $R_k^2\to 0$. Next, let us estimate I_k^3:

$$
\begin{aligned}
|I_k^3| &\le L^2\sum_{j=1}^{2}\max_{1\le l\le L}\Big[\|\kappa_j(x_l,\cdot)\|_{L^\infty(0,T)}^2\|\big(\widehat{u}_k-\widehat{u}+\widehat{\mu}_k*(\widehat{u}_k-\widehat{u})\big)(x_l,\cdot)\|_{L^2(0,T)}^2\Big] \\
&\quad + 2L^2\sum_{j=1}^{2}\max_{1\le l\le L}\Big[\|\kappa_j(x_l,\cdot)\|_{L^\infty(0,T)}^2\|\big((\widehat{\mu}_k-\widehat{\mu})*\widehat{u}\big)(x_l,\cdot)\|_{L^2(0,T)} \\
&\quad \times\|\big(\widehat{u}_k-\widehat{u}+\widehat{\mu}_k*(\widehat{u}_k-\widehat{u})\big)(x_l,\cdot)\|_{L^2(0,T)}\Big]+R_k^3 \\
&\le \tilde{C}_3\big(\|\widehat{u}_k-\widehat{u}\|_{\mathcal{U}(\Omega_T)}^2+\|\widehat{u}_k-\widehat{u}\|_{\mathcal{U}(\Omega_T)}\big)+R_k^3,
\end{aligned}
$$

where $\tilde{C}_3$ is a constant and

$$
R_k^3=L^2\sum_{j=1}^{2}\max_{1\le l\le L}\Big\{\|\kappa_j(x_l,\cdot)\|_{L^\infty(0,T)}^2\int_0^T\Big[\int_0^t(\widehat{\mu}_k-\widehat{\mu})(\tau)\widehat{u}(x_l,t-\tau)d\tau\Big]^2dt\Big\}.
$$

Here we also used the embedding $\mathcal{U}(\Omega_T)\hookrightarrow L^2((0,T);C[c,d])$ that holds in the case $n=1$. Since $\widehat{u}(x_l,t-\cdot)\in L^2(0,t)$ for all $t\in(0,T)$ we get $\int_0^t(\widehat{\mu}_k-\widehat{\mu})(\tau)\widehat{u}(x_l,t-$

$\tau)d\tau \to 0$ for all $t \in (0,T)$. Moreover, the sequence $|\int_0^t (\widehat{\mu}_k - \widehat{\mu})(\tau)\widehat{u}(x_l, t-\tau)d\tau|$ is bounded by a constant. Consequently, $R_k^3 \to 0$. Analogously we deduce the estimate

$$
\begin{gathered}
|I_k^4| \le \tilde{C}_4 \|\widehat{u}_k - \widehat{u}\|_{\mathcal{U}(\Omega_T)} + R_k^4, \quad \text{where } \tilde{C}_4 \text{ is a constant},\\
R_k^4 = 2L \sum_{j=1}^{2} \Big\| \sum_{l=1}^{L} \kappa_j(x_l,t)u(x_l,\cdot) - v_j \Big\|_{L^2(0,T)} \max_{1\le l\le L} \Big\{ \|\kappa_j(x_l,\cdot)\|_{L^\infty(0,T)} \\
\times \Big[\int_0^T \Big[\int_0^t (\widehat{\mu}_k - \widehat{\mu})(\tau)\widehat{u}(x_l,t-\tau)d\tau \Big]^2 dt \Big]^{1/2} \Big\},
\end{gathered}
$$

where $R_k^4 \to 0$.

Note that if we manage to show that $\|\widehat{u}_k - \widehat{u}\|_{\mathcal{U}(\Omega_T)} \to 0$ then the proof is complete. Indeed, in this case by virtue of $R_k^i \to 0$, $i = 1,2,3,4$, from the estimates of I_k^i we get $I_k^i \to 0$, $i = 1,2,3,4$ and due to (5.2) we obtain $J(z_k) \to J(z)$, which implies the statement of the theorem.

As in the proof of Theorem 3.1 we can show that $\widehat{u}$ and $\widehat{u}_k$ are the weak solutions of the following problems:

$$
\begin{gathered}
\widehat{u}_t = A\widehat{u} - \widehat{m} * A\widehat{u} + f + \phi_x \quad \text{in } \Omega_T, \qquad (5.4)\\
\widehat{u} = u_0 \quad \text{in } \Omega \times \{0\}, \qquad (5.5)\\
\widehat{u} = \widehat{g} \quad \text{in } \Gamma_{1,T}, \qquad (5.6)\\
-\nu_A \cdot \nabla\widehat{u} + \widehat{m} * \nu_A \cdot \nabla\widehat{u} = \vartheta\widehat{u} + \vartheta\widehat{\mu} * \widehat{u} + h + \nu\cdot\phi \quad \text{in } \Gamma_{2,T}, \qquad (5.7)\\
\widehat{u}_{k,t} = A_k\widehat{u}_k - \widehat{m}_k * A_k\widehat{u}_k + f + \phi_x \quad \text{in } \Omega_T, \qquad (5.8)\\
\widehat{u}_k = u_0 \quad \text{in } \Omega \times \{0\}, \qquad (5.9)\\
\widehat{u}_k = \widehat{g}_k \quad \text{in } \Gamma_{1,T}, \qquad (5.10)\\
-\nu_A \cdot \nabla\widehat{u}_k + \widehat{m}_k * \nu_A \cdot \nabla\widehat{u}_k = \vartheta\widehat{u}_k + \vartheta\widehat{\mu}_k * \widehat{u}_k + h + \nu\cdot\phi \quad \text{in } \Gamma_{2,T}, \qquad (5.11)
\end{gathered}
$$

where $A_k v = (a_{11}v_x)_x + a_k v$,

$$
\begin{gathered}
\widehat{m} = m - \widehat{\mu} + m * \widehat{\mu}\,, \quad \widehat{m}_k = m_k - \widehat{\mu}_k + m_k * \widehat{\mu}_k,\\
\widehat{g} = g + \mu * g\,, \quad \widehat{g}_k = g + \mu_k * g.
\end{gathered}
$$

We now show that $\widehat{m}_k \rightharpoonup \widehat{m}$. With any $\zeta \in L^2(0,T)$ we compute

$$
\begin{aligned}
&\langle \widehat{m}_k - \widehat{m}, \zeta\rangle = \langle m_k - m, \zeta\rangle - \langle \widehat{\mu}_k - \widehat{\mu}, \zeta\rangle + N_k^1,\\
&N_k^1 = \int_0^T \widehat{\mu}_k(\tau) \int_0^{T-\tau} (m_k - m)(s)\zeta(\tau+s)dsd\tau\\
&\quad + \int_0^T m(\tau) \int_0^{T-\tau} (\widehat{\mu}_k - \widehat{\mu})(s)\zeta(\tau+s)dsd\tau.
\end{aligned}
$$

We use the relations $m_k \rightharpoonup m$, $\widehat{\mu}_k \rightharpoonup \widehat{\mu}$ and treat the term N_k^1 similarly to the term N_k in (5.1) to get $N_k^1 \to 0$. As a result we get $\langle \widehat{m}_k - \widehat{m}, \zeta\rangle \to 0$, hence $\widehat{m}_k \rightharpoonup \widehat{m}$.

Subtracting the problem of $\widehat{u}$ from the problem of $\widehat{u}_k$ we see that $w_k := \widehat{u}_k - \widehat{u}$ is a weak solution of the problem

$$
\begin{gathered}
w_{k,t} = Aw_k - \widehat{m} * Aw_k + \tilde{f}_k + \tilde{\phi}_{k,x} \quad \text{in } \Omega_T, \qquad (5.12)\\
\widehat{u} = 0 \quad \text{in } \Omega \times \{0\}, \qquad (5.13)\\
\widehat{u} = \tilde{g}_k \quad \text{in } \Gamma_{1,T}, \qquad (5.14)
\end{gathered}
$$

$$-\nu_A \cdot \nabla w_k + \widehat{m} * \nu_A \cdot \nabla w_k = \vartheta w_k + \tilde{h}_k + \nu \cdot \tilde{\phi}_k \quad \text{in } \Gamma_{2,T}, \tag{5.15}$$

where

$$\begin{gathered}
\tilde{f}_k = (a_k - a)(\widehat{u}_k - \widehat{m}_k * \widehat{u}_k) - a(\widehat{m}_k - \widehat{m}) * \widehat{u}_k, \\
\tilde{\phi}_k = -a_{11}(\widehat{m}_k - \widehat{m}) * \widehat{u}_{k,x}, \quad \tilde{g}_k = (\mu_k - \mu) * g, \\
\tilde{h}_k = \vartheta[\widehat{\mu}_k * w_k + (\widehat{\mu}_k - \widehat{\mu}) * \widehat{u}].
\end{gathered}$$

To use the weak convergence $a_k \rightharpoonup a$ in forthcoming estimations we have to introduce the functions $\rho_k \in W_2^2(c,d)$ being the solutions of the following Neumann problems:

$$\rho_k'' - \rho_k = a_k - a \quad \text{in } (c,d)\,, \qquad \rho_k'(c) = \rho_k'(d) = 0.$$

Then $\rho_k(x) = \int_c^d G(x,y)(a_k - a)(y)dy$, $x \in (c,d)$, where

$$G(x,y) = \frac{1}{2(e^{c-d} - e^{d-c})} \begin{cases} (e^{c-y} + e^{y-c})(e^{d-x} + e^{x-d}) & \text{for } y < x \\ (e^{c-x} + e^{x-c})(e^{d-y} + e^{y-d}) & \text{for } y > x \end{cases}$$

is a Green function that satisfies the properties $G, G_x \in L^\infty(\Omega_T)$. The weak convergence $a_k \rightharpoonup a$ in $L^2(c,d)$ implies

$$\|\rho_k\|_{W_2^1(c,d)} \to 0. \tag{5.16}$$

Using ρ_k we rewrite the term $(a_k - a)(\widehat{u}_k - \widehat{m}_k * \widehat{u}_k)$ in $\tilde{f}_k$ as follows:

$$\begin{aligned}
&(a_k - a)(\widehat{u}_k - \widehat{m}_k * \widehat{u}_k) \\
&= [\rho_k'(\widehat{u}_k - \widehat{m}_k * \widehat{u}_k)]_x - \rho_k'(\widehat{u}_k - \widehat{m}_k * \widehat{u}_k)_x - \rho_k(\widehat{u}_k - \widehat{m}_k * \widehat{u}_k).
\end{aligned}$$

According to this relation we change the form of the problem for w_k as follows:

$$w_{k,t} = Aw_k - \widehat{m} * Aw_k + \overline{f}_k + \overline{\phi}_{k,x} \quad \text{in } \Omega_T, \tag{5.17}$$

$$\widehat{u} = 0 \quad \text{in } \Omega \times \{0\}, \tag{5.18}$$

$$\widehat{u} = \tilde{g}_k \quad \text{in } \Gamma_{1,T}, \tag{5.19}$$

$$-\nu_A \cdot \nabla w_k + \widehat{m} * \nu_A \cdot \nabla w_k = \vartheta w_k + \tilde{h}_k + \nu \cdot \overline{\phi}_k \quad \text{in } \Gamma_{2,T}, \tag{5.20}$$

where

$$\begin{gathered}
\overline{f}_k = -\rho_k'(\widehat{u}_k + \widehat{m}_k * \widehat{u}_k)_x - \rho_k(\widehat{u}_k + \widehat{m}_k * \widehat{u}_k) - a(\widehat{m}_k - \widehat{m}) * \widehat{u}_k, \\
\overline{\phi}_k = \rho_k'(\widehat{u}_k + \widehat{m}_k * \widehat{u}_k) - a_{11}(\widehat{m}_k - \widehat{m}) * \widehat{u}_{k,x}.
\end{gathered}$$

Let t be an arbitrary number in $[0,T]$. To estimate w_k we will use the projection operators P_t, defined in (3.22), and $\overline{P}_t w = \begin{cases} w & \text{in } \Omega_t \\ 0 & \text{in } \Omega_T \setminus \Omega_t \end{cases}$. for $w : \Omega_T \to \mathbb{R}$. Let w_k^t stand for the weak solution of problem (5.17)–(5.20) with $\overline{f}_k$, $\overline{\phi}_k$ and $\tilde{h}_k$ replaced by $\overline{P}_t \overline{f}_k$, $\overline{P}_t \overline{\phi}_k$ and $P_t \tilde{h}_k$, respectively. Then, due to the causality $w_k^t = w_k$ in Ω_t. Applying (3.11) for w_k^t we obtain

$$\begin{aligned}
\|w_k\|_{\mathcal{U}(\Omega_t)} &= \|w_k^t\|_{\mathcal{U}(\Omega_t)} \le \|w_k^t\|_{\mathcal{U}(\Omega_T)} \le \overline{C}_0 \Big[\|\overline{P}_t \overline{f}_k\|_{L^2((0,T);L^1(c,d))} \\
&\quad + \|\overline{P}_t \overline{\phi}_k\|_{(L^2(\Omega_T))^n} + \theta \|\tilde{g}_k\|_{\mathcal{T}(\Omega_T)} + \|P_t \tilde{h}_k\|_{L^2(\Gamma_{2,T})} \Big] \\
&= \overline{C}_0 \Big[\|\overline{f}_k\|_{L^2((0,t);L^1(c,d))} + \|\overline{\phi}_k\|_{L^2(\Omega_t)} + \theta \|\tilde{g}_k\|_{\mathcal{T}(\Omega_T)} + \|\tilde{h}_k\|_{L^2(\Gamma_{2,t})} \Big]
\end{aligned} \tag{5.21}$$

with a constant $\overline{C}_0$. Using the relation $a \in L^2(c,d)$, Cauchy inequality, the inequality (3.21), $g(x,0)=0$, the embedding $W_2^1(c,d) \hookrightarrow C[c,d]$ and $\widehat{u}_k = w_k + \widehat{u}$ we estimate:

$$
\begin{aligned}
\|\overline{f}_k\|_{L^2((0,t);L^1(c,d))} &\le \overline{C}_1\Big[\|(\widehat{m}_k - \widehat{m}) * \widehat{u}_k\|_{L^2((0,t);L^2(c,d))} \\
&\quad + \|\rho_k\|_{W_2^1(c,d)}(1+\|\widehat{m}_k\|_{L^2(0,T)})\|\widehat{u}_k\|_{\mathcal{U}(\Omega_t)}\Big] \\
&\le \overline{C}_1\Big[\int_0^t |(\widehat{m}_k - \widehat{m})(t-\tau)|\,\|w_k\|_{L^2(\Omega_\tau)}d\tau \\
&\quad + \|\rho_k\|_{W_2^1(c,d)}(1+\|\widehat{m}_k\|_{L^2(0,T)})\|w_k\|_{\mathcal{U}(\Omega_t)}\Big] + \overline{R}_k^1,
\end{aligned} \tag{5.22}
$$

$$
\begin{aligned}
\|\overline{\phi}\|_{L^2(\Omega_t)} &\le \overline{C}_2\Big[\|(\widehat{m}_k - \widehat{m}) * \widehat{u}_{k,x}\|_{L^2(\Omega_t)} \\
&\quad + \|\rho_k\|_{W_2^1(c,d)}(1+\|\widehat{m}_k\|_{L^2(0,T)})\|\widehat{u}_k\|_{L^2((0,T);C[c,d])}\Big] \\
&\le \overline{C}_2\Big[\int_0^t |(\widehat{m}_k - \widehat{m})(t-\tau)|\,\|w_{k,x}\|_{L^2(\Omega_\tau)}d\tau \\
&\quad + \|\rho_k\|_{W_2^1(c,d)}(1+\|\widehat{m}_k\|_{L^2(0,T)})\|w_k\|_{L^2((0,t);C[c,d])}\Big] + \overline{R}_k^2,
\end{aligned} \tag{5.23}
$$

$$
\|\tilde{g}_k\|_{\mathcal{T}(\Omega_T)} \le \overline{R}_k^3, \tag{5.24}
$$

$$
\begin{aligned}
\|\tilde{h}_k\|_{L^2(\Gamma_{2,t})} &\le \overline{C}_4\Big[\|\widehat{\mu}_k * w_k\|_{L^2((0,t);W_2^1(c,d))} + \|(\widehat{\mu}_k - \widehat{\mu}) * \widehat{u}\|_{L^2((0,t);W_2^1(c,d))}\Big] \\
&\le \overline{C}_4 \int_0^t |\widehat{\mu}_k(t-\tau)|\,\|w_k\|_{L^2((0,\tau);W_2^1(c,d))}d\tau + \overline{R}_k^4,
\end{aligned} \tag{5.25}
$$

where $\overline{C}_1, \overline{C}_2, \overline{C}_4$ are constants and

$$
\begin{aligned}
\overline{R}_k^1 &= \overline{C}_1\Big[\|(\widehat{m}_k - \widehat{m}) * \widehat{u}\|_{L^2(\Omega_T)} + \|\rho_k\|_{W_2^1(c,d)}(1+\|\widehat{m}_k\|_{L^2(0,T)})\|\widehat{u}\|_{\mathcal{U}(\Omega_T)}\Big], \\
\overline{R}_k^2 &= \overline{C}_2\Big[\|(\widehat{m}_k - \widehat{m}) * \widehat{u}_x\|_{L^2(\Omega_T)} + \|\rho_k\|_{W_2^1(c,d)}\big(1 \\
&\quad + \|\widehat{m}_k\|_{L^2(0,T)}\big)\|\widehat{u}\|_{L^2((0,T);C[c,d])}\Big], \\
\overline{R}_k^3 &= \|(\mu_k - \mu) * g\|_{L^2(\Omega_T)} + \|(\mu_k - \mu) * g_x\|_{L^2(\Omega_T)} + \|(\mu_k - \mu) * g_t\|_{L^2(\Omega_T)}, \\
\overline{R}_k^4 &= \overline{C}_4\|(\widehat{\mu}_k - \widehat{\mu}) * \widehat{u}\|_{L^2((0,T);W_2^1(c,d))}.
\end{aligned}
$$

By the weak convergence $\widehat{m}_k \rightharpoonup \widehat{m}$, $\mu_k \rightharpoonup \mu$, $\widehat{\mu}_k \rightharpoonup \widehat{u}$ in $L^2(0,T)$ and the relation $\|\rho_k\|_{W_2^1(c,d)} \to 0$ it holds

$$
\overline{R}_k^j \to 0, \quad j = 1,2,3,4. \tag{5.26}
$$

Indeed, to prove that $\|z_k * \hat{v}\|_{L^2(\Omega_T)} \to 0$, where z_k is one of the functions $\widehat{m}_k - \widehat{m}$, $\mu_k - \mu$ or $\widehat{\mu}_k - \widehat{\mu}$ and $\hat{v} \in L^2(\Omega_T)$ is one of the functions $\widehat{u}$, $\widehat{u}_x$, g, g_x or g_t, it is possible to use the dominated convergence theorem, again. More precisely,

$$
\|z_k * \hat{v}\|_{L^2(\Omega_T)} = \Big\{\int_0^T \int_c^d \Big[\int_0^t z_k(\tau)\hat{v}(x,t-\tau)d\tau\Big]^2 dx\,dt\Big\}^{1/2},
$$

where the component $\Big[\int_0^t z_k(\tau)\hat{v}(x,t-\tau)d\tau\Big]^2$ is bounded by an integrable in $x \in (c,d)$ function $\sup \|z_k\|^2_{L^2(0,T)}\|\hat{v}(x,\cdot)\|^2_{L^2(0,T)}$ and tends to zero for all $t \in (0,T)$ and a.e. $x \in (c,d)$, because $z_k \rightharpoonup 0$ and $\hat{v}(x, t - \cdot) \in L^2(0,T)$ for all $t \in (0,T)$ and a.e. $x \in (c,d)$. (The latter relation follows from $\hat{v} \in L^2(\Omega_T)$ and Tonelli's theorem.) Thus, $\|z_k * \hat{v}\|_{L^2(\Omega_T)} \to 0$.

As in proof of Theorem 3.1, we use the norms $\|w\|_\sigma = \sup_{0<t<T} e^{-\sigma t} \|w\|_{\mathcal{U}(\Omega_t)}$ with the weights $\sigma \geq 0$ in the space $\mathcal{U}(\Omega_T)$. Then in view of (5.22)–(5.25) from (5.21) we deduce

$$\begin{aligned}
\|w_k\|_\sigma &\leq \overline{C}_5 \Big[\sup_{0<t<T} \int_0^t e^{-\sigma(t-\tau)} r_k(t-\tau)\, e^{-\sigma\tau} \|w_k\|_{\mathcal{U}(\Omega_\tau)} d\tau \\
&\quad + \|\rho_k\|_{W_2^1(c,d)} (1 + \|\widehat{m}_k\|_{L^2(0,T)}) \|w_k\|_\sigma + \sum_{j=1}^4 \overline{R}_k^j \Big] \\
&\leq \overline{C}_5 \Big[\Big\{ \|e^{-\sigma t}\|_{L^2(0,T)} \|r_k\|_{L^2(0,T)} + \|\rho_k\|_{W_2^1(c,d)} (1 + \|\widehat{m}_k\|_{L^2(0,T)}) \Big\} \|w_k\|_\sigma \\
&\quad + \sum_{j=1}^4 \overline{R}_k^j \Big],
\end{aligned}$$

where $\overline{C}_5$ is a constant and $r_k = |\widehat{m}_k - \widehat{m}| + |\widehat{\mu}_k|$. Since $\|e^{-\sigma t}\|_{L^2(0,T)} \to 0$ as $\sigma \to \infty$, $\|\rho_k\|_{W_2^1(c,d)} \to 0$ and the sequences $\|r_k\|_{L^2(0,T)}$, $\|\widehat{m}_k\|_{L^2(0,T)}$ are bounded, there exist $\sigma_2 > 0$ and $K_2 \in \mathbb{N}$ such that

$$\|e^{-\sigma_2 t}\|_{L^2(0,T)} \|r_k\|_{L^2(0,T)} + \|\rho_k\|_{W_2^1(c,d)} (1 + \|\widehat{m}_k\|_{L^2(0,T)}) \leq \frac{1}{2\overline{C}_5}$$

for $k \geq K_2$. This, along with the previous inequality, implies

$$\|w_k\|_{\sigma_2} \leq 2\overline{C}_5 \sum_{j=1}^4 \overline{R}_k^j \quad \text{and hence} \quad \|w_k\|_{\mathcal{U}(\Omega_T)} \leq 2e^{\sigma_2 T} \overline{C}_5 \sum_{j=1}^4 \overline{R}_k^j$$

for $k \geq K_2$. Taking (5.26) into account we obtain the desired convergence: $\|\widehat{u}_k - \widehat{u}\|_{\mathcal{U}(\Omega_T)} = \|w_k\|_{\mathcal{U}(\Omega_T)} \to 0$. The theorem is proved. $\square$

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