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THREE-POINT THIRD-ORDER PROBLEMS WITH A SIGN-CHANGING NONLINEAR TERM

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Abstract. In this article we study a well-known boundary value problem

$$u'''(t) = f(t, u(t)), \quad 0 < t < 1,$$
$$u(0) = u'(1/2) = u''(1) = 0.$$

With $u'(\eta) = 0$ in place of $u'(1/2) = 0$, many authors studied the existence of positive solutions of both the positone problems with $\eta \geq 1/2$ and the semi-positone problems for $\eta > 1/2$. It is well-known that the standard method successfully applied to the semi-positone problem with $\eta > 1/2$ does not work for $\eta = 1/2$ in the same setting. We treat the latter as a problem with a sign-changing term rather than a semi-positone problem. We apply Krasnosel'skiĭ's fixed point theorem [4] to obtain positive solutions.

1. Introduction

We study the third-order nonlinear boundary-value problem

$$u'''(t) = f(t, u(t)), \quad 0 < t < 1, \tag{1.1}$$
$$u(0) = u'(1/2) = u''(1) = 0. \tag{1.2}$$

with a sign-changing nonlinearity.

Equation (1.1) satisfying the three-point condition

$$u(0) = u'(\eta) = u''(1) = 0, \tag{1.3}$$

with $\eta \geq 1/2$ has been studied by many authors [2, 7, 11]. We mention also relevant results in [1, 3], where, under nonlocal conditions involving Stieltjes integrals, the positone case was considered. A good theory of positive solutions for semi-positone problems with $\eta > 1/2$ is developed in [5, 8, 9, 10] (and the references therein). In particular, Yao [8] obtained a positive solution of the boundary value problem similar to (1.1), (1.3). The author assumed that the function $f : [0, 1] \times \mathbb{R}_+ \to \mathbb{R}$ satisfies the Carathéodory conditions and there exists a nonnegative function $h \in L_1[0, 1]$ such that $f(t, u) \geq h(t)$, $(t, u) \in [0, 1] \times \mathbb{R}_+$. Our paper is motivated by [8] where, we believe, the idea of a non-constant lower bound $-h(t)$ for the inhomogeneous term was originally used for the boundary value problem similar to (1.1), (1.2). The author refers to this type of problem as weakly semipositone.

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Prior to [8], similar semipositone problems have been solved effectively [10] only for $f : [0,1] \times \mathbb{R} \to [-M, \infty)$ due to selection of $h \equiv M > 0$. As in [8], in this paper, we need only $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$.

Regardless of the choice of h, as a first step, one translates the semipositone problem into a positone problem using the transformations

$$u \mapsto v - u_0 \quad \text{and} \quad f(\cdot, u) \mapsto f(\cdot, v - u_0) + h(\cdot),$$

where u_0 is a unique solution of the problem with the nonlinear term replaced with h. Subsequently, the positone problem is converted into an integral equation, which is shown to have one or, depending on conditions of f, several positive solutions. Finally, an important feature of this approach is that it requires the inequality $v(t) \geq u_0(t)$ to hold for a fixed point of the corresponding integral operator. This comparison depends on the properties of Green's function, or in particular, on the function appearing in the definition of a cone, and the solution u_0. The case of $\eta = 1/2$ stands alone since this type of approach used by many authors to study the case $\eta > 1/2$ does not readily apply to the case $\eta = 1/2$. The difficulty arises when we attempt to obtain the inequality $v(t) \geq u_0(t)$ for $\eta = 1/2$.

Since problem (1.1), (1.2) cannot be treated as a semipositone problem, we adopt a new set of assumptions and consider a sign-changing nonlinearity. We are unaware of any results on the case $\eta = 1/2$ with a sign-changing nonlinear term. Another benefit is that we can also obtain new results for the case $\eta > 1/2$ with a sign-changing nonlinearity by employing the concept of a sign-changing lower bound g_0. We think that it would not be difficult to extend our results to the case of f satisfying the Carathéodory conditions and even treat singularities as in [10]. Here we settle for a continuous sign-changing nonlinear term.

2. Properties of Green's function

Let $g_0 \in C[0,1]$. Then the differential equation

$$u'''(t) = g_0(t), \quad 0 < t < 1, \tag{2.1}$$

satisfying the boundary condition (1.2) has a unique solution

$$\begin{aligned} u_0(t) &= \frac{1}{2}\int_0^t (t-s)^2 g_0(s)\,ds - \frac{t^2}{2}\int_0^1 g_0(s)\,ds \\ &\quad + t\Big(\frac{1}{2}\int_0^1 g_0(s)\,ds - \int_0^{1/2}\big(\frac{1}{2} - s\big) g_0(s)\,ds\Big). \end{aligned}$$

Using Green's function

$$G(t,s) = \frac{1}{2}(t-s)^2 \chi_{[0,t]}(s) + \frac{1}{2}(t - t^2) - t\big(\frac{1}{2} - s\big)\chi_{[0,1/2]}(s), \tag{2.2}$$

for $(t,s) \in [0,1] \times [0,1]$, we have

$$u_0(t) = \int_0^1 G(t,s) g_0(s)\,ds.$$

Let

$$G_0(s) = G(1/2, s) = \frac{s^2}{2}\chi_{[0,1/2]}(s) + \frac{1}{8}\chi_{[1/2,1]}(s), \quad s \in [0,1].$$

We revisit the important properties [5] of (2.2) used in cone-theoretic methods:

$$q(t) G_0(s) \leq G(t,s) \leq G_0(s), \quad (t,s) \in [0,1] \times [0,1], \tag{2.3}$$

where

$$q(t) = 4(t - t^2). \tag{2.4}$$

Also,

$$L = \max_{t\in[0,1]} \int_0^1 G(t, s)\, ds = \frac{1}{12}, \tag{2.5}$$

and, for $0 < \alpha < 1/2$,

$$C = \int_\alpha^{1-\alpha} G_0(s)\, ds = \frac{1}{24}(2 - 4\alpha^3 - 3\alpha). \tag{2.6}$$

Note that if

$$\int_t^1 g_0(t)\, dt \ge 0, \quad t \in [0, 1], \tag{2.7}$$

then $u_0(t)$ is concave in $[0, 1]$. If, in addition,

$$u_0(1) = \frac{1}{2}\int_0^1 (1 - s)^2 g_0(s)\, ds - \int_0^{1/2} \big(\frac{1}{2} - s\big) g_0(s)\, ds \ge 0, \tag{2.8}$$

then $u_0(t) \ge 0$. Note that neither (2.7) nor (2.8) requires $g_0(t) \ge 0$ in all of $[0, 1]$. Moreover, if $g_0(t) \ge 0$ in $[0, 1]$ and $g_0(t) > 0$ in some $[\alpha, \beta] \subset [0, 1]$, then $u_0(1) > 0$.

This represents a difficulty due to the fact that one can not achieve the inequality $q(t) \ge \mu u_0(t)$ in $[0, 1]$ for any $\mu > 0$ (as $q(1) = 0$ while $u_0(1) > 0$). For this reason, the case $\eta = 1/2$ is forbidden in approaching (1.1), (1.3) as a semipositone problem.

If the identity takes place in (2.8), that is, $u_0(1) = 0$ is enforced, then we are in position to compare $q(t)$ and $u_0(t)$ in the next lemma.

Lemma 2.1. *Let $g_0 \in C[0, 1]$ satisfy (2.7) and suppose that the identity holds in (2.8). Then there exists a constant $\mu > 0$ such that*

$$q(t) \ge \mu u_0(t), \quad t \in [0, 1]. \tag{2.9}$$

Proof. Since the function $q - \mu u_0$ vanishes at the end-points of $[0, 1]$, it suffices to obtain $\mu > 0$ such that $-q''(t) \ge -\mu u_0''(t)$ in $[0, 1]$. That is,

$$8 \ge \mu \int_t^1 g_0(s)\, ds, \quad t \in [0, 1].$$

By (2.7), there exists $0 < \tau < 1$ and $\mu > 0$ such that

$$\mu \int_t^1 g_0(s)\, ds \le \mu \int_\tau^1 g_0(s)\, ds = 8. \tag{2.10}$$

$\square$

Suppose that the function f in (1.1) satisfies

(H1) $f \in C([0, 1] \times \mathbb{R}_+, \mathbb{R})$;

(H2) there exists a function $g_0 \in C[0, 1]$ such that
- (a) $f(t, z) + g_0(t) \ge 0$ in $[0, 1] \times \mathbb{R}_+$;
- (b) for all $t \in [0, 1]$, $\int_t^1 g_0(s)\, ds \ge 0$;
- (c)

$$\frac{1}{2}\int_0^1 (1 - s)^2 g_0(s)\, ds - \int_0^{1/2} \big(\frac{1}{2} - s\big) g_0(s)\, ds = 0.$$

Remark 2.2. It is easy to find a function g_0 satisfying (H2) (b) and (c). For example, one can take $g_0(t) = a(2t-1)$, $a > 0$. Of course, an example of $f(t,z)$ that fits (H2) (b) is also easy to obtain.

Remark 2.3. If the inequality (2.8) is replaced with the strict inequality, we cannot expect Lemma 2.1 to hold. So, in this paper, we need the identity in (H2) (c). If, instead of $u'(1/2) = 0$, we impose $u'(\eta) = 0$ with $\eta > 1/2$, then the problem (2.1), (1.3) has a unique solution

$$u_0(t) = \frac{1}{2}\int_0^t (t-s)^2 g_0(s)\,ds - \frac{t^2}{2}\int_0^1 g_0(s)\,ds + t\Big(\eta\int_0^1 g_0(s)\,ds - \int_0^\eta (\eta-s)g_0(s)\,ds\Big).$$

Again, the assumption (H1) (b) guarantees that u_0 is concave in $[0,1]$. So, if

$$u_0(1) = \frac{1}{2}\int_0^1 (1-s)^2 g_0(s)\,ds + \big(\eta - \frac{1}{2}\big)\int_0^1 g_0(s)\,ds - \int_0^\eta (\eta-s)g_0(s)\,ds \geq 0,$$

then $u(t) \geq 0$ in $[0,1]$. Similarly, the analogue of $q(t)$, in this case [5], is

$$p(t) = \frac{1}{\eta^2}(2\eta t - t^2).$$

Noting that $p(1) \neq 0$ and u_0, p are concave concave in $[0,1]$, we can easily obtain an analogue of Lemma 2.1 asserting the existence of $\mu > 0$ such that $p(t) \geq \mu u_0(t)$ in $[0,1]$. This would give a more general result than in [10, Lemma 2.1 (4)], which is derived for $g_0 \equiv M > 0$. This would also allow us to extend the results of [8] concerning an analogue of (1.1), (1.3), where $h(t)$, which serves the purpose of $g_0(t)$, is assumed to be nonnegative.

We modify the problem (1.1), (1.2) as follows. First, we define

$$f_p(t,z) = \begin{cases} f(t,z) + g_0(t), & (t,z) \in [0,1] \times [0,\infty), \\ f(t,0) + g_0(t), & (t,z) \in [0,1] \times (-\infty,0). \end{cases}$$

Next, we consider the equation

$$v'''(t) = f_p(t, v(t) - u_0(t)), \quad t \in (0,1), \tag{2.11}$$

under the boundary conditions (1.2). We can easily obtain the next lemma.

Lemma 2.4. *The function u is a positive solution of the boundary value problem* (1.1)*,* (1.2) *if, and only if, the function $v = u + u_0$ is a solution of the boundary value problem* (2.11)*,* (1.2) *satisfying $v(t) \geq u_0(t)$ in* $[0,1]$.

In the Banach space $\mathcal{B} = C[0,1]$ endowed with usual max-norm, we consider the operator

$$Tv(t) = \int_0^1 G(t,s) f_p(s, v(s) - u_0(s))\,ds, \tag{2.12}$$

where $G(t,s)$ is given by (2.2). By (H1), $T : \mathcal{B} \to \mathcal{B}$ is completely continuous.

Using the function q defined by (2.4), we introduce the cone

$$\mathcal{C} = \{v \in \mathcal{B} : v(t) \geq q(t)\|v\|,\ t \in [0,1]\}.$$

By (2.3), $T : \mathcal{C} \to \mathcal{C}$ and it is also easy to show that a fixed point of T is a solution of (2.11), (1.2). In particular,

$$v(t) \geq \gamma\|v\|, \quad t \in [\tau, 1-\tau], \tag{2.13}$$

where $\gamma = \min_{t\in[\alpha,1-\alpha]} q(t) = 4(\alpha - \alpha^2)$, and $\kappa = \max_{t\in[\alpha,1-\alpha]} q(t) = q(1/2) = 1$.

Theorem 2.5 ([4]). *Let $\mathcal{B}$ be a Banach space and let $\mathcal{C} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume that Ω_1, Ω_2 are open with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$T\colon \mathcal{C} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{C}$$

be a completely continuous operator such that either

(i) $\|Tu\| \le \|u\|$, $u \in \mathcal{C} \cap \partial\Omega_1$, *and* $\|Tu\| \ge \|u\|$, $u \in \mathcal{C} \cap \partial\Omega_2$, *or;*
(ii) $\|Tu\| \ge \|u\|$, $u \in \mathcal{C} \cap \partial\Omega_1$, *and* $\|Tu\| \le \|u\|$, $u \in \mathcal{C} \cap \partial\Omega_2$.

Then T has a fixed point in $\mathcal{C} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Positive solutions

To use Theorem 2.5, following [8] we introduce the "height" functions $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\phi(r) = \max\{f(t, z - u_0(t)) + g_0(t) : t \in [0,1],\ z \in [0,r]\},$$
$$\psi(r) = \min\{f(t, z - u_0(t)) + g_0(t) : t \in [\alpha, 1-\alpha],\ z \in [\gamma r, r]\}.$$

Now we present our main results.

Theorem 3.1. *Assume that* (H1) *and* (H2) *hold. Suppose that there exist $r, R > 0$ such that $\frac{1}{\mu} < r < R$, where $\mu > 0$ satisfies* (2.9), (2.10), *and*

(H3) $\phi(r) \le 12r$ *and* $\psi(R) \ge \frac{24R}{2-4\alpha^3-3\alpha}$.

Then the boundary-value problem (1.1), (1.2) *has at least one positive solution.*

Proof. Let

$$\Omega_1 = \{v \in B : \|v\| < r\}, \quad \Omega_2 = \{v \in B : \|v\| < R\}.$$

For $u \in \mathcal{C} \cap \partial\Omega_1$, we have $v(s) - u_0(s) \ge q(s)\|v\| - u_0(s) \ge (\mu r - 1)u_0(s) \ge 0$, $s \in [0,1]$. This implies that

$$f_p(s, v(s) - u_0(s)) = f(s, v(s) - u_0(s)) + g_0(s), \quad s \in [0,1].$$

In particular,

$$f(s, v(s) - u_0(s)) + g_0(s) \le \phi(r), \quad s \in [0,1],\ 0 \le v(s) \le r.$$

Thus, by (2.5) and (H3),

$$\begin{aligned}
\|Tv\| &= \max_{t\in[0,1]} \int_0^1 G(t,s) f_p(s, v(s) - u_0(s))\, ds \\
&\le \max_{t\in[0,1]} \int_0^1 G(t,s)\, ds\, \phi(r) \\
&= L\phi(r) = \frac{1}{12}\phi(r) \le r.
\end{aligned}$$

That is, $\|Tv\| \le \|v\|$ for all $v \in \mathcal{C} \cap \partial\Omega_1$.

Let $v \in \mathcal{C} \cap \partial\Omega_2$. Since $R > r$, we have $v(s) - u_0(s) \ge (\mu R - 1)u_0(s) \ge 0$, $s \in [0,1]$. Then, for all $s \in [\alpha, 1-\alpha]$, we have, recalling (2.13),

$$R \ge v(s) \ge q(s)\|v\| \ge \gamma R.$$

Hence

$$f_p(s, v(s) - u_0(s)) = f(s, v(s) - u_0(s)) + g_0(s) \ge \psi(R),$$

for $s \in [\alpha, 1-\alpha]$, $\gamma R \le v(s) \le R$. Then, by (2.6) and (H3),

$$\begin{aligned}
\|Tv\| &= \max_{t\in[0,1]} \int_0^1 G(t,s) f_p(s, v(s) - u_0(s))\, ds \\
&\ge \max_{t\in[0,1]} \int_\alpha^{1-\alpha} G(t,s) f_p(s, v(s) - u_0(s))\, ds \\
&\ge \max_{t\in[0,1]} \int_\alpha^{1-\alpha} q(t) G_0(s)\, ds\, \psi(R) \\
&= \max_{t\in[0,1]} q(t) \int_\alpha^{1-\alpha} G_0(s)\, ds\, \psi(R) \\
&= \kappa C \psi(R) \ge R.
\end{aligned}$$

That is, $\|Tv\| \ge \|v\|$ for all $v \in \mathcal{C} \cap \partial\Omega_2$.

By Theorem 2.5, there exists $v_0 \in \mathcal{C}$ with $u(t) = v_0(t) - u_0(t) \ge (\mu r - 1) u_0(t) \ge 0$ in $[0,1]$. By Lemma 2.4, u is a positive solution of the sign-changing problem (1.1), (1.2). □

Now we give an example of the right side of (1.1) satisfying the assumptions of Theorem 3.1.

Example. Let $f(t,z) = 6z^2 + 32(1-2t)$ for $z \ge 0$, $t \in [0,1]$. Then $f(t,z) + g_0(t) \ge 0$ with $g_0(t) = 32(2t-1)$. Of course, (H1) and (H2) hold and

$$\int_t^1 g_0(t)\, ds \le \int_{1/2}^1 g_0(t)\, ds = 8.$$

Hence we can choose $\mu = 1$ and note that $\mu r > 1$, if we choose $r = 2$. Then, recalling that $v(s) - u_0(s) \ge 0$,

$$f(t, v(s) - u_0(s)) + g_0(t) = 6(v(s) - u_0(s))^2 \le 6\|v\|^2 = 24 = 12r,$$

for all $v \in \mathcal{C} \cap \partial\Omega_1$. This shows that the first condition of (H3) is fulfilled.

It is easy to see that

$$\|u_0\| \le \max_{t\in[0,1]} \int_0^1 G(t,s) |g_0(s)|\, ds \le \max_{t\in[0,1]} \int_0^1 G(t,s)\, ds\, \|g_0\| = \frac{8}{3}.$$

Let now $\alpha = 1/4$ so that $C = 19/384$ and $\gamma = 4(\alpha - \alpha^2) = 3/4$. Then, for all $s \in [1/4, 3/4]$, $v \in \mathcal{C} \cap \partial\Omega_2$, where $R = 13$, we have

$$\begin{aligned}
f(t, v(s) - u_0(s)) + g_0(t) &= 6(v(s) - u_0(s))^2 \\
&\ge 6(\gamma\|v\| - \|u_0\|)^2 \\
&= 6\big(\frac{39}{4} - \frac{8}{3}\big)^2 = \frac{7225}{24} \\
&> \frac{4992}{19} = \frac{24R}{2 - 4\alpha^3 - 3\alpha}.
\end{aligned}$$

The above shows that the second part of (H_3) is also verified. Hence a solution v_0 exists in the cone and $2 \le \|v_0\| \le 13$.

The next result can be shown along the similar lines.

Theorem 3.2. *Assume that* (H1) *and* (H2) *hold. Suppose that there exist* $r, R > 0$ *such that* $\frac{1}{\mu} < r < R$, *where* $\mu > 0$ *satisfies* (2.9), (2.10), *and*

(H4) $\phi(R) \le 12R$ *and* $\psi(r) \ge \frac{24r}{2-4\alpha^3-3\alpha}$.

Then the boundary0value problem (1.1), (1.2) *has at least one positive solution.*

In conclusion of this paper presents a multiplicity result for (1.1), (1.2) which now is considered as a nonlinear eigenvalue problem. That is,

$$u'''(t) = \lambda f(t, u(t)), \quad 0 < t < 1, \tag{3.1}$$

subject to (1.2). The result including the assumptions and the method of proof echoes that of Ma [6], where a fourth order semipositone boundary-value problem with dependence on the first derivative was studied. The presence of the parameter $\lambda > 0$ provides an additional control on the growth of the right side. We introduce a new set of assumptions as follows:

(M1) there exists an interval $[\alpha, 1-\alpha] \subset (0,1)$ such that

$$\lim_{u\to\infty} \frac{f(t,u)}{u} = \infty,$$

uniformly in $[\alpha, 1-\alpha]$;

(M2) $f(t,0) > 0$, $t \in [0,1]$.

Our next result is a multiplicity criterion.

Theorem 3.3. *Assume that* (H1)*,* (H2)*,* (M1)*,* (M2) *hold. Then the boundary-value problem* (3.1)*,* (1.2) *has at least two positive solutions provided $\lambda > 0$ is small enough.*

Proof. We will construct open nonempty subsets $\Omega_i = \{v \in \mathcal{C} : \|v\| = R_i\}$, $i = 1, \dots, 4$. Now, we consider the operator

$$Tv(t) = \lambda \int_0^1 G(t,s) f_p(s, v(s) - \lambda u_0(s))\, ds,$$

where u_0 is the solution of $u''' = g_0$ subject to (1.2) and f_p as above. Let the $R_1 > 0$. Then

$$\|Tv\| = \max_{t\in[0,1]} \lambda \int_0^1 G(t,s) f_p(s, v(s) - \lambda u_0(s))\, ds \le \lambda L\phi(R_1) \le R_1$$

for all $v \in \mathcal{C} \cap \partial\Omega_1$, provided

$$\lambda \le \frac{L\phi(R_1)}{R_1}. \tag{3.2}$$

Let $v \in \mathcal{C} \cap \partial\Omega_2$, where $R_2 > R_1$. Then, by Lemma 2.1 with

$$\mu \max_{t\in[0,1]} \int_t^1 g_0(s)\, ds = 8.$$

Note that the equation in (M2) holds with f_p in place of f. Thus given $A > 0$, there exists $h \ge \frac{\gamma}{2} R_2$ such that $f_p(t,z) > Az$ for all $z \ge h$ and $t \in [\alpha, 1-\alpha]$. For every λ in (3.2), there exists a constant $A > 0$ such that

$$\frac{1}{2}\lambda C\gamma A \ge 1, \tag{3.3}$$

where C is given by by (2.6). For all $s \in [\alpha, 1-\alpha]$, we have

$$v(s) - \lambda u_0(s) \ge v(s) - \frac{\lambda}{\mu} q(s) = v(s) - \frac{\lambda}{\mu R_2} v(s) \ge \frac{1}{2} v(s) \ge \frac{\gamma}{2} R_2$$

provided

$$\lambda \le \frac{\mu R_2}{2}. \tag{3.4}$$

Hence

$$f_p(s, v(s) - \lambda u_0(s)) \ge A(v(s) - \lambda u_0(s)) \ge \frac{\gamma A}{2} R_2, \quad s \in [\alpha, 1-\alpha].$$

Then, by (3.3)), and recalling that $\kappa = 1$,

$$\begin{aligned}
\|Tv\| &= \max_{t\in[0,1]} \lambda \int_0^1 G(t,s) f_p(s, v(s) - \lambda u_0(s))\, ds \\
&\ge \lambda \max_{t\in[0,1]} \int_\alpha^{1-\alpha} q(t) G_0(s)\, ds\, \frac{\gamma A}{2} R_2 \\
&= \lambda \max_{t\in[0,1]} q(t) \int_\alpha^{1-\alpha} G_0(s)\, ds\, \frac{\gamma A}{2} R_2 \\
&= \lambda \kappa C \frac{\gamma A}{2} R_2 \ge R_2.
\end{aligned}$$

That is, $\|Tv\| \ge \|v\|$ for all $v \in \mathcal{C} \cap \partial\Omega_2$. As in Theorem 3.1, we have a solution v_1 such that $R_1 \le \|v_1\| \le R_2$ for every

$$0 < \lambda \le \lambda_0 = \min\big\{\frac{R_1}{L\phi(R_1)}, \frac{\mu R_2}{2}\big\}.$$

To make use of the assumption (M_2), we note that there exist $a, b > 0$ such that $f(t,z) \ge b$ for all $t \in [0,1]$ and $z \in [0,a]$ and introduce a "truncation" of f given by

$$f_t(t,z) = \begin{cases} f(t,z), & (t,z) \in [0,1] \times [0,a]), \\ f(t,a), & (t,z) \in [0,1] \times (a,\infty). \end{cases}$$

Consider now

$$u'''(t) = \lambda f_t(t, u(t)), \quad 0 < t < 1, \tag{3.5}$$

subject to (1.2). The operator, whose fixed point will be shown to be (a second) solution of (1.1), (1.2), is

$$Tv(s) = \lambda \int_0^1 G(t,s) f_t(s, v(s))\, ds.$$

Choose $R_3 < \min\{R_1, a\}$. Then, as in the proof of Theorem 3.1,

$$\|Tv\| \le \lambda L \phi(R_3),$$

where $\phi(R_3) = \max\{f(t,z) : t \in [0,1],\ z \in [0,R_3]\}$. So, if

$$\lambda < \min\big\{\frac{R_3}{L\phi(R_3)}, \lambda_0\big\}, \tag{3.6}$$

then $\|Tv\| \le \|v\|$ for all $v \in \mathcal{C} \cap \partial\Omega_3$. Choose λ according to (3.6). Since

$$\lim_{z\to 0^+} \frac{f_t(t,z)}{z} \ge \lim_{z\to 0^+} \frac{b}{z} = \infty$$

uniformly in $[0,1]$. Hence there exists $0 < R_4 < R_3$ such that

$$f_t(t,z) \ge Bz, \quad t \in [0,1],\ z \in [0, R_4],$$

where

$$\lambda BD \geq 1, \quad D = \max_{t\in[0,1]} \int_0^1 G(t,s)q(s)\,ds.$$

Then, for all $v \in \mathcal{C} \cap \partial\Omega_4$,

$$\begin{aligned}
\|Tv\| &= \max_{t\in[0,1]} \lambda \int_0^1 G(t,s) f_t(s, v(s))\,ds \\
&\geq \max_{t\in[0,1]} \lambda B \int_0^1 G(t,s)v(s)\,ds \\
&\geq \lambda B \max_{t\in[0,1]} \int_0^1 G(t,s)q(s)R_4\,ds \\
&= \lambda BDR_4 \\
&\geq \|v\|.
\end{aligned}$$

Thus, there exists a positive solution v_2 with $R_4 \leq \|v_2\| \leq R_3$ for every $\lambda > 0$ satisfying (3.6). Finally, since $R_3 < R_1$, the solutions are distinct. □

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References

[1] J. R. Graef, J. R. L. Webb; Third order boundary value problems with nonlocal boundary conditions, *Nonlinear Anal.* **71** (2009), 15421551.

[2] J. R. Graef, B. Yang; Positive solutions of a nonlinear third order eigenvalue problem, *Dynam. Systems Appl.* **15** (2006), no. 1, 97–110.

[3] G. Infante, P. Pietramala; A third order BVP subject to nonlinear boundary conditions, *Math. Bohem.* **135** (2010), 113121.

[4] M. A. Krasnosel'skiĭ; "Topological Methods in the Theory of Nonlinear Integral Equations", (English) Translated by A.H. Armstrong; A Pergamon Press Book, MacMillan, New York, 1964.

[5] K. Q. Lan; Eigenvalues of semi-positone Hammerstein integral equations and applications to boundary value problems, *Nonlinear Anal.* **71** (2009), 5979-5993.

[6] R. Ma; Multiple positive solutions for a semipositone fourth-order problem, *Hiroshima Math. J.* **33** (2013), 217-227.

[7] Y. Sun; Existence of triple positive solutions for a third-order three-point boundary-value problem, *J. Comp. Appl. Math.* **221** (2008), 194–201.

[8] Q. Yao; Positive solutions of a weak semipositone third-order three-point boundary value problem, *J. Math. Res. Exp.* **30** (2010), no. 1, 173–180.

[9] Q. Yao; Positive solutions of singular third-order three-point boundary value problems, *J. Math. Anal. Appl.* **354** (2009), 207–212.

[10] H. Yu, L. Haiyan, Y. Liu, Multiple positive solutions to third-order three-point singular semipositone boundary value problem, *Proc. Indian Acad. Sci. (Math. Sci.)* **114** (2004), no. 4, 409–422.

[11] L. Zhang, B. Sun, C. Xing; Existence of solutions to a third-order three-point boundary value problem, *Ann. Differential Equations* **28** (2012), no. 3, 368–372.

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